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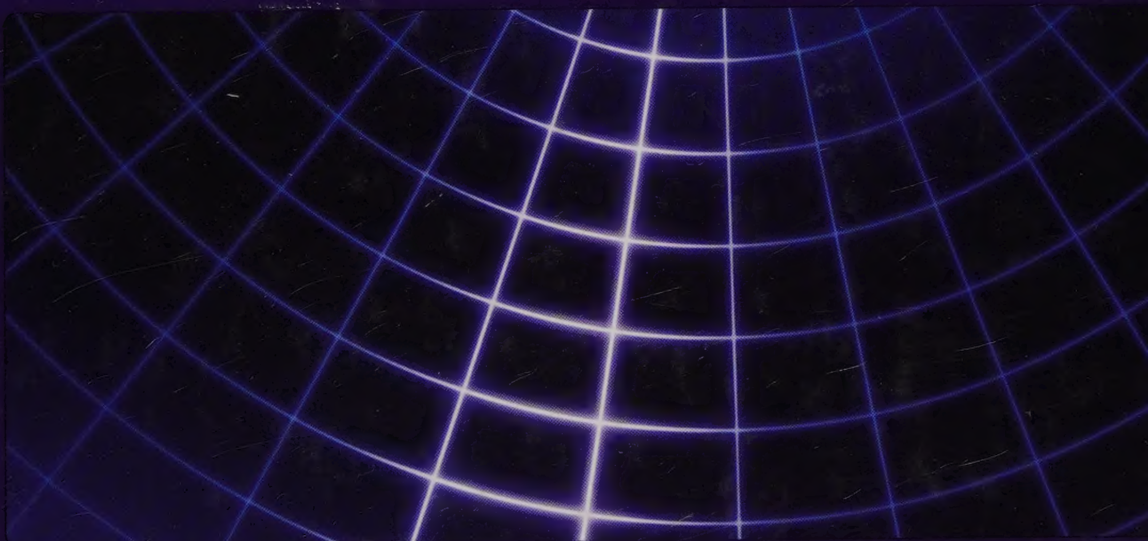

CHARLES G. DENLINGER

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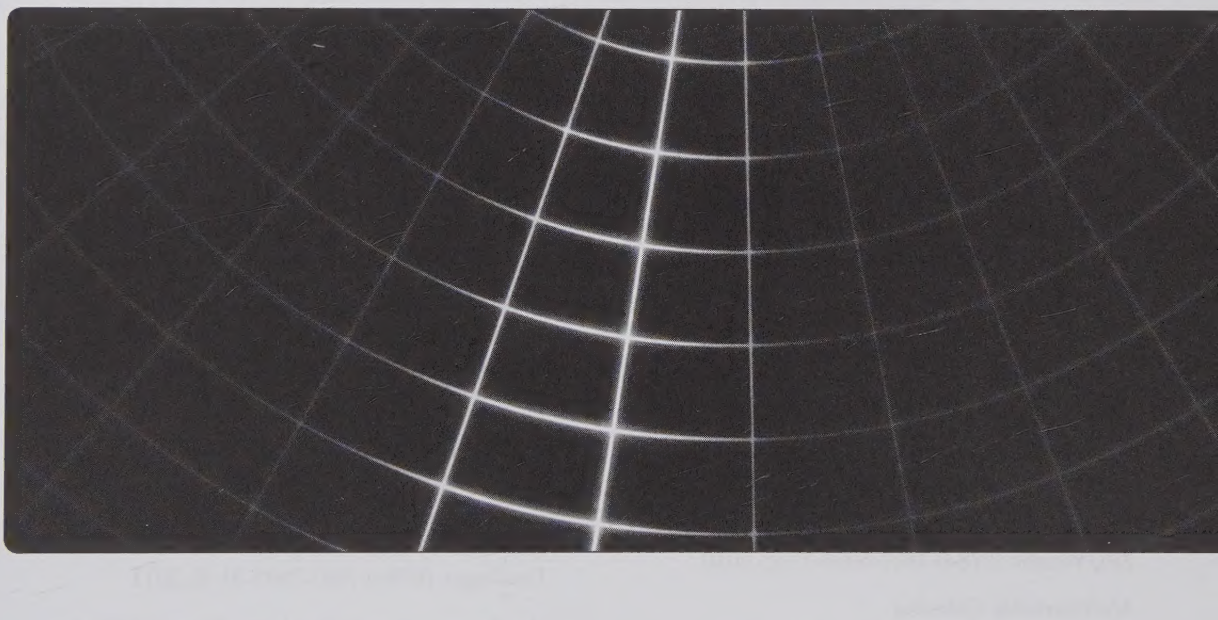

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Millersville University

JONES AND BARTLETT PUBLISHERS
Sudbury, Massachusetts
BOSTON TORONTO LONDON SINGAPORE

World Headquarters
Jones and Bartlett Publishers
40 Tall Pine Drive
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Cover and Title Page Design: Kristin E. Parker
Cover and Title Page Image: © Comstock Images/age fotostock
Printing and Binding: Malloy, Inc.
Cover Printing: John Pow Company

Library of Congress Cataloging-in-Publication Data
Denlinger, Charles G.
Elements of real analysis / Charles G. Denlinger.
p. cm.
Includes bibliographical references and index.
ISBN-13: 978-0-7637-7947-4 (hardcover)
ISBN-10: 0-7637-7947-4 (hardcover)
1. Mathematical analysis. 2. Functions of real variables. I. Title.
QA331.5.D456 2011
515—dc22

2009053784

6048
Printed in the United States of America
14 13 12 11 10 10 9 8 7 6 5 4 3 2 1

Dedicated to
Nick and Josh, and the memory of Tyler.

Contents

Preface

Elementary real analysis has earned its place as a core subject in the undergraduate mathematics curriculum. It deserves this status for several reasons. First, it develops the key concepts of calculus from a mature perspective. By laying a rigorous foundation for the theory of calculus and establishing its important results by logical deduction from a reasonably small set of assumptions, we organize the subject into a bona fide deductive system. In fact, we shall show that the entire subject follows from the properties of the real number system. Students generally have not considered this possibility; many find it surprising, even exciting, to be immersed in making this possibility a reality.

Second, because this course traces the powerful techniques of calculus back to their logical origins in the real number system, it is an essential part of the preparation of every mathematics teacher, particularly one who intends to teach calculus. Finally, a course in elementary real analysis opens the door to further study in real analysis, which is one of the cornerstone subjects in contemporary pure and applied mathematics. The concepts and techniques learned in this course will be explored further as part of any contemporary graduate-level program in mathematics. Indeed, the concepts of elementary real analysis belong in the repertoire of every mathematical scientist or teacher.

The motivation for writing yet another textbook in elementary real analysis comes from my many years of experience teaching the subject. My goal has been to make the subject accessible to as wide an audience as possible, without sacrificing content or rigor. At my university all students majoring in mathematics (including those preparing to be secondary school teachers) take one semester of elementary real analysis, usually in their junior or senior year. A second semester course is available as an elective. Students often report that the first real analysis course is one of the most challenging courses in the undergraduate mathematics curriculum. This is due partly to the nature of the material itself, and partly to the methodology that must be learned for success in the subject. To overcome this challenge the instructor and the textbook must meet students at their level. The elegant brevity of professional mathematical writing, so satisfying to the mathematician, is inappropriate for many students at this level. Unfortunately, the writing in most elementary real analysis books

tends toward this style. Many of my students find such textbooks indigestible, despite their excellence, so I decided to attempt to write one that my students would find more reader-friendly.

The resulting book is a straightforward, comprehensive presentation of the concepts and methodology of elementary real analysis, written at a level and in a style that can be understood by the typical undergraduate majoring in one of the mathematical sciences at a mainstream college or university. Its prerequisites are the usual core of courses in calculus and linear algebra, and at least one "transitional" course in which logic and proof techniques are taken seriously. The book is elementary in the sense that it stops short of the theory of Lebesgue measure and integral. It is also elementary in presentation. I have tried to keep the focus of the book on the central core of analysis. Material that is beyond this core is confined to "projects" or clearly labeled with an asterisk, "$*$."

This book is suitable for courses ranging in length from one term to one year. My colleagues and I have had considerable experience using preliminary versions of the text for a one-semester course covering the most essential topics (material not labeled with "$*$") in Chapters 1–7. Additional material has been added in these and later chapters to allow flexibility for instructors with differing priorities. For an outline of a specific one-semester course, see the "To the Instructor" section that follows. To cover the entire book at a reasonable pace a full year should be allowed.

FEATURES OF THIS TEXT

- Written at the undergraduate student's level; designed to be read. Exposition is often conversational, explaining both the details and the underlying motivation.

- Stresses the underlying ideas and unity of the subject; connects analysis with previously learned mathematics, and prepares students for graduate level analysis.

- Respects the deductive organization of mathematics. Analysis is developed as a deductive system based upon the axioms of the real number system.

- Proofs are written in a style appropriate for undergraduates to emulate in their homework rather than in the elegant style of the professional mathematician.

- Selected logical symbols (especially $\Rightarrow$, $\forall$, and $\exists$) are used frequently and consistently. They improve clarity of thought by calling attention to the presence of formal patterns of thinking that students might not otherwise recognize. A more complete rationale for the use of logical symbolism can be found in the "To the Instructor" section that follows.

- Efficiently organized; ideas introduced early are used later.
- Many illustrative examples.
- Generous exercise sets, including many routine exercises designed to develop student confidence.
- Every chapter begins with a rationale and suggestions for coverage.
- Every chapter contains clearly identified project-type exercises that advance student knowledge beyond the level of this book.
- Sequences are seen as a unifying theme, recurring as a useful tool throughout the course.
- Topological concepts and language are used extensively, because they help unify the subject.
- The Cantor set and Cantor's function are covered completely.
- Exponential and logarithm functions are defined rigorously in three separate contexts: in Section 5.6 in the context of continuous monotone functions; in Section 7.7 using the integral; and in Section 8.8 using infinite series. Similarly, trigonometric functions are defined and developed rigorously using the integral in Section 7.7 and using series in Section 8.8.
- Many surprising, even "pathological," examples appear throughout the text and exercises. Certain functions appear in chapter after chapter, forming a unifying chain of examples: Dirichlet-type functions, Thomae's function, the absolute value and related functions, and relatives of $\sin(1/x)$. Real analysis has historically derived much motivation from examples such as these.
- Many "applications" are shown to follow unexpectedly from the big ideas of the course. For example, the irrationality of e is derived from Taylor's theorem in Chapter 6; the irrationality of π and e^x for rational x is derived from the Fundamental Theorem of Calculus in Chapter 7.
- The core of Chapters 1–7 (material not labeled with a "$*$") can be learned by the typical student in a one-semester course. The "$*$" material and later chapters can be omitted in a one-term or one-semester course. Learning the "$*$" material will challenge the more talented student, and covering the entire book will require a second semester.
- Certain advanced topics are suitable for individual or group projects. They are clearly identified and accompanied by appropriate guidelines.
- A review of useful background material on logic, strategies of proof, sets, and functions appears in Appendices A and B.

AN OVERVIEW OF THE BOOK

Every student should read the "To the Student" section that follows. It provides some motivating rationale and some helpful words of advice. Instructors are advised to read "To the Instructor" for pedagogical rationale and suggestions for coverage.

As already suggested, this book develops elementary real analysis as a deductive system founded upon the axioms of the real number system. Thus, familiarity with the principles of logic and proof techniques will be helpful. Many universities provide a "transitional" course for this purpose. Appendix A provides a summary of the essential rules and notation of logic, and may be consulted as needed. Appendix B reviews the mathematics of sets and functions.

Chapter 1 is included here because without this (or equivalent) material it would be impossible to rigorously develop the subject of analysis. Nevertheless, much of it is not analysis; it is a *prelude* to analysis. Sections 1.1–1.4 and 1.7 can be skimmed lightly by those wanting to avoid getting bogged down in these preliminary issues. Sections 1.5 and 1.6 are the only sections of Chapter 1 that must be covered thoroughly. They discuss the Archimedean and completeness properties, which are the first topics with the characteristic flavor of analysis.

Chapter 2, on sequences, serves as the entrance into real analysis. It is of crucial importance in our development of the subject. The key concept of limit is introduced early and used extensively. The student learns the methodology of "epsilonics" by working through many examples, from concrete to abstract. I believe that the concept of limit is best learned in the context of sequences. In fact, throughout the remainder of the book the reader will find "sequential criteria" for various other concepts: cluster points, closed sets, compactness, denseness, limits of functions, continuity, uniform continuity, and even for integrals. Thus, sequences will serve as a unifying theme running through the course. Chapter 2 contains several powerful theorems, such as the monotone convergence theorem, Cantor's nested intervals theorem, the Bolzano-Weierstrass theorem for sequences, and Cauchy's convergence criterion. Countable and uncountable sets are also discussed in this chapter, since a countable set is simply the range of a sequence. Chapter 2 ends with an optional section on upper and lower limits.

Chapter 3 contains just enough topology of the real number system to allow us to convey the results of elementary real analysis with the force and clarity of contemporary topological language. Excessive generality and unnecessary vocabulary are avoided. The Cantor set and compactness are each given thorough coverage, but in optional sections. To get through the remaining chapters of this text the only definition of a compact set that one needs[1] is that it is closed and bounded.

---

1. Except in proving that Lebesgue's criterion is a sufficient condition for Riemann integrability (Theorem 7.9.7).

Chapter 4 focuses on limits of functions. The ε-δ techniques are discussed early and used extensively. The sequential criterion is an integral part of the presentation. One-sided limits and the use of infinity in limits are given full treatment, but the instructor may want to skim lightly over these sections, which the average student can handle as reading assignments.

Chapter 5 on continuous functions is a meaty chapter. The ε-δ techniques are discussed in full detail, and the methodology of sequences is shown to be a powerful supporting tool. One-sided continuity, discontinuities, and implications for monotone functions are discussed. Special emphasis is given to two powerful results about continuous functions: the continuous image of a compact set is compact, and the continuous image of an interval is an interval. Uniform continuity is treated in Section 5.4, but in a one-semester course this topic can be postponed until it is needed in Chapter 7 for proving that continuous functions are Riemann integrable.

A novel feature of this book is the (optional) "early" treatment of exponential and logarithm functions, in line with the trend to introduce these functions early in the calculus sequence. Thus, Section 5.6 provides a rigorous development of these functions and is somewhat novel in this regard. In order to provide the necessary background for these functions I felt it necessary to precede this material with a section on monotone functions, continuity, and inverses. This, in turn, provided the perfect opportunity to discuss Cantor's function. Chapter 5 ends with the Baire category theorem, and a proof that the set of discontinuities of a real function must be an F_σ set.

Chapter 6 is a comprehensive treatment of differentiability and the derivative. All the usual rules for differentiation are proved rigorously. This includes rules for general power, exponential, and logarithm functions, which were introduced in Chapter 5. The relationship between the derivative, monotonicity, and extreme values is explored carefully, as is the intermediate value property of derivatives. Rolle's theorem, the mean value theorem, and many of their applications, are given full attention. Taylor's theorem is seen as a mean-value-type theorem, and L'Hôpital's rule is derived from Cauchy's mean value theorem.

In Chapter 7 the Riemann integral is defined using the Darboux sum approach. It is my view that, for students at this level, this is still the most appropriate integral and its most natural definition. Riemann's criterion is proved and used to establish the integrability of monotone and continuous functions. The integral is shown to be a limit of Riemann sums, and the first and second fundamental theorems of calculus are proved. A novel feature is the proof that regular partitions are sufficient for Riemann integrability. In an optional section the exponential, logarithmic, and trigonometric functions are defined and developed using the integral. In the final (optional) section, Lebesgue's criterion for Riemann integrability is proved.

Chapter 8 is a standard presentation of the theory of infinite series of real numbers, including Cauchy product series, the theory of power series, real analytic functions, and a little on double series. All the customary convergence

tests are discussed in detail, as well as Raabe's test, Dirichlet's test, and Abel's test. A little more attention is given to the consequences of absolute convergence than is customary. The chapter concludes with a project that develops the elementary functions from infinite series.

Chapter 9 focuses on the notion of convergence of sequences and series of functions. Uniform convergence is contrasted with pointwise convergence, and the implications of uniform convergence in calculus are explored. By discussing the notion of convergence of functions in the setting of normed vector spaces we prepare the student for further courses in modern analysis. The chapter culminates in proofs of two famous results of Weierstrass: the existence of an everywhere continuous, nowhere differentiable function, and his celebrated "polynomial approximation" theorem. An historical appreciation of Weierstrass himself is included. A concluding section points students toward interesting topics for further study.

THE ROLE OF RIGOR

A few words need to be said about the role of rigor in this course. While intuition is a powerful motivator of progress in mathematics, it is not reliable as a guarantor of truth. Indeed, many of the results of analysis run counter to a beginning student's intuition. Some notable examples are the countability of the rational numbers versus the uncountability of the irrational numbers; the existence of functions that are continuous on the irrational numbers and discontinuous on the rational numbers, versus the impossibility of functions that are continuous on the rational numbers and discontinuous on the irrational numbers; and the existence of functions that are continuous everywhere and differentiable nowhere. Contrary to the suspicions of many students, rigor plays a very practical, even indispensable, role in analysis.

ACKNOWLEDGMENTS

I am very grateful to Millersville University for providing support of various kinds during the gestation period of this book, but especially for a sabbatical leave during which the bulk of Chapters 1–7 were completed. I owe a special debt to my colleagues in the mathematics department at Millersville for their encouragement and moral support. Dr. Robert Smith, then Chairman and now Dean, could not have been more supportive. Special thanks go to Drs. Ximena Catepillan and Antonia Cardwell, who class-tested the manuscript as it evolved, and provided valuable critique.

My most heartfelt appreciation goes to the hundreds of students who studied real analysis in my classes over the years. Their enthusiasm for learning real analysis and their willingness to work hard toward that end have been a source of deep satisfaction. Unfortunately, I shall have to leave them unnamed, as to name a few of them would be an injustice to those not named. I have benefited

from several outstanding mathematics teachers and mentors who have had lasting impact on my life, especially E. Mae Shenk of Hempfield High School, Bruce M. Tyndall of Elizabethtown College, and John J. Masterson of Michigan State University.

I am deeply indebted to the many anonymous reviewers who provided valuable feedback while this book was under development. Their thoughtful and candid criticism was most helpful in removing some serious flaws and in strengthening the book. I would also like to thank Amy Rose, Production Director at Jones and Bartlett Publishers, and her outstanding staff for the superb work they have done with my manuscript. It has been a pleasure to work with them.

A special thanks to Gloria, without whose patient understanding and emotional support it would not have been possible to complete this project.

REFERENCING EXERCISES BY NUMBER

Throughout the book, references to Exercises are given in modified decimal form. For example,

- "Exercise 5.4.11" refers to Exercise 11 in Section 5.4.
- "Exercise 1.6-A.5" refers to Exercise 5 in Exercise Set 1.6-A.
- "Exercise B.2.9" refers to Exercise 9 in Section 2 of Appendix B.

To the Student

OUR SUBJECT IS CALLED "ANALYSIS"

Analysis is the mathematical subject that underlies and extends the theory of calculus. It is a deep and extensive subject that has been under development for centuries. It has itself evolved into a number of distinct fields of study, two of which may be called *real analysis* and *complex analysis*, according to whether the underlying number system is taken to be the real number system or the complex number system. In this course we focus on *real* analysis, although most of what we discuss finds use in all areas of analysis.

Because analysis has its origins in calculus, it will look somewhat familiar to you. However, you will be exploring the subject at a level much deeper than that of your introductory calculus courses. This course will demand careful, critical thought. Indeed, it is designed to help you gain what mathematicians call "mathematical maturity."

FOUNDATIONS DO MATTER

You are already familiar with some of the powerful results of analysis; you have seen them and used them in your calculus courses. However, you most likely did not prove all of these results rigorously from a small set of beginning assumptions. Consequently, your understanding of why they are true may be somewhat clouded in mystery. How can you be sure that all this theory is really "true?" In what sense can these results be proved?

We cannot answer these questions by looking "forward." We must look backward, and trace the subject back to its logical (but not necessarily historical[2]) origins. After we have laid a secure foundation for the subject, and have reconstructed its core framework by rigorous logical deduction, we will have a fresh understanding of the analysis we once naively thought we knew. We will also push forward toward new and deeper results.

Thus, the course brings a shift of emphasis: from the development of mathematical techniques and applications, to a critique of the subject itself. We shall

2. For insights on the historical development of analysis, consult references [34], [37], [56], [57], [58], [59], [60], [61], [66], [75], [88], and [127].

take a critical, even skeptical approach. Mere passive acceptance will not do! In fact, we will be so critical that we will not consider any statement of analysis to be fully reliable until we have a firm justification of it (which we call a "proof"). For this justification, we are forced to look backward to the foundations of the subject. In one sense, you are asked to forget what you learned about calculus (if you haven't already) and build upon a new foundation.

WHY PROOFS ARE IMPORTANT

"Proof" can be an intimidating word to many students of mathematics who would rather just be told what is true. The reason why proof is so important in mathematics is found in the very nature of "mathematical truth" itself, as understood in the western intellectual tradition. In this tradition, a body of mathematics is not just a collection of disconnected "facts" that are accepted because they seem to be true. Rather, these facts must be connected together and organized according to the "deductive method." Mathematicians go to great lengths to isolate some of the facts that they can regard as basic (coming at the very beginning) and then go to even greater lengths to show that all the remaining facts can be derived from the basic ones by the process of logical deduction. The "facts" that come at the beginning are then really assumptions (axioms). The remaining facts, as they are deduced one-by-one from these assumptions, are called "theorems." The process of deducing (or deriving) a theorem is called "proof."

What, then, does a proof prove? The answer is not as obvious as it may seem at first. A theorem is ultimately derived from the axioms set forth at the beginning of a mathematical subject. Thus, the truth of a theorem is really contingent upon the truth of the axioms. If the axioms are all true, then any theorem that is derived from them by valid logical deduction must also be true. But the proof of a theorem cannot assure us that the axioms upon which the proof rests are themselves true. Thus, a "proof" does not guarantee that a theorem is true. A proof guarantees only that **if** the axioms are all true, then the theorem is true. In other words, *a proof of a theorem proves that the axioms are sufficiently strong to guarantee the theorem.* Thus, a theorem is really a statement about the axioms. For this reason axioms serve as the foundation of a mathematical subject.

OUR PLAN OF ATTACK

In Chapter 1 we set forth a few basic assumptions from which the entire subject of analysis can be derived by the process of logical deduction. In later chapters we carry out that process as far as the constraints of time permit. We reach a natural culmination point with the "fundamental theorem of calculus," in Chapter 7. Readers who continue to the end of the book will reach another culmination point in Chapter 9, exploring some fascinating consequences of uniform convergence.

You will recall that calculus is built upon the concept of limit. Thus, Chapter 2 takes up that idea. Along with many other analysts, I believe that limits are best understood and appreciated if they are introduced in the context of infinite sequences. You will probably be quite amazed at the power of this approach to limits. In fact, you might come to agree with me that Chapter 2 is the key chapter in the whole course. Chapters 2 and 3 contain many important ideas upon which the whole subject of analysis depends.

In Chapter 4 we take up limits of functions, and in Chapter 5 we define and derive deep results about the related notion of continuous functions. The notion of continuity is subtle yet far more powerful than you might expect. Chapter 6 at last looks like "calculus;" in it the familiar rules and properties of differentiable functions are derived using the techniques of limits. In particular, we see why the mean value theorem is so important, and we see Taylor's Theorem in its proper context as a type of mean value theorem. All along, we continue to rely on the techniques developed in Chapter 2.

Chapter 7 is a fresh start. In it we develop the Riemann integral from scratch, using only the properties of the real number system developed in Chapter 1 until we come to the point where we wish to relate it to the previously learned concepts of continuity and the derivative. The fundamental theorem of calculus is a natural high point, tying together the major themes of the course, but once again supported by the basic concepts of limits established in Chapter 2.

In Chapter 8 you will renew your acquaintance with infinite series, a topic you previously studied in calculus. This time you will examine series in greater depth and hopefully will find them more interesting and useful than you had previously believed. We will uncover some quite interesting facts and techniques involving series.

In Chapter 9 functions are considered as objects themselves, or points in a "function space." Considering sequences and series of functions (rather than of numbers) leads to many powerful results in higher level analysis. We conclude this chapter with two especially intriguing results: the existence of continuous, nowhere differentiable functions, and the fact that every continuous function can be approximated with any prescribed accuracy by a polynomial.

REVIEW MATERIAL IN APPENDICES

The statements and proofs of analysis often involve a high degree of complexity and subtlety. You will find these statements and proofs much less intimidating if you have a working knowledge of the principles of formal logic. Familiarity with the rules of logic will minimize confusion and improve the clarity of the statements of analysis. Many students acquire this familiarity in a "transitional" or proof-oriented mathematics course.

Some of the actual symbolism of "symbolic logic" is used in this book. The purpose of this symbolism is not to confuse, but to clarify mathematical

statements and proofs by calling the reader's attention to the presence of formal logical patterns. In particular, the symbols $\Rightarrow$ ("implies"), $\forall$ ("for all"), $\exists$ ("there exists"), and $\ni$ ("such that") are used frequently for this purpose. You are advised to get used to these symbols as soon as possible.

You can review the basic principles of logic in the first two sections of Appendix A. If you are not familiar with this material, time spent learning it will repay you richly. The third section of Appendix A outlines some common strategies of proof.

Nearly all significant results of analysis are expressed in terms of sets and/or functions. The facts you need to know about them are reviewed in Appendix B, which you can consult as needed.

WORDS OF ADVICE FROM THE AUTHOR: SEVEN RULES FOR SUCCESS IN THIS COURSE

Elementary real analysis is not an easy subject. In fact, it is one of the most challenging courses in the undergraduate curriculum. While calculus is one of the most applicable areas of mathematics, analysis is highly theoretical in spirit and makes uncompromising demands for rigor.

This book is student-oriented. It is designed to be readable, and therefore to be read. It represents my best attempt to make the subject as understandable as possible without compromising rigor. I offer these words of advice to those who really want to succeed:

1. Read the book, word-by-word, page-by-page, except where your instructor may chart an alternative path for you. Do not skip over the reading and head straight for the exercises, as you might have done in your calculus courses! If you do, you will miss much of the course.

2. Some of the material is marked with an asterisk, "$*$." Let your instructor decide how much of that to cover.

3. Study the proofs. Tear them apart and examine them critically until you are sure that you understand them completely. Ask for help where you do not understand. No one can claim to understand analysis who does not understand its theorems and proofs. They serve as models of the kind of thinking required to develop new results in analysis. Your instructor may require that you learn some of the proofs well enough to explain them to your classmates or to do them on examinations.

4. Make sure you understand the definitions. Learn (even memorize) them! This is a far more serious issue than most students realize. Definitions are the place to start when proving results about a new concept.

5. Learning mathematics is not a spectator sport! You learn mathematics by *doing* mathematics. You cannot expect to learn analysis by reading

this book as you would read a newspaper or a novel. You must "read" this book with pencil and paper. Write out the key steps yourself as you progress through the text, working out the details as you go. (This may be called "active reading.") It will keep your attention focused and facilitate your learning.

6. Treat the exercises sets as a continuation of the learning begun in class or in the text. They are carefully graded so that you learn as you progress through an exercise set. There are plenty of exercises; generally you can get along fine by doing only every other one and omitting some of the later ones. If you can't get anywhere on an exercise after much effort, find out from someone else how to do it and then work it through several times until you can do it by yourself, without help.

7. In mathematics, as in other branches of human knowledge, truth is communicated in sentences. Even in mathematics, a sentence must have a subject and a predicate and obey all the laws of grammar. For example, the subject of the sentence $x^2 + 3x - 7 = 11$ is "x" and the predicate is "$=$." Please remember, when you write your own proofs or solutions of exercises, to express your ideas clearly and in complete sentences. Sloppy writing is often a sign of sloppy thinking. An idea poorly expressed is often poorly understood.

To the Instructor

When Newton and Leibniz[3] invented the calculus in the late seventeenth century, they did not use delta-epsilon proofs. It took a hundred and fifty years to develop them. ... [It] is no wonder that a modern student finds the rigorous basis of calculus difficult.

Delta-epsilon proofs are first found in the works of Augustin-Louis Cauchy (1789–1867).

Cauchy, followed by Riemann and Weierstrass,[4] gave the calculus a rigorous basis.

—Judith V. Grabiner [56]

Many reasons can be given to explain why the standard introductory real analysis course has come to assume its present form, with its reliance upon rigorous proofs based on the ε-δ definition of limit. As you know, analysis had a brilliant history of remarkable achievements for almost two centuries without the benefit of a rigorous foundation. Mathematicians such as the Bernoullis, Euler, Lagrange, Laplace, Fourier, and Gauss[5] made many brilliant discoveries without seeming to need rigor as defined by today's standards. The development of the theory of Fourier series is often cited as one of the principal motivators of the change in attitude toward rigor. Careful and critical study of Fourier series required a more rigorous understanding of such basic concepts as function, limit, continuity, and convergence. It was also in this connection that a proper definition of the definite integral was needed, which led to a rigorous development of the Riemann integral, and ultimately to the Lebesgue integral and still other generalized integrals. Rigor thus came to be seen as essential to the core of analysis, not just as an afterthought.

3. Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716).

4. Karl Weierstrass (1815–1897) and Bernard Riemann (1826–1866).

5. Jacob and Johann Bernoulli (1654–1705 and 1667–1748), Leonard Euler (1707–1783), Joseph Louis Lagrange (1736–1863), Pierre Simon de Laplace (1749–1827), Joseph Fourier (1768–1830), and Carl Friedrich Gauss (1777–1855).

In her essay [56] quoted at the beginning of this section, Grabiner points to another factor contributing to the increased attention paid to rigor in the works of Cauchy, Riemann, Weierstrass, and their contemporaries. That was "the need to teach." As the number of royal courts with sufficient affluence to employ resident mathematicians declined, the number of mathematicians was on the increase. Forced to look elsewhere for employment, mathematicians found that they were needed to teach in technical high schools and universities. In Grabiner's words, "Teaching forces one's attention to basic questions," and "provide[s] a catalyst for the crystallization of the foundations of the calculus .... "

In our own times, real analysis is being taught to an ever-widening audience. Since calculus is universally taught in American high schools, it is now crucially important that teachers certified to teach secondary school mathematics have a firm understanding of at least the elements of real analysis. Thus many colleges and universities (such as my own) are requiring all their mathematics majors to take a course in elementary real analysis. This book is written to fulfill the need for a textbook appropriate for such a course.

Teaching elementary real analysis can be quite challenging. Typical undergraduates find the concepts and techniques of analysis difficult. The pleasure you find in the subject, and the enthusiasm with which you teach it, may be met by somewhat disappointing results. This book was written out of my desire to improve those results by making the material more understandable to the average student. I am assuming that your typical student is not too much different from mine.

As teachers we need to recognize that in order for students to grow they must start from where they are. Mathematical maturity will come, but rarely does it appear full-blown from the start. This book takes a more patient approach. I ask for your patience too; I believe you will like the results.

STYLE OF PROOFS

Beauty is in the eye of the beholder! That includes mathematical beauty. What the professional mathematician finds so satisfying in an elegant proof may be inaccessible to a novice undergraduate. A student may be left completely in the dark by a proof presented in the slick style of a professional mathematician. Students need to see proofs presented in a style that they can understand and emulate in their own proofs. In this way they are much more likely to become comfortable reading and constructing proofs.

Because proofs in this book are written for digestion by students, you might initially find them somewhat tedious and over-detailed. They are designed to facilitate gradual but steady growth, which is a more reasonable expectation than instantaneous maturation. Features of this style include:

- The hypotheses of a theorem are stated at the beginning of each proof. This may seem redundant, as they are already stated in the theorem itself.

But it causes the student to recognize the importance of the hypotheses, and to rethink what they actually say. Sometimes alternative forms of the hypotheses are more useful than the original forms.

- Proofs are *written*, using proper English sentence structure, grammar, and punctuation. I feel we must be responsible to our students for modeling proper communication style.
- Even though the proofs are in paragraph form, reasons for each "step" are given as often as practical. Here, judgment is exercised to avoid choking a proof with unnecessary detail.
- The end of a proof is clearly indicated with the customary symbol, ■. (We frequently use the symbol □ to denote the end of an example or thought, to set it off from the beginning of a new thought.)
- The sophistication of proofs escalates as the book progresses. Proofs later in the book tend to be more streamlined than those coming earlier.

USE OF LOGICAL SYMBOLS

Formal logical symbolism makes its first appearance in Chapter 1 and is used consistently after that. The justifications for this are historical, pedagogical, and practical—stemming from calculus' need to make use of the concept of "infinity." Calculus as we know it could not exist without effectively harnessing this concept. Faced with the task of making the concept respectable, real analysts of the past two centuries came up with a brilliant two-step solution. First, nineteenth century mathematicians discovered how to treat "infinity" with complete precision using only the tools of finite mathematics: by using inequalities and logical quantifiers. Then, by developing symbols for the logical operations and rules for their use, twentieth century mathematicians made it possible to invoke the relevant logical principles systematically and clearly, without the "fog" that often beclouds the "words only" approach. My own intellectual development as an undergraduate was profoundly affected by my first encounter with elementary symbolic logic. I was overwhelmed by its power and economy. My eyes were opened to how mathematical ideas can be expressed and how proofs flow.

Thus, you can understand why I resort to logical symbols in putting across the concepts of analysis. In particular, I find it beneficial to use symbols for the two *logical connectives*, $\Rightarrow$ and $\Leftrightarrow$, and the two *quantifiers*, $\forall$ and $\exists$. A correct understanding of "and," "or," and DeMorgan's laws is also very helpful, but we do not need to use their symbols. Quantifiers are ubiquitous in analysis, and it is unthinkable to try to learn the subject without coming to grips with them in some way. For example, the definition of limit involves three quantifiers: $\forall$, $\exists$, and a second $\forall$. I often leave the third quantifier unstated in order to give the reader a break, but its presence must be recognized when negating that

definition. Its presence is crucial, for example, in the proof of Theorem 4.1.9, the sequential criterion for limits of functions.

My students have never complained that the use of symbols impedes their progress in any way. On the contrary, students learn them easily and put them to use quickly in mastering complex concepts and proofs. A brief exposition of the logical concepts and symbols used in this book is found in Appendix A. Of crucial importance is a secure understanding of the negation of the various logical forms, especially the negation of quantified statements.

Finally, the use of logical symbols is not an excuse for sloppy writing or incorrect grammar. In fact, these symbols are subject to the usual rules of grammar and syntax. Used properly, they reinforce these rules. As instructors, we cannot condone careless use of symbols any more than careless thinking of any kind.

WARNING

While this book began life as a textbook for a one-semester course, it contains much more material than can be covered in one semester. The additional material is of two types: enrichment topics included in Chapters 1–7 to provide optional extended study or projects to challenge more advanced students, and additional chapters or sections intended to provide enough material for a second course. In teaching a one-semester course it is important to avoid getting bogged down in the optional material in the earlier chapters. For example, while the entire Chapter 1 is essential to the logical development of the Real Number system, only Sections 1.5 and 1.6 must be covered in detail; the others can be skimmed.

The book incorporates several measures to ensure instructors choose material appropriate for a one-semester course. Each chapter, and some sections, begin with comments in a box, giving advice on which sections must be covered in detail and which can be skimmed or omitted. Theorems, examples, results, proofs, and exercises that are optional are clearly marked with an asterisk, "$*$." In addition, the following table suggests one way to cover the core material of Chapters 1–7 in one semester.

SUGGESTED ONE-SEMESTER COURSE

The material identified in the table below[6] as "core" corresponds to my own one-semester course. Where a number appears in both columns, the number on the right estimates the amount of additional time that may be required to completely cover the optional material.

6. Based on the author's experience teaching a one-semester course meeting three 50-minute periods per week.

Section(s)	Description	Number of Days (Core)	(Optional)
1.1–1.4	Fields; ordered fields; natural and rational nos.	1.5	2.5
1.5	Archimedean fields; density of rationals.	1	
1.6	Suprema/infima; completeness property.	1.5	.5
1.7	Existence and uniqueness of the real number field.	0	1
2.1	Limits of sequences; the basics.	1.5	
2.2	Algebra of limits of sequences.	2	.5
2.3	Inequalities and limits.	1	
2.4	Divergence to ∞.	.5	.5
2.5	Monotone sequences.	3	
2.6	Subsequences; cluster points of sequences.	1	1
2.7	Cauchy sequences.	1	1
2.8	Countable and uncountable sets.	0	2
2.9	Upper and lower limits.	0	2
3.1	Neighborhoods; open sets.	1.5	
3.2	Closed sets; cluster points; closure of a set.	1.5	
3.3	Compact sets.	0	2–3
3.4	The Cantor set; Cantor-like sets.	0	2
4.1	Limits of functions: definition and ε-δ proofs.	1.5	
4.2	Algebra of limits of functions.	2	.5
4.3	One-sided limits.	.5	.5
4.4	Infinity in limits.	1	1
5.1	Continuity: definition and ε-δ proofs.	1.5	
5.2	Discontinuities; monotone functions.	.5	1.5
5.3	Continuity on compact sets and intervals.	2	.5
5.4	Uniform continuity (can be postponed to Sect 7.2).	1	1
5.5	Monotonicity, continuity, inverses; Cantor function.	0	2–3
5.6	Exponential, powers, and logarithms	0	2
5.7	Sets of points of discontinuity.	0	2
6.1	The derivative; differentiability.	1	
6.2	Rules for differentiation.	1	1
6.3	Relative extrema; monotone functions.	1	1
6.4	Mean value-type theorems.	1.5	1
6.5	Taylor's theorem.	1.5	1
6.6	L'Hôpital's rules.	0	2

7.1	Refresher on suprema & infima.	0	.5
7.2	Riemann integral defined via Darboux sums.	2	
7.3	The integral as a limit of Riemann sums.	1	1
7.4	Basic existence and additivity theorems.	1	1–2
7.5	Algebraic properties of the integral.	1	1
7.6	Fundamental theorem of Calculus.	2	1
7.7	Elementary transcendental functions.	0	2–3
7.8	Improper integrals.	0	1
7.9	Lebesgue's criterion for Riemann integrability.	0	1–2
	Subtotal:	40	42.5–48.5
	Leeway:	2	
	TOTAL:	**42**	

Chapter 1

The Real Number System

Sections 1.1–1.4 are optional, containing background on ordered fields and the rational numbers. Students "know" these "facts" but may not have proved them unless they have had a course in abstract algebra. They can be outlined in one day or covered completely in four. Sections 1.5 and 1.6 on the Archimedean and completeness properties, respectively, are essential. Section 1.7 is an optional section, outlining a proof of the uniqueness of the complete ordered field.

We begin our study of analysis with an investigation of the real number system because, ultimately, the entire subject of real analysis rests upon this system. Every result presented in this course is derived from the properties of the real number system. This may seem like an exaggerated claim to you. If so, I ask for your patience. Before the end of the course you will understand that this is no exaggeration.

There is no universal agreement on what constitutes the best approach to the study of the real number system. There are at least three popular viewpoints. The "constructive" view insists that any proper description of the real number system must start with the most primitive number system of all, the *natural number* system, and *construct* the real number system by a strictly rigorous (and tedious) process. A second, "descriptive," view suggests that it is better to begin directly with the real number system itself, listing its fundamental axioms and deriving all of its properties from them. A third, "pragmatic," view downplays the importance of a rigorous development of the real number system, holding that it is enough to cover only those aspects of the number system that seem especially interesting or important in light of their usefulness in analysis. In this book we take the second, descriptive approach. It is both

efficient and honest. The constructive approach would detain us too long, and the pragmatic approach is neither rigorous nor intellectually honest. Readers interested in the constructive approach may consult references[1] [10], [11], [25], [37], [44], [63], [74], [82], [96], [117], and [129] listed in the Bibliography.

Much of the methodology in Sections 1.1–1.4 is not really characteristic of analysis. It more closely resembles algebra than analysis. This material appears here because it provides us with a rigorous starting point. You are perhaps already familiar with most of these results, having covered them in a course in proof theory or abstract algebra. You may only need to review such results briefly. But the methods used in the proofs here are important enough for you to pay careful attention to, at least once in your life. Remember, we are laying a foundation that must be secure enough to allow us to *prove* all the results of calculus.

LOGICAL SYMBOLS USED IN THIS TEXT

Certain principles of logic are used frequently in real analysis. I have found that students who recognize and understand these principles learn the subject more easily than those who don't. To help the reader be aware of the presence of important logical patterns we use standard logical symbols. In particular, we shall make frequent use of the following symbols:

- **Implication:** $P \Rightarrow Q$

  If P and Q are statements, then "$P \Rightarrow Q$" is the statement "P implies Q," which is equivalent to each of the following:

if P, then Q	if P, Q
Q if P	P only if Q

- **Bi-implication:** $P \Leftrightarrow Q$ (often written "P iff Q")

  If P and Q are statements, then "$P \Leftrightarrow Q$" is the statement "P if and only if Q," which is equivalent to the conjunction "$P \Rightarrow Q$ and $Q \Rightarrow P$."

- **Universal Quantifier:** $\forall x \in A,\ P(x)$

  If $P(x)$ is a statement about x, and A is a set, then "$\forall x \in A,\ P(x)$" is the statement, "For all x in the set A, $P(x)$ is true." If the set A is understood without writing it, we sometimes write simply "$\forall x,\ P(x)$." Variations of this usage will be self-explanatory when we use them.

- **Existential Quantifier:** $\exists\, x \in A \ni P(x)$

  If $P(x)$ is a statement about x, and A is a set, then "$\exists\, x \in A \ni P(x)$" is the statement, "There exists (at least one) x in the set A such that $P(x)$ is true." Variations of this usage will be self-explanatory.

1. The numbers in square brackets refer to entries in the Bibliography, which follows Appendix C.

A more complete discussion of these logical symbols and others, along with important rules for their use, are found in Appendix A. For rationale explaining their importance in real analysis, see "Use of Logical Symbols" in the Preface. Appendix A includes much additional material on logic and methods of proof, while Appendix B reviews the concepts and notation of sets and functions.

1.1 The Field Properties

The real number system can be completely described in four words: it is "**THE COMPLETE ORDERED FIELD.**" Each of these four words deserves careful examination. We shall do so in reverse order. We begin by defining the word "field" in this section, and the remaining words in subsequent sections.

The real number system is first of all a "field." This, of course, requires a formal definition.

Definition 1.1.1 A "**field**" is a set F together with two binary operations, denoted "+" (called addition) and "$\cdot$" (called multiplication), which behave according to the following axioms:

ADDITION AXIOMS:

(A0) $\forall x, y \in F$, $\exists$ unique element $x + y \in F$ called the "sum" of x and y.
(A1) $\forall x, y \in F$, $x + y = y + x$. (commutative property of +)
(A2) $\forall x, y, z \in F$, $x + (y + z) = (x + y) + z$. (associative property of +)
(A3) $\exists$ element $0 \in F \ni \forall x \in F$, $x + 0 = x$. (existence of a zero element)
(A4) $\forall x \in F$, $\exists u \in F \ni x + u = 0$. (existence of additive inverses)

MULTIPLICATION AXIOMS:

(M0) $\forall x, y \in F$, $\exists$ unique element $x \cdot y \in F$ called the "product" of x and y.
(M1) $\forall x, y \in F$, $x \cdot y = y \cdot x$. (commutative property of $\cdot$)
(M2) $\forall x, y, z \in F$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. (associative property of $\cdot$)
(M3) $\exists$ element $1 \in F \ni 1 \neq 0$ and $\forall x \in F$, $x \cdot 1 = x$. (existence of an identity element)
(M4) $\forall x \in F \ni x \neq 0$, $\exists u \in F \ni x \cdot u = 1$. (existence of a multiplicative inverse of every nonzero element)

DISTRIBUTIVE AXIOM:

(D) $\forall x, y, z \in F$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$. (distributive property)

Comment on Multiplicative Notation: The symbol "$\cdot$" is usually not written. As in elementary algebra, we usually write xy instead of $x \cdot y$.

Some number systems are fields, but many are not. The following exercises will help you see the difference.

EXERCISE SET 1.1-A

In Exercises 1–10, a specific set is given and two operations on that set are specified. On the basis of your previous experience, tell which of the field axioms are satisfied in the given mathematical system. Justify your answers briefly.

1. The set $\mathbb{N}$ of **natural numbers**,[2] with ordinary $+$ and $\cdot$.

2. The set $\mathbb{Z}$ of **integers**,[2] with ordinary $+$ and $\cdot$.

3. The set $\mathbb{Q}$ of **rational numbers**,[2] with ordinary $+$ and $\cdot$.

4. The set $\mathbb{Z}$ of **integers**,[2] with the operations: $a + b =$ the units digit of the ordinary sum $a + b$, and $a \cdot b = 1 +$ the ordinary product ab. For example, $23 + 184 = 7$ and $4 \cdot 8 = 33$.

5. The set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ with $(x, y) + (u, v) = (x + u, y + v)$, the ordinary vector sum, and $(x, y) \cdot (u, v) = xu + yv$, the ordinary dot product. [Here, $\mathbb{R}$ denotes the set of **real numbers**.[2]]

6. The set $\mathbb{R}^2$ with $(x, y) + (u, v) = (x + u, y + v)$, the ordinary vector sum, and $(x, y) \cdot (u, v) = (xu, yv)$, the "pairwise" product.

7. The set $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ with $(x, y, z) + (u, v, w) = (x + u, y + v, z + w)$, the ordinary vector sum, and $(x, y, z) \cdot (u, v, w) = (yw - zv, -xw + zu, xv - yu)$, the ordinary cross product.

8. The set of real numbers $F = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$, with the ordinary addition and multiplication of real numbers.

9. The set $\{0, 1, 2, 3, 4\}$ with the operations $+$ and $\cdot$ defined by the tables:

+	**0**	**1**	**2**	**3**	**4**
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

2. The number systems $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers), and $\mathbb{R}$ (real numbers) will be defined formally later in this chapter. In these exercises, assume that these number systems have the properties usually ascribed to them.

10. The set $\{0, 1, 2, 3\}$ with the operations $+$ and $\cdot$ defined by the tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

11. **(Project)** Consider the set $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$ of "**complex numbers**," where we define

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc),$$

using the usual addition and multiplication of elements of $\mathbb{R}$.

(a) Prove that the set $\mathbb{C}$ is a field.

(b) Prove that the set $\{(a, 0) : a \in \mathbb{R}\}$ is a subfield of $\mathbb{C}$.

(c) In what sense is this subfield essentially "the same" as $\mathbb{R}$. [Having done Part (b) you know that the identity element of $\mathbb{C}$ is $1 = (1, 0)$. Continue, showing that $\forall a \in \mathbb{R}$, a and $(a, 0)$ can be regarded as "the same."]

(d) Let $i = (0, 1)$ and prove that $i^2 = -1$, where we are identifying -1 with $(-1, 0)$ as justified in (c) above.

(e) Explain how it is true that $\forall a, b \in \mathbb{R}$, the complex number (a, b) can be written in the form $a + bi$.

CONSEQUENCES OF THE FIELD AXIOMS

We now proceed to show that the familiar algebraic rules governing addition, subtraction, multiplication, and division hold in any field.

Theorem 1.1.2 *In any field F, the cancellation laws hold:*

(a) *if $x + y = x + z$ (or $y + x = z + x$), then $y = z$;*

(b) *if $xy = xz$ (or $yx = zx$) and $x \neq 0$, then $y = z$.*

Proof. (a) Suppose $y + x = z + x$. Then, using (A3) and (A4), $\exists\, u \in F \ni x + u = 0$, and (give reasons):

$$\begin{aligned} y = y + 0 &= y + (x + u) \\ &= (y + x) + u \\ &= (z + x) + u \\ &= z + (x + u) \\ &= z + 0 = z. \end{aligned}$$

Thus, $y = z$.

(b) Exercise 1. ■

Theorem 1.1.3 (*Uniqueness of Identities and Inverses*) *In any field F,*

(a) *there is only one element with the property of* 0 *described in (A3);*

(b) *there is only one element with the property of* 1 *described in (M3);*

(c) $\forall x \in F$, *there is only one element in F with the property of u described in (A4);*

(d) $\forall x \neq 0$ *in F, there is only one element in F with the property of u described in (M4).*

Proof. (a) Suppose 0 and $0'$ are elements that satisfy (A3). Then

$$\forall x \in F,\ x + 0 = x, \text{ and } \forall x \in F,\ x + 0' = x, \text{ so}$$
$$0' + 0 = 0', \text{ and } 0 + 0' = 0.$$

Thus, $0' = 0$, since $0' + 0 = 0 + 0'$.

(b) Exercise 2.

(c) Let $x \in F$. Suppose u and v are elements of F, which satisfy the property of u described in (A4). Then

$$\begin{aligned} x + u &= 0, \text{ and} \\ x + v &= 0. \end{aligned}$$

Thus, $x + u = x + v$, and so by the cancellation law [Theorem 1.1.2 (a)], $u = v$.

(d) Exercise 3. ■

Notation for Inverses: Since by Theorem 1.1.3, inverses are unique, we usually denote them with special symbols. The **additive inverse** of an element $x \in F$ described in (A4) is usually denoted "$-x$". Similarly, we usually write "$\frac{1}{x}$" or "x^{-1}" to represent the **multiplicative inverse** of x described in (M4).

Theorem 1.1.4 (*Properties of Identities and Inverses*) *In any field F, the following properties hold:*

(a) $-0 = 0$;

(b) $\forall x \in F$, $-(-x) = x$;

(c) $1^{-1} = 1$ *and* $(-1)^{-1} = -1$;

(d) $\forall x \in F$, $x \cdot 0 = 0$;

(e) $xy = 0 \Leftrightarrow$ *either* $x = 0$ *or* $y = 0$;

(f) *if* $x \neq 0$, *then* $x^{-1} \neq 0$ *and* $(x^{-1})^{-1} = x$;

(g) *if* $x, y \neq 0$, *then* $xy \neq 0$ *and* $(xy)^{-1} = x^{-1}y^{-1}$;

(h) $\forall x \in F$, $(-1)x = -x$;

(i) $\forall x, y \in F$, $(-x)y = -(xy) = x(-y)$;

(j) $(-1)(-1) = 1$;

(k) $\forall x, y \in F$, $(-x)(-y) = xy$.

Proof. (a) Observe that $0 + 0 = 0$. This says that 0 is an additive inverse of 0. Theorem 1.1.3 (c) says that there is only one additive inverse of 0; namely, -0. Thus, $-0 = 0$.

(b) Observe that $(-x) + x = x + (-x) = 0$ by (A1) and (A4). This says that x is an additive inverse of $(-x)$. Theorem 1.1.3 (c) says that $-x$ has only one additive inverse; namely, $-(-x)$. Thus, $-(-x) = x$.

(c) Exercise 4.

(d) Let $x \in F$. Then (give reasons)

$$x \cdot 0 + x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 = x \cdot 0 + 0.$$

That is, $x \cdot 0 + x \cdot 0 = x \cdot 0 + 0$. Applying Theorem 1.1.2 (a), (cancellation), we have $x \cdot 0 = 0$.

(e) This is a two-part proof.

Part 1: First we prove the "$\Rightarrow$" part. Let $xy = 0$. Suppose $x \neq 0$. Then by (M4), $\exists x^{-1} \in F$, and

$$x^{-1}(xy) = x^{-1} \cdot 0 = 0.$$

By the associative law, this implies

$$\begin{aligned} (x^{-1}x)y &= 0, \quad \text{i.e.,} \\ 1 \cdot y &= 0 \\ y &= 0. \end{aligned}$$

Thus, $x \neq 0 \Rightarrow y = 0$; that is, either $x = 0$ or $y = 0$.

Part 2: For the "$\Leftarrow$" part, observe that if either $x = 0$ or $y = 0$, then $xy = 0$ by Part (d) and Axiom (M1).

(f) Exercise 5.

(g) Exercise 6.

(h) Note that $x+(-1)x = 1 \cdot x+(-1)x = [1+(-1)] \cdot x = 0 \cdot x = x \cdot 0 = 0$. Thus, $x + (-1)x = 0$. This says that $(-1)x$ is an additive inverse of x. Theorem 1.1.3 (c) says that x has only one additive inverse; namely, $-x$. Thus, $(-1)x = -x$.

(i) Exercise 7.

(j) Exercise 8.

(k) Exercise 9. ■

SUBTRACTION AND DIVISION

Results (c) and (d) of Theorem 1.1.3 make it possible to define subtraction and division in an arbitrary field F, as follows:

Definition 1.1.5 (Subtraction) $\forall x, y \in F$, define $x - y = x + (-y)$.

Definition 1.1.6 (Division) $\forall x, y \in F$, if $y \neq 0$, define $x \div y = x \cdot y^{-1}$.

Note that by this definition, $x \div y = x \cdot \left(\frac{1}{y}\right)$. We can also denote this using "fraction" notation, $\frac{x}{y}$.

Theorem 1.1.7 *(Properties of Subtraction) In any field F,*

(a) $\forall x \in F,\ 0 - x = -x;$

(b) $\forall x, y, z \in F,\ x(y - z) = xy - xz,$ *and* $(x - y)z = xz - yz;$

(c) $\forall x, y \in F,\ -(x + y) = -x - y;$

(d) $\forall x, y \in F,\ -(x - y) = y - x.$

Proof. (a) Exercise 10.

(b) Let $x, y, z \in F$. Then (give reasons, where asked)

$$\begin{aligned} x(y - z) &= x[y + (-z)] && \text{by definition of } y - z \\ &= xy + x(-z) && \text{Why?} \\ &= xy + (-xz) && \text{Why?} \\ &= xy - xz && \text{Why?} \end{aligned}$$

(c) Exercise 11.

(d) Exercise 12. ■

Theorem 1.1.8 *(Properties of Division and Fractions) In any field F,*

(a) $\forall x \in F$, *if* $x \neq 0$, *then* $0 \div x = 0$;

(b) $\forall x \in F$, $x \div 1 = x$; *if* $x \neq 0$, *then* $1 \div x = x^{-1}$;

(c) $\forall x \in F$, *if* $x \neq 0$, *then* $(-x)^{-1} = -x^{-1}$ $\left[\text{that is, } -\left(x^{-1}\right)\right]$;

(d) *If* $y \neq 0$, *then* $\dfrac{x}{y} = 0 \Leftrightarrow x = 0$;

(e) *If* $b, c \neq 0$, *then* $\dfrac{a}{b} = \dfrac{ac}{bc}$;

(f) *If* $b, d \neq 0$, *then* $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$;

(g) *If* $b, d \neq 0$, *then* $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$;

(h) *If* $b \neq 0$, *then* $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$;

(i) *If* $a, b \neq 0$, *then* $\left(\dfrac{a}{b}\right)^{-1} = \dfrac{b}{a}$.

(j) *If* $a \neq 0$, *then the equation* $ax + b = 0$ *has the unique solution* $x = -\dfrac{b}{a}$.

Proof. (a) Exercise 13.
(b) Exercise 14.
(c) Exercise 15.
(d) Let $y \neq 0$ in F. Then $y^{-1} \neq 0$ by Theorem 1.1.4 (f). By definition, $\frac{x}{y} = x \cdot y^{-1}$. Thus,

$$\begin{aligned}
\frac{x}{y} = 0 &\Leftrightarrow x \cdot y^{-1} = 0 \\
&\Leftrightarrow x = 0 \text{ or } y^{-1} = 0 \text{ by Theorem 1.1.4 (e)} \\
&\Leftrightarrow x = 0, \qquad \text{since } y^{-1} \neq 0.
\end{aligned}$$

(e) Suppose $b, c \neq 0$. Then, since $c \cdot c^{-1} = 1$,

$$\begin{aligned}
\frac{ac}{bc} &= (ac) \cdot (bc)^{-1} \text{ by Definition 1.1.6} \\
&= (ac)(c^{-1}b^{-1}) \text{ (give reasons)} \\
&= a(c \cdot c^{-1})b^{-1} = a(1)b^{-1} = \frac{a}{b}.
\end{aligned}$$

(f) Exercise 16.
(g) Exercise 17.

(h) Exercise 18.
(i) Exercise 19.
(j) Exercise 20. ■

EXERCISE SET 1.1-B

1. Prove Theorem 1.1.2 (b).
2. Prove Theorem 1.1.3 (b). [Hint: see how Part (a) was proved.]
3. Prove Theorem 1.1.3 (d). [Hint: see how Part (c) was proved.]
4. Prove Theorem 1.1.4 (c). [Hint: compare with the proof of (a).]
5. Prove Theorem 1.1.4 (f). [Hint: use Part (d) and Theorem 1.1.3 (d), and compare with the proof of (b).]
6. Prove Theorem 1.1.4 (g). [Hint: use Part (e) and prove that $x^{-1}y^{-1}$ is a multiplicative inverse of xy.]
7. Prove Theorem 1.1.4 (i). [Hint: apply (h).]
8. Prove Theorem 1.1.4 (j).
9. Prove Theorem 1.1.4 (k).
10. Prove Theorem 1.1.7 (a).
11. Prove Theorem 1.1.7 (c).
12. Prove Theorem 1.1.7 (d).
13. Prove Theorem 1.1.8 (a).
14. Prove Theorem 1.1.8 (b).
15. Prove Theorem 1.1.8 (c).
16. Prove Theorem 1.1.8 (f).
17. Prove Theorem 1.1.8 (g).
18. Prove Theorem 1.1.8 (h).
19. Prove Theorem 1.1.8 (i).
20. Prove Theorem 1.1.8 (j). [Note that this requires a two-part proof: first, that $-\dfrac{b}{a}$ is a solution and then, that it is the *only* solution.]

1.2 The Order Properties

In working with real numbers we make frequent use of the concepts of less than and greater than. But the field properties make no mention of "<" or ">." To make these concepts available, we must make additional assumptions about our field. We must assume that it is an "ordered" field, which we now define.

Definition 1.2.1 A field F is said to be an **ordered field** with respect to a particular subset $\mathcal{P} \subseteq F$ if the subset $\mathcal{P}$ satisfies the following axioms:

ORDER AXIOMS:
(O1) $\forall x, y \in \mathcal{P}$, $x + y \in \mathcal{P}$ ($\mathcal{P}$ is "closed" under $+$)
(O2) $\forall x, y \in \mathcal{P}$, $x \cdot y \in \mathcal{P}$ ($\mathcal{P}$is "closed" under $\cdot$)
(O3) $\forall x \in F$, one and only one of the following holds:

$$x \in \mathcal{P}, -x \in \mathcal{P}, \text{ or } x = 0 \quad \text{(the "law of trichotomy")}$$

While these three axioms do not appear very promising, they are entirely sufficient to allow us to define the relations "<" and ">" and derive the properties usually associated with them. First, we observe that (O1) – (O3) allow us to define "positive" and "negative."

Definition 1.2.2 If $x \in \mathcal{P}$ we say that x is **positive**, and if $-x \in \mathcal{P}$ we say that x is **negative**. Thus, the law of trichotomy says that every element of an ordered field is either positive, negative, or zero, but not more than one of these.

Definition 1.2.3 Given x, y in an ordered field F, we say that x and y **have the same sign** if $x, y \in \mathcal{P}$, or $-x, -y \in \mathcal{P}$. We say that x and y **have opposite signs** if $-x, y \in \mathcal{P}$, or $x, -y \in \mathcal{P}$.

Definition 1.2.4 ("Greater than," "Less than," etc.) We define the symbols $<$, $>$, $\leq$, and $\geq$ in an ordered field F as follows:

$x < y$ if $y - x \in \mathcal{P}$;
$x > y$ if $y < x$;
$x \leq y$ if $x < y$ or $x = y$;
$x \geq y$ if $x > y$ or $x = y$.

The inequalities " <" and " >" are called "strict" inequalities, to distinguish them from " $\leq$" and " $\geq$."

Theorem 1.2.5 (*Trivia*) *Let x and y be elements of an ordered field F. Then,*

(a) $x > 0$ *iff* $x \in \mathcal{P}$; $x < 0$ *iff* $-x \in \mathcal{P}$.

(b) *One and only one of the following holds: $x < y$, $x > y$, or $x = y$.* *(Alternate form of the law of trichotomy)*

(c) *$x \le y$ iff $x \not> y$; $x \ge y$ iff $x \not< y$.*

(d) *If $x \le y$ and $y \le x$, then $x = y$.* *(Anti-symmetric property)*

Proof. (a) Exercise 2.
(b) Exercise 3.
(c) Exercise 4.
(d) Suppose $x \le y$ and $y \le x$. For contradiction, suppose $x \ne y$. Then our hypotheses become $x < y$ and $y < x$. But this contradicts the alternate law of trichotomy, above. Therefore, $x = y$. ■

Theorem 1.2.6 (*Combinations of Positive and Negative Elements*) *In any ordered field F,*

(a) *The sum of two negative elements is negative.*

(b) *The product of two negative elements is positive.*

(c) *The square of any nonzero element is positive.*

(d) *The product of a positive element and a negative element is negative.*

(e) *$\forall x, y \in F$, if $xy > 0$, then x and y have the same sign.*

(f) *$\forall x, y \in F$, if $xy < 0$, then x and y have opposite signs.*

Proof. (a) Suppose x and y are negative elements. By definition, this means that $-x \in \mathcal{P}$ and $-y \in \mathcal{P}$. Thus, by Axiom (O1), $(-x) + (-y) \in \mathcal{P}$. That is, $-(x+y) \in \mathcal{P}$. But by definition of "negative," this means that $x+y$ is negative.
(b) Exercise 5.
(c) Suppose $x \ne 0$. By the law of trichotomy, either $x \in \mathcal{P}$ or $-x \in \mathcal{P}$.
Case 1 ($x \in \mathcal{P}$): By Axiom (02), $x^2 \in \mathcal{P}$. That is, x^2 is positive.
Case 2 ($-x \in \mathcal{P}$): By Axiom (02), $(-x)^2 \in \mathcal{P}$. But $(-x)^2 = x^2$.
In either case, x^2 is positive.
(d) Exercise 6.
(e) Exercise 7.
(f) Exercise 8. ■

Corollary 1.2.7 $1 > 0$. [*Thus,* $-1 < 0$.]

Proof. Exercise 9. ■

Theorem 1.2.8 (*Algebraic Properties of Inequalities*) *For any ordered field F, the following properties hold $\forall x, y, z \in F$:*

(a) *If $x < y$, and $y < z$, then $x < z$.* (*Transitive property*)

(b) *$x < y$ iff $x + z < y + z$; similarly, $x < y$ iff $x - z < y - z$.* (*That is, the same element of F can be added to, or subtracted from, both sides of an inequality.*)

(c) *If $z > 0$, then $x < y \Rightarrow xz < yz$.* (*That is, if both sides of an inequality are multiplied by the same* positive *element, the inequality is preserved.*)

(d) *If $z < 0$, then $x < y \Rightarrow xz > yz$.* (*That is, if both sides of an inequality are multiplied by the same* negative *element, the inequality reverses.*)

(e) *If $x, y > 0$, then $x < y \Leftrightarrow x^2 < y^2$.*

Proof. (a) Suppose $x < y$ and $y < z$. By definition, this means $y - x \in \mathcal{P}$ and $z - y \in \mathcal{P}$. Then

$$\begin{aligned} z - x &= z + (-y + y) - x \\ &= (z - y) + (y - x) \in \mathcal{P} \text{ by Axiom (O1).} \end{aligned}$$

That is, $x < z$.

(b) Exercise 10.

(c) Suppose $z > 0$ and $x < y$. By definition, this means $z \in \mathcal{P}$ and $y - x \in \mathcal{P}$. Then, by Axiom (O2), $(y - x)z \in \mathcal{P}$. That is, $yz - xz \in \mathcal{P}$. By definition, this means $xz < yz$.

(d) Exercise 11.

(e) Suppose $x, y > 0$.

First, the "$\Rightarrow$" part. Suppose $x < y$. Since $x > 0$, we may multiply both sides of this inequality by x, and have

$$x^2 < xy.$$

Also, we may multiply both sides of the inequality $x < y$ by y and have

$$xy < y^2.$$

Now, applying the transitive property [Part (a) above],

$$x^2 < y^2.$$

The "$\Leftarrow$" part is Exercise 12. ∎

Corollary 1.2.9 (a) *If $x > 0$, then $x^{-1} > 0$. [Also, if $x < 0$, $x^{-1} < 0$.]*

(b) *If both sides of an inequality are divided by the same positive element, the inequality is preserved.*

(c) *If both sides of an inequality are divided by the same negative element, the inequality is reversed.*

Proof. Exercise 13. ∎

Theorem 1.2.10 (*Further Algebraic Properties of Inequalities*) *In any ordered field F, the following properties hold:*

(a) $0 < x < y \Leftrightarrow 0 < y^{-1} < x^{-1}$.

(b) *If $x < y$ and $u < v$, then $x + u < y + v$.*

(c) *If $0 < x < y$ and $0 < u < v$, then $xu < yv$ and $\dfrac{x}{v} < \dfrac{y}{u}$.*

(d) *If $x < y$, then $x < \dfrac{x+y}{2} < y$.*

Proof. (a) First, the "$\Rightarrow$" part. Suppose $0 < x < y$. By Corollary 1.2.9, $x^{-1} > 0$ and $y^{-1} > 0$. Then $(xy)^{-1} = x^{-1}y^{-1} > 0$. Multiplying both sides of the inequality $x < y$ by $x^{-1}y^{-1}$, we have

$$\begin{aligned} x(x^{-1}y^{-1}) &< y(x^{-1}y^{-1}), \text{ i.e.,} \\ (xx^{-1})y^{-1} &< (yy^{-1})x^{-1}, \text{ i.e.,} \\ y^{-1} &< x^{-1}. \end{aligned}$$

The "$\Leftarrow$" part is Exercise 14.

(b) Exercise 15.

(c) Suppose $0 < x < y$ and $0 < u < v$. Since $u > 0$, we may multiply both sides of the inequality $x < y$ by u and have $xu < yu$.

Since $y > 0$, we may multiply both sides of the inequality $u < v$ by y and have $yu < yv$. Combining this inequality with that of the previous paragraph and the transitive property, we have $xu < yv$.

The proof of the second claim is Exercise 16.

(d) Exercise 17. ■

Theorem 1.2.11 (*The Large and Small of It*)

(a) *An ordered field has no largest element (and no smallest element).*

(b) *An ordered field has no smallest positive element (and no largest negative element).*

(c) *In any ordered field, $\mathcal{P}$ (and consequently F itself) is an infinite set.*

Proof. (a) Let F be an ordered field. Let $x \in F$. Can x be the largest element of F? Adding x to both sides of the inequality $1 > 0$, we have $x+1 > x$. Thus, x cannot be the largest element of F. Hence, F cannot have a largest element.

(b) Exercise 18.

(c) Suppose F is an ordered field. Then $1 \in F$ and, since F is closed under addition, F must contain the elements

$$1, \quad 1+1, \quad 1+1+1, \quad 1+1+1+1, \quad \cdots \quad .$$

By Corollary 1.2.7, $1 \in \mathcal{P}$, and by Axiom (O1), each element in this list is in $\mathcal{P}$. Moreover, by Theorem 1.2.8,

$$1 < (1+1) < (1+1+1) < (1+1+1+1) < \quad \cdots \quad .$$

By the transitive property, each successive element in the above list is larger than all previous elements in the list. So, the list above contains no duplicates; all elements are different.

Since there is no end to the number of times we can add 1, this list must contain an infinite number of different elements. Therefore, $\mathcal{P}$ must be an infinite set. ■

EXERCISE SET 1.2-A

1. Which of the fields found in Exercise Set 1.1-A are ordered fields with respect to some natural choice of subsets $\mathcal{P}$. [In each case, try to find a natural set $\mathcal{P}$ of "positive" elements.]
2. Prove Theorem 1.2.5 (a).
3. Prove Theorem 1.2.5 (b).
4. Prove Theorem 1.2.5 (c). [Hint: apply Axiom (O3).]
5. Prove Theorem 1.2.6 (b).
6. Prove Theorem 1.2.6 (d).
7. Prove Theorem 1.2.6 (e)
8. Prove Theorem 1.2.6 (f).
9. Prove Corollary 1.2.7.
10. Prove Theorem 1.2.8 (b).
11. Prove Theorem 1.2.8 (d).
12. Prove the "$\Leftarrow$" part of Theorem 1.2.8 (e).
13. Prove Corollary 1.2.9.
14. Prove the "$\Leftarrow$" part of Theorem 1.2.10 (a).
15. Prove Theorem 1.2.10 (b).

16. Prove the second claim of Theorem 1.2.10 (c).

17. Prove Theorem 1.2.10 (d).

18. Prove Theorem 1.2.11 (b). [Hint: for $x \in \mathcal{P}$, proceed as in the proof of Part (a), using $\frac{x}{2}$ in place of $x + 1$.]

19. For the field $F = \{a + b\sqrt{2}\colon a, b \in \mathbb{Q}\}$ defined in Exercise 1.1-A.8, let $\mathcal{P}' = \{a + b\sqrt{2}\colon a > b\sqrt{2}\}$, where the "$>$" symbol here refers to the ordinary "$>$" relation in $\mathbb{R}$. Prove that F is an ordered field with respect to $\mathcal{P}'$.

20. Give an example of a field F that can be ordered with respect to two *different* subsets, $\mathcal{P}$ and $\mathcal{P}'$.

21. Prove that the field $\mathbb{C}$ of **complex numbers**[3] *cannot* be ordered; that is, it is impossible to find a subset $\mathcal{P}$ of $\mathbb{C}$ satisfying the axioms (O1)–(O3). [Hint: Find one of the properties established in the theorems above that cannot hold in $\mathbb{C}$ regardless of the choice of subset $\mathcal{P} \subseteq \mathbb{C}$.]

22. Prove that the field defined in Exercise 1.1-A.9 cannot be ordered.

ABSOLUTE VALUE

Definition 1.2.12 Suppose F is an ordered field. For each $x \in F$, we define the **absolute value** of x to be

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Theorem 1.2.13 (***Basic Properties of Absolute Value***) *Let F be an ordered field. Then, $\forall x, y \in F$,*

(a) $|x| \geq 0$;

(b) $|-x| = |x|$;

(d) $-|x| \leq x \leq |x|$;

(d) $|x - y| = |y - x|$;

(e) $|xy| = |x||y|$.

---

3. See Exercise 1.1-A.11

Proof. (a) Exercise 1.

(b) By Definition 1.2.12,

$$\begin{aligned}
|-x| &= \begin{cases} -x & \text{if } -x \geq 0 \\ -(-x) & \text{if } -x < 0 \end{cases} \\
&= \begin{cases} -x & \text{if } -(-x) \leq 0 \\ -(-x) & \text{if } -(-x) > 0 \end{cases} \\
&= \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \\
&= |x|.
\end{aligned}$$

(c) Let $x \in F$. Then either $x \geq 0$ or $x < 0$.

Case 1 ($x \geq 0$): Then $0 \leq x = |x|$, so $-|x| \leq 0 \leq x \leq |x|$. Thus, $-|x| \leq x \leq |x|$.

Case 2 ($x < 0$): Then $|x| = -x$, so $-|x| = x < 0 \leq |x|$. Thus, $-|x| \leq x \leq |x|$.

In either case, $-|x| \leq x \leq |x|$.

(d) Exercise 1.

(e) Exercise 1. ■

Theorem 1.2.14 (***Absolute Value Inequalities***) *Let $a \geq 0$ be a fixed non-negative element in an ordered field F. Then $\forall x, y \in F$,*

(a) $|x| < a \Leftrightarrow -a < x < a$.

(b) $|x| > a \Leftrightarrow x > a$ *or* $x < -a$.

(c) $|x - y| < a \Leftrightarrow y - a < x < y + a$.

Proof. (a) Exercise 2.

(b) First, the "$\Rightarrow$" part. Suppose $|x| > a$.

Case 1 ($x \geq 0$): Then $|x| = x$, so $x > a$.

Case 2 ($x < 0$): Then $|x| = -x$, so $-x > a$, which is equivalent to $x < -a$.

Since either Case 1 or Case 2 must be true, either $x > a$ or $x < -a$.

Now, for the "$\Leftarrow$" part. Suppose $x > a$ or $x < -a$.

Case 1 ($x > a$): Then $x > 0$, so $|x| = x$, so $|x| > a$.

Case 2 ($x < -a$): Then $-x > a$. But in this case, $x < 0$, so $|x| = -x$, from which it follows that $|x| > a$.

In either case, $|x| > a$.

(c) Exercise 2. ■

Theorem 1.2.15 (***Triangle Inequalities***) *For all x, y in an ordered field F,*

(a) $|x+y| \leq |x| + |y|$;

(b) $|x| - |y| \leq |x-y|$;

(c) $||x| - |y|| \leq |x-y|$;

(d) $||x| - |y|| \leq |x+y|$.

Proof. (a) First, recall that $-|x| \leq x \leq |x|$, and $-|y| \leq y \leq |y|$. Thus, by Theorem 1.2.10 (b),

$$-|x| - |y| \leq x + y \leq |x| + |y|, \text{ or}$$
$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

But $|x| + |y|$ is a nonnegative element of F. Thus, by Theorem 1.2.14 (a), $|x+y| \leq |x| + |y|$.

(b) By Part (a), $|x| = |(x-y) + y| \leq |x-y| + |y|$. Subtracting $|y|$ from both sides, we have $|x| - |y| \leq |x-y|$.

(c) Exercise 3.

(d) Exercise 3. ■

INTERVALS

Definition 1.2.16 (Intervals) Let F be an ordered field. We first define closed intervals and then extend this definition to define arbitrary intervals.

(a) $\forall a, b \in F$, we define the **closed interval** $[a, b]$ to be the set

$$[a, b] = \{x \in F : a \leq x \leq b\}.$$

Note that we do not require that $a < b$ in this definition. Thus, for example,

$$[a, a] = \{a\}, \text{ and}$$
$$[2, 1] = \varnothing, \text{ since } 2 \leq x \leq 1 \text{ is impossible.}$$

(b) In general, an **interval** in F is any subset $I \subseteq F$ such that $[a, b] \subseteq I$ whenever $a, b \in I$.

That is, an interval is a set that always contains the entire closed interval between any two of its points. It thus would make sense to say that an interval is a "convex set" in an ordered field. The following theorem specifies exactly which sets are intervals.

Theorem 1.2.17 (***Intervals***) *In an ordered field F, the following sets are intervals:*

(a) $[a, b] = \{x \in F : a \leq x \leq b\}$; *(This could be $\{a\}$ or $\varnothing$.)*

(b) $(a, b) = \{x \in F : a < x < b\}$; *(This could be $\varnothing$.)*

(c) $(a, b] = \{x \in F : a < x \leq b\}$; (*This could be* $\varnothing$.)

(d) $[a, b) = \{x \in F : a \leq x < b\}$; (*This could be* $\varnothing$.)

(e) $(-\infty, b) = \{x \in F : x < b\}$;

(f) $(-\infty, b] = \{x \in F : x \leq b\}$;

(g) $(a, +\infty) = \{x \in F : x > a\}$;

(h) $[a, +\infty) = \{x \in F : x \geq a\}$;

(i) $(-\infty, +\infty) = F$.

(Intervals of the form (b), (e), (g), and (i) are called **open intervals.**)

Proof. See Exercise 4. ∎

EXERCISE SET 1.2-B

1. Prove Theorem 1.2.13 (a), (d), and (e).
2. Prove Theorem 1.2.14 (a) and (c).
3. Prove Theorem 1.2.15 (c) and (d).
4. Prove that the sets described in Theorem 1.2.17 (c) and (g) are intervals.
5. **Intervals:** Let I denote an interval, and $x \in I$. Prove that $I = \bigcup \{[y, z] : y, z \in I\}$.
6. Prove that $\forall x, y \in$ ordered field F, $\max\{x, y\} = \dfrac{x + y + |x - y|}{2}$ and $\min\{x, y\} = \dfrac{x + y - |x - y|}{2}$.
7. Prove that $\forall x, y \in$ ordered field F, $\min\{x, y\} = -\max\{-x, -y\}$.
8. Prove that in any ordered field, $0 \leq x < y \Rightarrow \dfrac{x}{1 + x} < \dfrac{y}{1 + y}$.
9. Prove that in any ordered field, $\dfrac{|x + y|}{1 + |x + y|} \leq \dfrac{|x|}{1 + |x|} + \dfrac{|y|}{1 + |y|}$.
10. **(Project)**

    (a) Three elements a, b, c of a field form an **arithmetic progression** if their successive differences are equal: $b - a = c - b$. Prove that $b = \dfrac{a + c}{2}$. [b is called the **arithmetic mean** of a and c.]

(b) Three positive elements a, b, c of an ordered field form a **geometric progression** if their successive quotients are equal: $\frac{b}{a} = \frac{c}{b}$. Prove that $b = \sqrt{ac}$, if this square root exists.[4] [b is called the **geometric mean** of a and c.]

(c) Three positive elements a, b, c of an ordered field form a **harmonic progression** if their reciprocals form an arithmetic progression: $\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$. Prove that $b = \frac{2ac}{a+c}$. [b is called the **harmonic mean** of a and c.]

(d) Three positive elements a, b, c of an ordered field form a **quadratic progression** if their squares form an arithmetic progression: $b^2 - a^2 = c^2 - b^2$. Prove that $b = \sqrt{\frac{a^2+c^2}{2}}$, if this square root exists[4]. [b is called the **quadratic mean** of a and c.]

(e) Prove that in any ordered field, if $0 < a \leq b$, and if the indicated square roots exist,[4] then $a \leq \frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}} \leq b$.

1.3 Natural Numbers

In our development of ordered fields, we have not seen any familiar numbers except 0 and 1. As we shall now see, all ordered fields must contain lots of familiar numbers. First, they must all contain the "natural numbers," $1, 2, 3, \cdots, n, n+1, \cdots$. Before we can prove that, we must define the natural numbers in a rigorous way.

Definition 1.3.1 An **inductive subset** of an ordered field F is a subset $A \subseteq F$ with the properties:

(i) $1 \in A$, and
(ii) $\forall x \in F,\ x \in A \Rightarrow x + 1 \in A$.

Note that any ordered field F contains at least two inductive sets, for both $\mathcal{P}$ and F are inductive subsets of F. We shall see that there are many more.

4. Square roots of positive elements do not necessarily exist in ordered fields. In the real number system, however, they always exist (See Theorems 1.6.12, 2.5.11, 5.3.13, and Exercise 1.6-B.6).

Theorem 1.3.2 *The intersection[5] of any collection of inductive subsets of F is inductive.*

Proof. Let $\mathcal{C}$ denote a collection of inductive sets. We shall prove that $\cap\, \mathcal{C}$ is an inductive set.

(i) For all $C \in \mathcal{C}$, $1 \in C$, since C is an inductive set. Therefore, $1 \in \cap\, \mathcal{C}$.

(ii) Suppose $x \in \cap\, \mathcal{C}$. Let $C \in \mathcal{C}$. Then $x \in C$. Since C is inductive, $x + 1 \in C$. Since this is true for all $C \in \mathcal{C}$, $x + 1 \in \cap\, \mathcal{C}$. ∎

We are now ready to define the natural numbers in any ordered field.

Definition 1.3.3 The set of **natural numbers** of an ordered field F is the intersection of all the inductive subsets of F. In symbols,

$$\mathbb{N}_F = \cap\, \mathcal{S},$$

where $\mathcal{S}$ denotes the collection of **all** inductive subsets of F.

We call the elements of $\mathbb{N}_F$ the **natural numbers** of F.

Theorem 1.3.4 *The set of natural numbers is the smallest inductive subset of F, in the sense that $\mathbb{N}_F$ is an inductive set and every inductive subset of F contains $\mathbb{N}_F$ as a subset.*

Proof. By Theorem 1.3.2, $\mathbb{N}_F$ is an inductive set. Now, suppose C is an inductive subset of F. Then $C \in \mathcal{S}$, the collection of all inductive subsets of F. Thus, $\cap\, \mathcal{S} \subseteq C$, since the intersection of a collection of sets is a subset of any one of the sets in the collection. That is, $\mathbb{N}_F \subseteq C$. ∎

Theorem 1.3.5 *For any ordered field F,*

(a) *All natural numbers of F are positive.*

(b) *1 is the smallest natural number of F. That is, $\forall n \in \mathbb{N}_F$, $n \geq 1$.*

(c) *$\forall n \in \mathbb{N}_F$, if $n > 1$, then $n - 1 \in \mathbb{N}_F$.*

Proof. (a) Recall that the set $\mathcal{P}$ is an inductive subset of F. Therefore, by Theorem 1.3.4, $\mathbb{N}_F \subseteq \mathcal{P}$. Thus, all elements of $\mathbb{N}_F$ are positive.

(b) Let $A = \{x \in F : x \geq 1\}$. Then

(i) $1 \in A$, and

(ii) Suppose $x \in A$. Then $x \geq 1$. Hence, $x + 1 \geq 1$;

That is, $x + 1 \in A$. Therefore $x \in A \Rightarrow x + 1 \in A$.

---

5. For a review of sets and intersections, see Appendix B.1.

Therefore, A is an inductive set. By Theorem 1.3.4, $\mathbb{N}_F \subseteq A$. That is, all natural numbers in F are ≥ 1.

(c) Let $A = \{n \in \mathbb{N}_F : n > 1 \Rightarrow n - 1 \in \mathbb{N}_F\}$. Then

(i) $1 \in A$, since $1 \not> 1$.

(ii) Suppose $x \in A$. Then $x > 1 \Rightarrow x - 1 \in \mathbb{N}_F$. Consider $x + 1$. Since $x \in \mathbb{N}_F$, $x \geq 1$, so, $x + 1 > 1$ and $(x + 1) - 1 = x \in \mathbb{N}_F$. Thus,

$$x + 1 > 1 \Rightarrow (x + 1) - 1 \in \mathbb{N}_F.$$

Thus, $x + 1 \in A$.

Therefore, A is an inductive set. By Theorem 1.3.4, $\mathbb{N}_F \subseteq A$. That is, $\forall n \in \mathbb{N}_F$, if $n > 1$, then $n - 1 \in \mathbb{N}_F$. ∎

A surprising result of this axiomatic approach is that it enables us to **prove** the principle of mathematical induction. This important technique of proof is itself a theorem within the theory of ordered fields. Most students regard the principle of mathematical induction as an axiom of logic that must be assumed; it may come as a surprise to you that it is actually a theorem.

Theorem 1.3.6 (***The Principle of Mathematical Induction***) *Let F be an ordered field. Suppose that $\forall n \in \mathbb{N}_F$, $p(n)$ is a proposition about n. If*

(1) *$p(1)$ is true, and*

(2) *$\forall k \in \mathbb{N}_F$, $p(k) \Rightarrow p(k + 1)$,*

then $\forall n \in \mathbb{N}_F$, $p(n)$ is true.

Proof. Suppose $p(n)$ is as described in the hypotheses. Let $A = \{x \in \mathbb{N}_F : p(x) \text{ is true}\}$. Then

(i) $1 \in A$, by (1).

(ii) Suppose $x \in A$. Then $x \in \mathbb{N}_F$ and $p(x)$ is true. Thus, by (2), $p(x + 1)$ is true. That is, $x + 1 \in A$. Therefore, $x \in A \Rightarrow x + 1 \in A$.

Therefore, A is an inductive set. By Theorem 1.3.4, $\mathbb{N}_F \subseteq A$. That is, $\forall n \in \mathbb{N}_F$, $p(n)$ is true. ∎

In the remainder of this section, we put the principle of mathematical induction to good use, as we derive some additional results about natural numbers.

Theorem 1.3.7 *Let F be an ordered field.*

(a) *$\forall m, n \in \mathbb{N}_F$, if $m < n$, then $n - m \in \mathbb{N}_F$.*

(b) *$\forall n \in \mathbb{N}_F$, there is no natural number between n and $n + 1$.*

***Proof.** (a) $\forall n \in \mathbb{N}_F$ we let $p(n)$ denote the proposition

$$p(n) : \forall m \in \mathbb{N}_F, m < n \Rightarrow n - m \in \mathbb{N}_F.$$

*An asterisk before a theorem, proof, or other item in this chapter indicates that the item is challenging and can be omitted, especially in a one-semester course.

Then:

(1) $p(1)$ is true, since $\nexists m \in \mathbb{N}_F \ni m < 1$.

(2) Suppose $p(k)$ is true. That is,

$$m < k \Rightarrow k - m \in \mathbb{N}_F.$$

To prove $p(k+1)$, suppose $m < k+1$. Then $m - 1 < k$, so by $p(k)$,

$$k - (m-1) \in \mathbb{N}_F.$$

That is, $(k+1) - m \in \mathbb{N}_F$. But that means

$$m < (k+1) \Rightarrow (k+1) - m \in \mathbb{N}_F.$$

That is, $p(k+1)$ is true. We have thus proved that $p(k) \Rightarrow p(k+1)$.

Therefore, by the principle of mathematical induction, $\forall n \in \mathbb{N}_F$, $p(n)$ is true.

(b) Let $n \in \mathbb{N}_F$. Suppose, for contradiction, that $\exists m \in \mathbb{N}_F \ni n < m < n+1$. Subtracting n from all sides of these inequalities, we then have

$$0 < m - n < 1.$$

By (a), $m - n \in \mathbb{N}_F$. But, by Theorem 1.3.5 (b), there is no natural number less than 1. Thus, we have a contradiction. Therefore, $\nexists m \in \mathbb{N}_F \ni n < m < n+1$. ■

Theorem 1.3.8 *In any ordered field,*

(a) $\mathbb{N}_F$ *is closed under addition.*

(b) $\mathbb{N}_F$ *is closed under multiplication.*

(d) $\mathbb{N}_F$ *is **not** closed under subtraction or division.*

***Proof.** (a) Let $n \in \mathbb{N}_F$ be fixed. $\forall m \in \mathbb{N}_F$ we let $p(m)$ denote the proposition $p(m) : n + m \in \mathbb{N}_F$. Then:

(1) $p(1)$ is the statement $n+1 \in \mathbb{N}_F$, which is true since $\mathbb{N}_F$ is an inductive set.

(2) Suppose $p(k)$ is true. Then $n + k \in \mathbb{N}_F$. Since $\mathbb{N}_F$ is an inductive set,

$$(n+k) + 1 \in \mathbb{N}_F; \text{ i.e.,}$$
$$n + (k+1) \in \mathbb{N}_F.$$

That is, $p(k+1)$ is true. We have thus proved that $p(k) \Rightarrow p(k+1)$.

Therefore, by the principle of mathematical induction, $\forall m \in \mathbb{N}_F$, $p(m)$ is true. That is,

$$\forall m \in \mathbb{N}_F, n + m \in \mathbb{N}_F.$$

Therefore, $\mathbb{N}_F$ is closed under addition.

(b) Let $n \in \mathbb{N}_F$ be fixed. $\forall m \in \mathbb{N}_F$ we let $p(m)$ denote the proposition $p(m) : nm \in \mathbb{N}_F$. Then:

(1) $p(1)$ is the statement $n \cdot 1 \in \mathbb{N}_F$, which is obviously true.

(2) Suppose $p(k)$ is true. Then $nk \in \mathbb{N}_F$. Now,

$$n(k+1) = nk + n,$$

and since $\mathbb{N}_F$ is closed under addition, $n(k+1) \in \mathbb{N}_F$. Therefore, $p(k+1)$ is true. We have thus proved that $p(k) \Rightarrow p(k+1)$.

Therefore, by the principle of mathematical induction, $\forall m \in \mathbb{N}_F$, $p(m)$ is true. That is, $\forall m \in \mathbb{N}_F$, $nm \in \mathbb{N}$, which means $\mathbb{N}_F$ is closed under multiplication.

(c) Exercise 1. ■

ORDINARY NATURAL NUMBERS

The "ordinary natural numbers" 1, 2, 3, 4, ... exist in any ordered field in the following sense: suppose $a \in F$, an ordered field. We define

$$\begin{aligned} 1 &= \text{the multiplicative identity of } F \\ 2 &= 1+1 \\ 3 &= 2+1 \\ 4 &= 3+1 \\ \vdots \; &\;\; \vdots \end{aligned}$$

We shall use the symbols 1, 2, 3, ... to represent the elements of $\mathbb{N}_F$, regardless of the ordered field F. Thus, every ordered field contains the ordinary natural numbers, or at least a copy of them. Another way of saying this is that the natural numbers may be considered as "embedded" in any ordered field. For this reason, **we shall hereafter discontinue using the symbol $\mathbb{N}_F$, and use instead the generic symbol $\mathbb{N}$ to represent the set of all natural numbers**, regardless of the ordered field F in which they occur.

Theorem 1.3.9 (***Alternate Principle*[6] *of Mathematical Induction***)
Suppose that $\forall n \in \mathbb{N}$, $p(n)$ is a proposition about n such that

(1) *$p(1)$ is true, and*

(2) *$\forall k \in \mathbb{N}$, if $p(m)$ is true for all natural numbers $m < k$ in $\mathbb{N}$ then $p(k)$ is true.*

Then $\forall n \in \mathbb{N}$, $p(n)$ is true.

6. This is often called "strong mathematical induction," although it is equivalent to ordinary mathematical induction rather than stronger than it.

***Proof.** Suppose $p(n)$ satisfies conditions (1) and (2) above. Let $q(1)$ denote $p(1)$, and for $k \geq 2$ let $q(k)$ denote the statement "$p(m)$ is true for all natural numbers $m < k$." Then,

(i) $q(1)$ is true;

(ii) $\forall k \in \mathbb{N}$, $q(k) \Rightarrow p(m)$ is true for all natural numbers $m < k$
$\Rightarrow p(k)$ is true by condition (2) above
$\Rightarrow p(k)$ is true for all natural numbers $m < k+1$
$\Rightarrow q(k+1)$.

Therefore, by the principle of mathematical induction (1.3.6), $\forall n \in \mathbb{N}$, $q(n)$ is true, from which it follows that $\forall n \in \mathbb{N}$, $p(n)$ is true. ■

The following property is closely related to the principle of mathematical induction, and is one of the principle characteristics of the set of natural numbers.

Theorem 1.3.10 (***Well-Ordering Property***) *Every nonempty set of natural numbers has a smallest element.*

***Proof.** Let A be a nonempty subset of $\mathbb{N}$. For contradiction, suppose that A does not have a smallest element. Let

$$S = \mathbb{N} - A.$$

Then

(1) $1 \notin A$, since if $1 \in A$ it would be the smallest element of A. So, $1 \in S$.

(2) Suppose $1, 2, \cdots, k \in S$. Then, $1, 2, \cdots, k \notin A$. Thus, $k+1 \notin A$, since if $k \in A$ it would be the smallest element of A. Thus, $k \in S$.

By (1), (2) and the alternate principle of mathematical induction, $\forall n \in \mathbb{N}$, $n \in S$. Therefore, $S = \mathbb{N}$, and so $A = \varnothing$. Contradiction. Therefore, A must have a smallest element. ■

The Principle of Mathematical Induction, as presented in Theorem 1.3.6, establishes the truth of $p(n)$, for all integers n starting with $n = 1$. However, there is no reason why we must start with $n = 1$. We could use **any** natural number as a starting point. The following theorem expresses this fact.

Theorem 1.3.11 (***Principle of Mathematical Induction For $n \geq n_0$***) *Suppose $n_0 \in \mathbb{N}$ and $\forall$ natural numbers $n \geq n_0$, $p(n)$ is a proposition about n. If*

(1) *$p(n_0)$ is true, and*

(2) *$\forall k \in \mathbb{N} \ni k \geq n_0$, $p(k) \Rightarrow p(k+1)$,*

then $\forall n \geq n_0$ in $\mathbb{N}$, $p(n)$ is true.

*__Proof.__ Exercise 18. ■

Similarly, the Alternate Principle of Mathematical Induction can be rewritten to start with any natural number.

MATHEMATICAL INDUCTION AS A METHOD OF DEFINITION

When we wish to define a quantity $f(n)$, for all natural numbers, mathematical induction is often useful. This method of definition is often called "recursive" definition, especially in computer science. Consider the following example:

Definition 1.3.12 We define $a^n, \forall\, n \in \mathbb{N}$, as follows:
(1) $a^1 = a$;
(2) $\forall k \in \mathbb{N}$, $a^{k+1} = a \cdot a^k$.

EXERCISE SET 1.3

1. Prove Theorem 1.3.8 (c).

2. Prove that $\forall n \in \mathbb{N}$, $0 < \dfrac{1}{n^2} \le \dfrac{1}{n} \le 1$; if $n > 1$, then $0 < \dfrac{1}{n^2} < \dfrac{1}{n} < 1$.

   In Exercises 3–22, use mathematical induction to prove the given equation, statement, or inequality, $\forall n \in \mathbb{N}$.

3. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

5. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

6. $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

7. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$.

8. $n(n+1)(n+2)$ is divisible by 3.

9. $n^5 - n$ is divisible by 5.

10. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$.

11. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n} = \dfrac{3}{2} - \dfrac{1}{2}\left(\dfrac{1}{3}\right)^n$.

12. **Finite Geometric Sums:** If $r \neq 1$, $a + ar + ar^2 + ar^3 + \cdots + ar^n = \dfrac{a - ar^{n+1}}{1-r}$.

13. $2^n \leq (n+1)!$.

14. **Bernoulli's Inequality:** for any fixed $x > -1$, $(1+x)^n \geq 1 + nx$.

15. For any fixed $x \geq 0$, $(1+x)^n \geq 1 + nx + \frac{1}{2}n(n-1)x^2$.

16. $13^n - 6^n$ is divisible by 7.

17. $2^{2n-1} + 1$ is divisible by 3.

18. Prove Theorem 1.3.11.

19. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1})$. [Hint: start with $n = 2$ and use Theorem 1.3.11.]

20. $\forall m, n \in \mathbb{N}$, $a^m a^n = a^{m+n}$. [Hint: let m be a fixed natural number, and use mathematical induction on n; see Definition 1.3.12.]

21. $\forall m, n \in \mathbb{N}$, $(a^m)^n = a^{mn}$. [See hint for Exercise 20.]

22. $\forall n \in \mathbb{N}$, $a^n b^n = (ab)^n$.

23. **Binomial Coefficients:**

    (a) **Factorials**: $\forall n \in \mathbb{N}$, we use mathematical induction to define n **factorial** (denoted $n!$): $1! = 1$, and $(n+1)! = n!(n+1)$. We also define $0! = 1$. Show that $\forall n \in \mathbb{N}$, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

    (b) $\forall k \leq n$ in $\mathbb{N}$, we define the **binomial coefficient** $\binom{n}{k}$ by the formula $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Since $0!$ makes sense, we define $\binom{n}{0}$ by letting $k = 0$ in this definition. Verify the familiar formulas, $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{k} = \binom{n}{n-k}$, and $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. Relate these identities to the famous "Pascal's triangle."

    (c) Prove that under this definition, $\binom{n}{k}$ is always a natural number. [Use mathematical induction on n and the last identity in (b).]

24. **The Binomial Theorem:** Recall the summation notation, $\sum\limits_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$, and its obvious extensions to sums starting with $k = 0$ or $k =$ any other natural number. Use the results of Exercise 23 to prove the binomial theorem,

$$\forall n \in \mathbb{N},\ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

1.4 Rational Numbers

We have seen in Section 1.3 that every ordered field contains the natural numbers. Since fields are closed under subtraction, one would expect every ordered field to contain all the "integers" (natural numbers, negatives of natural numbers, and zero). Further, since fields are closed under division, except by 0, one would expect every ordered field to contain all the "rational numbers" (quotients of integers). We begin by defining integers and rational numbers.

Definition 1.4.1 The set of **integers** of an ordered field F is the set

$$\mathbb{Z}_F = \{x \in F\colon x \in \mathbb{N}, \text{ or } -x \in \mathbb{N}, \text{ or } x = 0\}.$$

Thus, the set of integers of F consists of natural numbers, additive inverses of natural numbers, and 0.

Definition 1.4.2 The set of **rational numbers** of an ordered field F is the set

$$\mathbb{Q}_F = \{x \in F\colon \exists\, m, n \in \mathbb{Z}_F \ni n \neq 0, \text{ and } x = \tfrac{m}{n}\}.$$

That is, the rational numbers of F are quotients of integers of F, commonly called "fractions." One of the most significant features of the rational numbers of F is that they form a field within the field F; i.e., a "subfield" of F. This is summarized in the following theorem.

Theorem 1.4.3 *For any ordered field F with positive subset $\mathcal{P}$, the set $\mathbb{Q}_F$ of rational elements of F is an ordered field relative to the same operations $+$ and $\cdot$ used in F and the positive set $\mathcal{P}' = \mathcal{P} \cap \mathbb{Q}_F$. ($\mathbb{Q}_F$ will be called the* ***rational subfield of F****.)*

***Proof.** Let F be an ordered field. We must show that $\mathbb{Q}_F$, with the same $+$ and $\cdot$ used in F and the positive set $\mathcal{P}'$, satisfies axioms (A0)–(A4), (M0)–(M4), (D), and (O1)–(O3).

(A0) Let $x, y \in \mathbb{Q}_F$. Then $\exists\, m, n, m', n' \in \mathbb{Z}_F \ni n, n' \neq 0$, $x = \frac{m}{n}$ and $y = \frac{m'}{n'}$. Thus,

$$x + y = \frac{m}{n} + \frac{m'}{n'} = \frac{mn' + m'n}{nn'} \in \mathbb{Q}_F. \text{ (Justify)}$$

(A1) This property is "inherited" from F. (Explain what that means)

(A2) Inherited from F. (Explain)

(A3) $0 \in \mathbb{Q}_F$, since $0 = \frac{0}{1}$. This element of $\mathbb{Q}_F$ satisfies the condition specified in (A3).

(A4) Let $x \in \mathbb{Q}_F$. Then $\exists\, m, n \in \mathbb{Z}_F \ni n \neq 0$ and $x = \frac{m}{n}$. Then $-x = \frac{-m}{n} \in \mathbb{Q}_F$ (Justify). This element $-x$ satisfies the property required by (A4).

(M0) Let $x, y \in \mathbb{Q}_F$. Then $\exists\, m, n, m', n' \in \mathbb{Z}_F \ni n, n' \neq 0$, $x = \frac{m}{n}$ and $y = \frac{m'}{n'}$. Thus,

$$x \cdot y = \frac{m}{n} \cdot \frac{m'}{n'} = \frac{mm'}{nn'} \in \mathbb{Q}_F. \text{ (Justify)}$$

(M1) This property is "inherited" from F. (Explain)

(M2) Inherited from F. (Explain)

(M3) $1 \in \mathbb{Q}_F$, since $1 = \frac{1}{1}$. This element of $\mathbb{Q}_F$ satisfies the condition specified in (M3).

(M4) Let $x \in \mathbb{Q}_F \ni x \neq 0$. Then $\exists\, m, n \in \mathbb{Z}_F \ni m, n, \neq 0$, and $x = \frac{m}{n}$. Then $x^{-1} = \frac{n}{m} \in \mathbb{Q}_F$ (Justify). This element x^{-1} satisfies the property required by (M4).

(D) Inherited from F. (Explain)

(O1)–(O3) are also inherited from F. (Explain) ■

Definition 1.4.4 If an ordered field F contains an element that is not a rational number (by our definition) then such an element is called an **irrational element** of F.

Theorem 1.4.5 *There is no element of $\mathbb{Q}_F$ whose square is 2.*

Proof. For contradiction, suppose $\exists\, x \in \mathbb{Q}_F \ni x^2 = 2$. Then $\exists\, m, n \in \mathbb{Z}_F \ni n \neq 0$ and $x = \frac{m}{n}$. Without loss of generality, we may assume that m and n have no prime factors in common. Then we have

$$\left(\frac{m}{n}\right)^2 = 2, \text{so}$$
$$m^2 = 2n^2.$$

Thus, m^2 is divisible by 2. By Exercise 3 below, this means that m is divisible by 2. That is, $\exists\, k \in \mathbb{Z}_F \ni m = 2k$. Thus, the above equation becomes

$$(2k)^2 = 2n^2, \text{ so}$$
$$4k^2 = 2n^2, \text{ so } 2k^2 = n^2.$$

But this means that n^2 is even, which means that n is also even. Thus the numerator, m, and the denominator, n, are both divisible by 2. Contradiction. Hence, $\nexists\, x \in \mathbb{Q}_F \ni x^2 = 2$. ■

SIMPLIFYING THE NOTATION

To simplify notation, we shall discontinue using the notation $\mathbb{Z}_F$ and $\mathbb{Q}_F$, and from now on simply use $\mathbb{Z}$ and $\mathbb{Q}$ to denote these sets. We can do this without ambiguity, because the symbols $1, 2, 3, \ldots$ mean the same in all ordered fields.

In this sense, every ordered field contains the familiar natural numbers, integers, and rational numbers.

EXERCISE SET 1.4

In each of the following, assume that F is an ordered field. Prove the stated property.

1. $\mathbb{Z}$ satisfies all the properties of an ordered field except (M4). [You may assume (A0) and (M0); see Exercise 13 below.]

2. $\mathbb{Q}$ is the "smallest" ordered field, in the sense that every ordered field F contains $\mathbb{Q}$ as a subset.

3. Let $n \in \mathbb{Z}$. If n^2 is divisible by 2, then n is also divisible by 2. [Hint: any integer is either even (of the form $2k$, where $k \in \mathbb{Z}$) or odd (of the form $2k+1$, where $k \in \mathbb{Z}$).]

4. There is no element of $\mathbb{Q}$ whose square is 3. [Hint: Use suitable modifications of Exercise 3 and Theorem 1.4.5.]

   In Exercises 5–10, assume that F is an **ordered field with at least one irrational element**.

5. If x is rational and y is irrational, then $x + y$ is irrational.

6. If $x \neq 0$ is rational and y is irrational, then xy is irrational.

7. If x is irrational, then so are $-x$ and x^{-1}.

8. The set of irrationals is not closed under addition.

9. The set of irrationals is not closed under multiplication.

10. F contains infinitely many irrational elements.

11. The Principle of Mathematical Induction, and its alternative forms, can use *any integer* n_0 as the "starting point."(Theorem 1.3.11 establishes this for *natural numbers* only.)

12. Prove that $\mathbb{Z}$ satisfies axioms (A0) and (M0).

13. **(Project) Integer exponents:** In Definition 1.3.12 we used mathematical induction to define a^n, $\forall a$ in an ordered field F and all $n \in \mathbb{N}$. For $a \neq 0$ we define $a^0 = 1$ and $a^{-n} = 1/a^n$. Prove that with this definition the familiar "laws of exponents" hold: if a, b are nonzero elements of an ordered field F, then $\forall m, n \in \mathbb{Z}$,

    (a) $a^m a^n = a^{m+n}$ (b) $(a^m)^n = a^{mn}$
    (c) $a^m / a^n = a^{m-n}$ (d) $(ab)^n = a^n b^n$

    [See Exercises 1.3.20–1.3.22.]

1.5 The Archimedean Property

In this section, we discuss a property so natural that it may seem hard to imagine an ordered field without it. Indeed, an ordered field without this property would not be acceptable as a number system. We would like to use our number system to record measurements, for example, in geometry. We often let the number 1 stand for a unit of measurement. Then, given any positive quantity x, we expect to be able to reach beyond x by "counting off" sufficiently many 1's. An ordered field with this property will be called an Archimedean ordered field. Not every ordered field has this property.

Definition 1.5.1 An ordered field F is **Archimedean** if it satisfies the **Archimedean property:** $\forall x \in F, \exists n \in \mathbb{N} \ni n > x$.

That is, for every element x in F, there is a natural number larger than x. The field of rational numbers is an example of an Archimedean ordered field (see Exercise 1). The next theorem shows other, equivalent, forms of this property.

Theorem 1.5.2 *Let F be an ordered field. The following properties are equivalent to the Archimedean property*[7] *in F:*

(a) $\forall x > 0, \exists n \in \mathbb{N} \ni n > x.$

(b) *If* $a > 0$, *then* $\forall x \in F, \exists n \in \mathbb{N} \ni na > x.$

(c) $\forall \varepsilon > 0, \exists n \in \mathbb{N} \ni \dfrac{1}{n} < \varepsilon.$

Proof. Our strategy will be to prove (a) $\Rightarrow$ Archimedean prop. $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(1) (a) $\Rightarrow$ Archimedean property. (Exercise 2)

(2) Archimedean property $\Rightarrow$ (b). (Exercise 3)

(3) To prove (b) $\Rightarrow$ (c), suppose (b) is true. Let $\varepsilon > 0$. Then $\dfrac{1}{\varepsilon} > 0$. Hence, by (b) with $a = 1$ and $x = \dfrac{1}{\varepsilon}$, $\exists n \in \mathbb{N} \ni n \cdot 1 > \dfrac{1}{\varepsilon}$. Then $n\varepsilon > 1$, so $\dfrac{1}{n} < \varepsilon$.

Therefore, $\forall \varepsilon > 0, \exists n \in \mathbb{N} \ni \dfrac{1}{n} < \varepsilon$. That is, (c) is true.

(4) (c) $\Rightarrow$ (a). (Exercise 4) ■

7. Since these three properties are equivalent to the Archimedean property, any one of them may be called the Archimedean property.

Theorem 1.5.3 *Every positive element of an Archimedean ordered field can be located between a unique pair of successive natural numbers. That is,*

$$\forall x > 0, \exists \text{ unique } n \in \mathbb{N} \ni n-1 \leq x < n.$$

***Proof.** Part 1 (Existence): Let F be an Archimedean ordered field, and $x > 0$. By the Archimedean property, the set $S = \{n \in \mathbb{N} : x < n\}$ is nonempty. Hence, by the well-ordering property (Theorem 1.3.10) the set S has a smallest element, say n_0. Then $x < n_0$ and $n_0 - 1 \notin S$, since n_0 is the smallest element of S. Hence, $x \nless n_0 - 1$. That is, $n_0 - 1 \leq x$. Therefore, we have $n_0 - 1 \leq x < n_0$.

Part 2 (Uniqueness): Suppose $\exists m, n \in \mathbb{N} \ni$

$$n-1 \leq x < n \text{ and } m-1 \leq x < m.$$

That is, $x < n \leq x+1$ and $x < m \leq x+1$. Multiplying through the second inequality by -1, we have

$$x < n \leq x+1 \text{ and } -x-1 \leq -m < -x.$$

Adding these two inequalities, we have

$$-1 < n-m < 1.$$

But there is only one integer between -1 and 1; namely, 0. Therefore, $n-m = 0$. That is, $n = m$.

Therefore, there is only one natural number n such that $n-1 \leq x < n$. ■

Corollary 1.5.4 *Every element of an Archimedean ordered field can be located between a unique pair of successive integers. That is,*

$$\forall x \in F, \ \exists \text{ unique } n \in \mathbb{Z} \ni n-1 \leq x < n.$$

Proof. Exercise 5. ■

Corollary 1.5.5 *Between any two elements greater than one unit apart in an Archimedean ordered field, there is an integer. That is, if $y - x > 1$, then $\exists$ integer $n \ni x < n < y$.*

***Proof.** Suppose F is an Archimedean ordered field, $x, y \in F$, and $y - x > 1$. By Corollary 1.5.4, $\exists$ integer $n \ni$

$$n-1 \leq x < n.$$

Adding 1 to both sides of these inequalities,

$$n \leq x+1 < n+1.$$

By putting together pieces of these inequalities, we have (justify each step)

$$x < n \leq x+1 < y.$$

Thus, $\exists$ integer $n \ni x < n < y$. ■

DENSE SETS IN ORDERED FIELDS

Definition 1.5.6 A set S is **dense** in an ordered field F if

$$\forall a < b \text{ in } F,\ \exists x \in S \ni a < x < b.$$

Theorem 1.5.7 (a) (***Denseness of the Rationals***) *The rational numbers form a dense set in any Archimedean ordered field.*

(b) (***Denseness of the Irrationals***) *In any Archimedean ordered field with at least one irrational element, the irrational elements form a dense subset.*

Proof. Suppose F is an Archimedean ordered field, and $a < b$ in F.

(a) Since $b - a > 0$, Theorem 1.5.2 (b) tells us that $\exists n \in \mathbb{N} \ni n(b-a) > 1$. Then $nb - na > 1$, so by Corollary 1.5.5, $\exists m \in \mathbb{Z} \ni$

$$na < m < nb.$$

Hence,

$$a < \frac{m}{n} < b.$$

Thus, the rational numbers form a dense subset of F.

(b) Exercise 6. ■

Theorem 1.5.8 *If S is a dense set in an ordered field F, then between any two elements of F there are* ***infinitely many*** *elements of S.*

Proof. Exercise 8. ■

As a consequence of Exercise 1.4.5, an Archimedean ordered field F with one irrational element must contain at least as many irrational as rational elements. Moreover, Theorem 1.5.7 says that they must be densely scattered in F. A more surprising result is that in the real number system, the irrational numbers far outnumber the rational numbers. The sense in which this is true, and a proof of that result, is found in Section 2.8.

The following theorem continues in the vein of Theorem 1.5.2. It establishes a method used frequently in analysis to prove both inequalities and equalities.

Theorem 1.5.9 (***Forcing Principle***) *Suppose F is an Archimedean ordered field, and $x, a, b \in F$.*

(a) *If $\forall \varepsilon > 0$, $x \le \varepsilon$, then $x \le 0$.*

(b) *If $\forall \varepsilon > 0$, $x \le a + \varepsilon$, then $x \le a$.*

(c) *If $\forall \varepsilon > 0$, $|x| \le \varepsilon$, then $x = 0$.*

(d) *If $\forall \varepsilon > 0$, $|a - b| \le \varepsilon$, then $a = b$.*

Proof. (a) Suppose that $\forall \varepsilon > 0$, $x \leq \varepsilon$. For contradiction, suppose $x \nleq 0$. Then $x > 0$, so $\frac{1}{2}x > 0$. Then, taking $\varepsilon = \frac{1}{2}x$, $x \leq \frac{1}{2}x$. Contradiction. Therefore, $x \leq 0$.

(b) Exercise 9.

(c) Exercise 10.

(d) Exercise 11. ■

Although it may be hard to imagine, there are non-Archimedean ordered fields. Such fields are not often encountered, but can be constructed without great difficulty. See Exercise 13 below.

EXERCISE SET 1.5

1. Prove that the ordered field of rational numbers is Archimedean.
2. Prove Theorem 1.5.2, (a) $\Rightarrow$ Archimedean Property.
3. Prove Theorem 1.5.2, Archimedean Property $\Rightarrow$ (b).
4. Prove Theorem 1.5.2, (c) $\Rightarrow$ (a).
5. Prove Corollary 1.5.4.
6. Prove Theorem 1.5.7 (b).
7. Prove that any ordered field F, whether or not it is Archimedean, is "**dense in itself**;" that is, between any two elements of F, there exists another element of F.
8. Prove Theorem 1.5.8.
9. Prove Part (b) of the "forcing principle" (Theorem 1.5.9).
10. Prove Part (c) of the "forcing principle" (Theorem 1.5.9).
11. Prove Part (d) of the "forcing principle" (Theorem 1.5.9).
12. Prove the following extension of the "forcing principle" (Theorem 1.5.9):

    (a) If $\forall \varepsilon > 0$, $x \geq -\varepsilon$, then $x \geq 0$.

    (b) $\forall \varepsilon > 0$, $x \geq a - \varepsilon$, then $x \geq a$.

13. **(Project) A non-Archimedean ordered field:** Recall that a *polynomial* (in one variable) is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \cdots a_n$ are (constant) real numbers. If $a_n \neq 0$, then n is called the "degree" of $p(x)$, and the coefficient a_n is called the "leading coefficient" of $p(x)$. A function of the form $R(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x)$ is not the zero polynomial, is called a *rational expression* (in x).

(a) Let F denote the set of all rational expressions in x, together with the usual operations of addition and multiplication of rational expressions. Prove that F is a field.

(b) Let $\mathcal{P}$ denote the set of all rational expressions $\frac{p(x)}{q(x)}$, where the leading coefficients of $p(x)$ and $q(x)$ are either both positive or both negative. Prove that $\mathcal{P}$ satisfies the order axioms (O1)–(O3), and thus F, together with $\mathcal{P}$, is an ordered field.[8]

(c) Identify the "natural numbers" in this ordered field.

(d) Prove that there are rational expressions in F larger than all the natural numbers in F, and thus prove that this ordered field is non-Archimedean.

1.6 The Completeness Property

The final defining characteristic of the real number system is that it is a "complete" ordered field. The completeness property is the most difficult of all the properties to describe. To do so, we first need to make some remarks and observations about "bounded" sets, and then define "suprema" and "infima" of sets.

BOUNDED SETS IN ORDERED FIELDS

Definition 1.6.1 Suppose that F is an ordered field, $A \subseteq F$, and $u \in F$. We say that

(1) u is an **upper bound** for A if $\forall x \in A$, $x \leq u$.

(2) u is a **lower bound** for A if $\forall x \in A$, $x \geq u$.

(3) u is a **maximum** (or **greatest**) **element** for A if $u \in A$, and $\forall x \in A$, $x \leq u$. The notation we use to express this is $u = \max A$.

(4) u is a **minimum** (or **least**) **element** for A if $u \in A$, and $\forall x \in A$, $x \geq u$. The notation we use to express this is $u = \min A$.

---

8. See Definition 1.2.1.

If A has an upper bound we say that A is **bounded above**; if A has a lower bound we say that A is **bounded below**. If A is bounded above and below, we say that A is **bounded**.

Theorem 1.6.2 (a) *A set can have more than one upper bound and more than one lower bound.*

(b) *A set cannot have more than one maximum nor more than one minimum element.*

(c) *Every nonempty finite*[9] *set has both a maximum element and a minimum element.*

Proof. (a) Exercise 3.

(b) Part 1: Suppose $u = \max A$ and $v = \max A$. Then

$$\begin{aligned}&(1)\ u \in A, \text{ and } \forall x \in A,\ x \le u;\\ &(2)\ v \in A, \text{ and } \forall x \in A,\ x \le v.\end{aligned}$$

In (1) we may take $x = v$, and so $v \le u$. In (2) we may take $x = u$, and so $u \le v$. Therefore, $u = v$.

Part 2: Suppose $u = \min A$ and $v = \min A$. Show $u = v$ (Exercise 4).

(c) Suppose S is a nonempty finite subset of F. That is, $\exists n \in \mathbb{N} \ni$

$$S = \{x_1, x_2, \cdots x_n\}.$$

Part 1: To prove that S has a *maximum* element, we shall use mathematical induction on n.

(1) Any set with only one element, $S = \{x_1\}$, has a maximum element; $x_1 = \max S$.

(2) Suppose that any set with k elements has a maximum element. Let S be a set with $k+1$ elements, say

$$S = \{x_1, x_2, \cdots x_{k+1}\}.$$

Then the set $T = \{x_1, x_2, \cdots, x_k\}$ has a maximum element; say $t = \max T$. By the Law of Trichotomy, we may let

$$u = \max\{t, x_{k+1}\}.$$

Then $u \in S$, and $\forall x \in S$, $x \le u$. Therefore, $u = \max S$. Thus, any set with $k+1$ elements has a maximum element.

Therefore, by mathematical induction, any finite set has a maximum element.

Part 2: Prove that S has a *minimum* element (Exercise 5). Therefore, any finite set has a minimum element. ■

An *infinite* set, even when bounded, may or may not have a maximum or a minimum element. For example, an "open interval" (a, b) has neither a minimum element nor a maximum element. (Explain.) Yet although a and b

9. For a rigorous definition of "finite" set, see Definition 2.8.2.

are not minimum or maximum elements of (a, b), there is something special about a and b. In technical language, we say that a is the *greatest lower bound of* (a, b) and b is the *least upper bound of* (a, b). We now define these technical terms. Examples and exercises will clarify them further.

SUPREMA AND INFIMA

Definition 1.6.3 Suppose that F is an ordered field and $A \subseteq F$. We say that an element $u \in F$ is

(1) a **least upper bound ("supremum")** of A if u is an upper bound for A and $\forall$ upper bounds v for A, $u \leq v$. The notation we use is $u = \sup A$.

(2) a **greatest lower bound ("infimum")** of A if u is a lower bound for A and $\forall$ lower bounds v for A, $u \geq v$. The notation we use is $u = \inf A$.

We now justify the assertion that a is the *greatest lower bound of* (a, b) and b is the *least upper bound of* (a, b).

Theorem 1.6.4 *Let $a < b$ in an ordered field F. Then*

$$a = \inf(a, b), \text{ and } b = \sup(a, b).$$

Proof. Part 1. First we prove that $a = \inf(a, b)$.

(i) By definition of (a, b), a is a lower bound of (a, b).

(ii) Suppose that v is a lower bound of (a, b). We must prove that $a \geq v$. For contradiction, suppose $a \not\geq v$. That is, $a < v$. Since v is a lower bound for (a, b) and $\frac{a+b}{2} \in (a, b)$, we have

$$a < v \leq \frac{a+b}{2} < b.$$

Let $c = \frac{a+v}{2}$. Then $a < c < v < b$. Thus, $c \in (a, b)$ and $c < v$. But v is a lower bound for (a, b). Contradiction. Therefore, $a \geq v$.

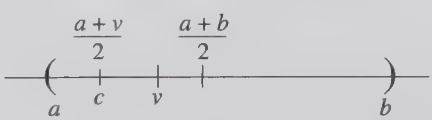


Figure 1.1

Part 2. Next, prove that $b = \sup(a, b)$. Exercise 6. ■

It may be helpful to think of $\inf A$ and $\sup A$ as "substitutes" for $\min A$ and $\max A$ in instances when A is a bounded set that does not have a maximum or minimum. It should also be observed that although $\min A$ and $\max A$ must

be members of A, $\inf A$ and $\sup A$ do not necessarily belong to A. In fact, we must be very careful when we read that a set A "has" an *inf* or a *sup* not to conclude that these are elements of A.

Theorem 1.6.5 (a) *A set cannot have more than one greatest lower bound.*

(b) *A set cannot have more than one least upper bound.*

(c) *If a set has a minimum (or maximum) element, then that element is the greatest lower bound (or least upper bound) of A.*

(d) *If a set* ***contains*** *a greatest lower bound (or least upper bound) then that element is the minimum (or maximum) element of A.*

Proof. (a) Let u and u' be greatest lower bounds of A. Then

(i) $u = \inf A$ and u' is a lower bound of $A \Rightarrow u \geq u'$.

(ii) $u' = \inf A$ and u is a lower bound of $A \Rightarrow u' \geq u$.

(iii) By (i) and (ii), $u = u'$. Therefore, A cannot have more than one greatest lower bound.

(b) Exercise 7.

(c) Exercise 8.

(d) Exercise 9. ■

Theorem 1.6.6 (ε ***Criterion for Supremum***) *Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Then $u = \sup A \Leftrightarrow \forall \varepsilon > 0$,*

(a) *$\forall x \in A$, $x < u + \varepsilon$, and*

(b) *$\exists x \in A \ni x > u - \varepsilon$.*

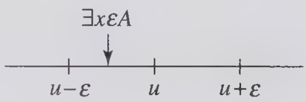


Figure 1.2

Proof. Let F be an ordered field, $A \subseteq F$, and $u \in F$.

(1) First, the ($\Rightarrow$) part. Suppose $u = \sup A$. Let $\varepsilon > 0$.

(i) Let $x \in A$. Then $x \leq \sup A = u < u + \varepsilon$. Thus, $\forall x \in A$, $x < u + \varepsilon$.

(ii) $u - \varepsilon < u$, which is the *least* upper bound for A, so $u - \varepsilon$ cannot be an upper bound for A. Hence, $\exists x \in A \ni x > u - \varepsilon$.

(2) To prove the ($\Leftarrow$) part, suppose $u \in F$ satisfies conditions (a) and (b) of the hypotheses.

(i) Let $x \in A$. By (a), $\forall \varepsilon > 0$, $x < u + \varepsilon$. Thus, by the "forcing principle" [Theorem 1.5.9 (b)], $x \leq u$. Thus, $\forall x \in A$, $x \leq u$. That is, u is an upper bound for A.

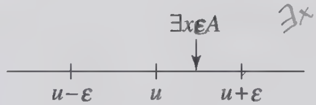
(ii) Suppose that v is an upper bound for A. We need to prove that $u \leq v$. For contradiction, suppose $u > v$. Let $\varepsilon = u - v$. Then $\varepsilon > 0$, so by hypothesis (b), $\exists x \in A \ni$

$$x > u - \varepsilon$$
$$x > u - (u - v)$$
$$x > v.$$

But v is an upper bound for A, and $x \in A$. Contradiction. Therefore, $u \leq v$.
By (i) and (ii) together, $u = \sup A$. ■

Theorem 1.6.7 (*ε Criterion for Infimum*) *Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Then $u = \inf A \Leftrightarrow \forall \varepsilon > 0$,*

(a) *$\forall x \in A$, $x > u - \varepsilon$, and*
(b) *$\exists x \in A \ni x < u + \varepsilon$.*



Typo
∃x∈A

Figure 1.3

Proof. Exercise 11. ■

EXERCISE SET 1.6-A

1. Assume that real numbers exist and behave according to the familiar rules of algebra. For each of the following sets of real numbers, tell whether or not the given set is bounded above. For those that are, give three different upper bounds and find the least upper bound.

(a) $(-1, 3]$ (b) $[-1, 3)$
(c) $\{1, 2, 3, 4\}$ (d) $\{5\}$
(e) $(-\infty, 0]$ (f) $(0, +\infty)$
(g) $\varnothing$ (h) $\left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$
(i) $\left\{\dfrac{1}{x} : x > 0\right\}$ (j) $\left\{\dfrac{1}{x} : 1 < x < 2\right\}$
(k) $\left\{2 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$ (l) $\left\{1 + \dfrac{1}{2^n} : n \in \mathbb{N}\right\}$
(m) $\left\{\dfrac{x+1}{x} : x > 2\right\}$ (n) $\left\{\sin\left(\dfrac{n\pi}{2}\right) : n \in \mathbb{N}\right\}$

2. For each of the sets listed in Exercise 1, tell whether or not the given set is bounded below. For those that are, give three different lower bounds and find the greatest lower bound.

3. Prove Theorem 1.6.2 (a).

4. Prove Theorem 1.6.2 (b), Part 2.

5. Finish proving Theorem 1.6.2 (c), that S has a minimum element.

6. Complete the proof of Theorem 1.6.4 by proving Part 2.

7. Prove Theorem 1.6.5 (b).

8. Prove Theorem 1.6.5 (c).

9. Prove Theorem 1.6.5 (d).

10. Prove that $\mathbb{N}$ has no maximum element. Does it have a supremum? A minimum? An infimum?

11. Prove Theorem 1.6.7. [Note: The "forcing principle" (Theorem 1.5.9) must be "massaged" a little to make it apply here. See Exercise 1.5.12.]

12. Prove that in an ordered field F, u is a lower bound of a set $A \Leftrightarrow -u$ is an upper bound of the set $-A = \{-a : a \in A\}$. [Thus, A is bounded below iff $-A$ is bounded above.]

13. Let F be an Archimedean ordered field, $A \subseteq F$, and $u \in F$. Prove the modified **ε-criterion for supremum**: if $u \notin A$, $u = \sup A \Leftrightarrow \forall \varepsilon > 0$,

    (a) $\forall x \in A$, $x < u + \varepsilon$, and

    (b) $\exists$ infinitely many $x \in A \ni x > u - \varepsilon$.

14. State and prove a modified **ε-criterion for** $v = \inf A$ if $v \notin A$. [See Exercise 13.]

THE COMPLETENESS AXIOM

Definition 1.6.8 An ordered field F is **complete** if it satisfies the

Completeness property (C): every nonempty subset of F that has an upper bound in F has a least upper bound in F. (See also Theorem 1.6.12 below.)

Caution: recall that the word "has" here does not mean "contains." The supremum of A need not be a member of A.

As we shall shortly see, there are common fields that are *not* complete. But first, we develop a few properties of complete ordered fields.

Theorem 1.6.9 *Any complete ordered field is Archimedean.*

Proof. Suppose F is a complete ordered field. For contradiction, suppose F is not Archimedean. That is, it is not true that $\forall x > 0$, $\exists n \in \mathbb{N}_F \ni n > x$. Equivalently, $\exists x > 0$ in $F \ni \forall n \in \mathbb{N}_F$, $n \le x$.

This means that the set $\mathbb{N}_F$ is a nonempty set in F with an upper bound, x. Then, since F is complete,

$$\exists\, x_0 = \sup \mathbb{N}_F.$$

Now, $\forall n \in \mathbb{N}_F$, $n+1 \in \mathbb{N}_F$, so $n+1 \le x_0$, which implies that $n \le x_0 - 1$. In summary,

$$\forall n \in \mathbb{N}_F,\ n \le x_0 - 1.$$

This says that $x_0 - 1$ is an upper bound for $\mathbb{N}_F$; but x_0 is the least upper bound for $\mathbb{N}_F$. Contradiction. Therefore, F is Archimedean. ■

Theorem 1.6.10 *If an ordered field F is complete, then $\exists x \in F \ni x^2 = 2$.*

***Proof.** Suppose F is a complete ordered field. Define the following subsets of F:

$$A = \{x > 0 : x^2 < 2\} \text{ and } B = \{x > 0 : x^2 > 2\}.$$

Then A and B are nonempty, since $1 \in A$ and $2 \in B$. Also, A is bounded above. Since F is complete, we may let $u = \sup A$. Then $u \ge 1$, since $1 \in A$.

We first establish some preliminary results:

1. Suppose y satisfies $0 < y < u$. Then $y < \sup A$, so $\exists a \in A \ni y < a \le u$, and so $y^2 < a^2 < 2$. (See Theorem 1.2.8 (e).) Thus, $y \in A$.

2. Suppose y satisfies $y > u$. Then let $u < t < y$. (Such a t exists; for example, $t = \frac{u+y}{2}$.) Since $t > \sup A$, $t \notin A$; so $t^2 \ge 2$. Then $y^2 > t^2 \ge 2$. Thus, $y \in B$.

3. Summarizing (1) and (2), every positive number less than u lies in A, and every number greater than u lies in B.

We now prove that $u^2 = 2$. We shall use the forcing principle [Theorem 1.5.9 (d)]. Let $\varepsilon > 0$. Choose $\delta \in F$ to be any element such that $0 < \delta < \min\left\{u, \dfrac{\varepsilon}{4u}\right\}$. Then $0 < \delta < u$ and $0 < \delta < \dfrac{\varepsilon}{4u}$, so by (3) above, $u - \delta \in A$ and $u + \delta \in B$. Thus,

$$(u - \delta)^2 < 2 < (u + \delta)^2.$$

Multiplying through by -1, we have

$$-(u + \delta)^2 < -2 < -(u - \delta)^2.$$

But we also know from Theorem 1.2.8 (e) that

$$(u-\delta)^2 < u^2 < (u+\delta)^2.$$

Adding these two inequalities together, we have

$$\begin{gathered}(u-\delta)^2 - (u+\delta)^2 < u^2 - 2 < (u+\delta)^2 - (u-\delta)^2.\\ -4u\delta < u^2 - 2 < 4u\delta\\ |u^2-2| < 4u\delta\\ |u^2-2| < 4u \cdot \frac{\varepsilon}{4u} \quad \left(\text{since } \delta < \frac{\varepsilon}{4u}\right)\\ |u^2-2| < \varepsilon.\end{gathered}$$

Since this holds $\forall \varepsilon > 0$, we have $u^2 = 2$ by the "forcing principle." ■

Corollary 1.6.11 *The ordered field $\mathbb{Q}$ of rational numbers is* not *complete.*

Proof. By Theorem 1.4.5, $\mathbb{Q}$ has no element whose square is 2. By Theorem 1.6.10, a complete ordered field must have an element whose square is 2. Therefore, $\mathbb{Q}$ cannot be complete. ■

The completeness property makes no mention of nonempty sets with lower bounds. The curious student will ask whether, in a complete ordered field, such sets must have greatest lower bounds. The next theorem answers that question.

Theorem 1.6.12 *In any complete ordered field, every nonempty set that has a lower bound in F has a greatest lower bound in F.*

Proof. Exercise 1. ■

THE REAL NUMBER SYSTEM, $\mathbb{R}$

It turns out that there is one, and essentially only one, complete ordered field. This is a deep and fundamental result in the foundations of mathematics, and we shall discuss it briefly in Section 1.7. Meanwhile, we take advantage of this result to define the real number system.

Definition 1.6.13 The **real number system** is **the complete ordered field**. It is denoted $\mathbb{R}$. Its elements are called **real numbers**.

Finally, we admit the symbols $+\infty$ and $-\infty$ into our system, to be used for the supremum and infimum of unbounded sets.

SETS WITH NO INFIMUM OR NO SUPREMUM; THE SYMBOLS $-\infty$ AND $+\infty$

In view of Definition 1.6.8 and Theorem 1.6.12, the only sets in a complete ordered field that have no infimum or no supremum are sets that either are empty, have no lower bounds, or have no upper bounds. Accordingly, we make the following:

Definition 1.6.14 ($-\infty$ and $+\infty$ as infimum and supremum): Let $\mathbb{R}$ denote the complete ordered field,

(a) If a set $A \subseteq \mathbb{R}$ has no lower bound, we say that $\inf A = -\infty$;

(b) If a set $A \subseteq \mathbb{R}$ has no upper bound, we say that $\sup A = +\infty$;

(c) Since every real number is a lower bound of $\varnothing$, the empty set has no greatest lower bound, so we define $\inf \varnothing = +\infty$.

(d) Since every real number is an upper bound of $\varnothing$, the empty set has no least upper bound, so we define $\sup \varnothing = -\infty$.

It is important to remember that $-\infty$ and $+\infty$ are not real numbers, but are merely symbols used to conveniently convey certain carefully defined situations (here, to denote that a set is unbounded below or above). In addition, we adopt the universal convention that all real numbers x obey the inequalities

$$-\infty < x < +\infty.$$

In analysis, the symbols $-\infty$ and $+\infty$ are frequently used in inequalities and equations, whenever it is convenient. But these symbols are always used with the understanding that they do not represent real numbers.

EXERCISE SET 1.6-B

In Exercises 1–5 below, assume that $A, B \subseteq F$, a complete ordered field.

1. Prove Theorem 1.6.12. (See Exercise 1.6-A.12.)

2. Suppose $A \subseteq B$ and $A \neq \varnothing$. Prove that

   (a) if B is bounded below, then so is A, and $\inf A \geq \inf B$.

   (b) if B is bounded above, then so is A, and $\sup A \leq \sup B$.

3. Suppose A is a nonempty set of F that is bounded above in F, and let B be the set of all upper bounds of A. Prove that B has a minimum element, and $\sup A = \min B$.

4. Suppose A is a nonempty set of F that is bounded below in F, and let B be the set of all lower bounds of A. Prove that B has a maximum element, and $\sup A = \max B$.

5. Suppose A and B are nonempty subsets of F bounded above in F. Prove that $A \cup B$ is bounded above, and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.

6. Modify the proof of Theorem 1.6.10 to prove that if $a > 0$ in a complete ordered field F, then $\exists x \in F \ni x^2 = a$.

7. Let X be a nonempty set and suppose $f, g : X \to \mathbb{R}$ are functions whose ranges are bounded sets of real numbers. Prove that
   (a) $\sup\{f(x)+g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$.
   (b) $\inf\{f(x)+g(x) : x \in X\} \geq \inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\}$.

1.7 *"The" Complete Ordered Field

In this section we take a closer look at Definition 1.6.13. In Sections 1.1–1.6 we explored successively the properties of fields, ordered fields, Archimedean ordered fields, and complete ordered fields. We defined a complete ordered field as a set F together with two operations, $+$ and $\cdot$, which satisfy fifteen axioms: (A0)–(A4), (M0)–(M4), (D), (O1)–(O3), and (C). We subsequently showed that a complete ordered field, if there is one, must have the properties we expect our real number system to have.

Philosophically, two questions of great significance remain to be addressed:

- **QUESTION #1: Is there a complete ordered field?**

  Whenever a concept is defined by specifying the properties one wishes it to possess, one runs the danger of being overly prescriptive. We may have listed so many properties that it is impossible for anything to exist that possesses all these properties. The properties may even be contradictory. Consider an example from everyday life: Were you to search for a mate who possessed **all** the properties that a perfect mate would have, it is doubtful that you would ever find such a person! Your search might well end in frustration: sooner or later you could reach the conclusion that you have been overprescriptive. That is precisely what lies behind Question #1, in the case of complete ordered fields. We must determine whether we have laid down so many axioms that nothing exists that satisfies them!

The standard way of dealing with this question is to build up, or "construct," a complete ordered field from basic building blocks whose existence is not questioned, using a process that is beyond reproach. Then, in effect, we can say that since we accept the existence of the basic building blocks, and do not question the construction process, we must accept the existence of the complete ordered field that is constructed by this process.

This constructive process has been carried out to the satisfaction of mathematicians, starting with the natural numbers $1, 2, 3, \cdots$ as the basic building blocks, and using formal logic and set theory as the process. Such a construction process, however, is deep and more complicated than appropriate for this course. It would take several weeks of hard and tedious effort. Nevertheless, an enterprising student may wish to investigate this process as a project. The

classic reference[10] is [82]; other constructions may be found in Chapter 1 of Part B of [10], as well as in [11], [25], [37], [44], [63], [74], [96], [117], and [129].

An easier way to deal with Question #1 is to merely assume the following as an axiom:

AXIOM: There exists a complete ordered field.

In effect, this is the approach we take in this book. We simply assume this axiom, realizing that, as we have just discussed, it would be possible to *construct* a complete ordered field, starting only with a set of axioms for the natural number system. This decision seems appropriate for this course.

- **QUESTION #2: Is there more than one complete ordered field?** Students of mathematics become very familiar with defining a concept by specifying its properties. For example, we are familiar with the process of defining groups, vector spaces, rings, and so on. Anything that satisfies certain properties is a "group," for example. We ordinarily expect there to be many different examples that satisfy these properties. In fact, because there are many different groups, we feel it is necessary to categorize them into classes of look-alikes. We use the concept of **isomorphism** to express this alikeness. Two algebraic systems are **isomorphic** as groups, for example, if they are both groups and are indistinguishable in the sense that one of the groups is merely a relabeling of the other. The concern raised by Question #2 is this: When we categorize complete ordered fields into classes according to isomorphism, do we get more than one class? We should be quite surprised if we get only one isomorphism class. This has never happened for other algebraic systems that we have encountered. (Think of groups, rings, fields, vector spaces, etc.)

It can be shown by algebraic methods that **any two complete ordered fields are isomorphic**! Of course that statement requires clarification. We shall not prove this fact in this course, but will state it as a formal theorem and provide a guideline for proving it as a project.

Theorem 1.7.1 (*Uniqueness of the Complete Ordered Field*) *If F and F' are any two complete ordered fields, then $\exists$ a 1-1 correspondence $f : F \to F' \ni$* PROJECT

(1) $\forall x, y \in F,\ f(x+y) = f(x) + f(y)$;

(2) $\forall x, y \in F,\ f(xy) = f(x) \cdot f(y)$;

(3) $f(0_F) = 0_{F'}$;

10. The numbers in square brackets refer to entries in the Bibliography, which follows Appendix B.

(4) $f(1_F) = 1_{F'}$;

(5) $\forall x \in F,\ f(-x) = -f(x)$;

(6) $\forall x \in F,\ f(x^{-1}) = f(x)^{-1}$;

(7) *For each natural number* $n_F \in \mathbb{N}_F$, *and the corresponding natural number* $n_{F'} \in \mathbb{N}_{F'}$, $f(n_F) = n_{F'}$;

(8) *For each rational number* $\dfrac{n_F}{m_F} \in \mathbb{Q}_F$, *and the corresponding rational number* $\dfrac{n_{F'}}{m_{F'}} \in \mathbb{Q}_{F'}$, $f\left(\dfrac{n_F}{m_F}\right) = \dfrac{n_{F'}}{m_{F'}}$;

(9) $\forall x, y \in F,\ x < y \Leftrightarrow f(x) < f(y)$;

(10) *For all* $A \subseteq F$,

   (a) A *is bounded above in* $F \Leftrightarrow$ *the set* $f(A) = \{f(x) : x \in A\}$ *is bounded above in* F'. *Moreover,* $f(\sup A) = \sup f(A)$;

   (b) A *is bounded below in* $F \Leftrightarrow$ *the set* $f(A) = \{f(x) : x \in A\}$ *is bounded below in* F'. *Moreover,* $f(\inf A) = \inf f(A)$.

*To prove this theorem is beyond the scope of this book. However, it is not too difficult, and would make a rewarding project for an enterprising student. A few remarks about how to proceed with the proof will suffice for our purposes. First, there is much redundancy in the statement of the theorem. Properties (3)–(8) are easily derived from properties (1) and (2), and property (10) is derivable from (9). Thus, all we need to do is show how to construct a 1-1 correspondence $f : F \to F'$ satisfying properties (1), (2), and (9).

Step 1. Define $f : \mathbb{N}_F \to \mathbb{N}_{F'}$ by $f(n_F) = n_{F'}$.

Step 2. Extend f to $f : \mathbb{Z}_F \to \mathbb{Z}_{F'}$ by

$$\begin{cases} f(n_F) & = n_{F'}, \quad \forall n_F \in N_F \\ f(0_F) & = 0_{F'} \\ f(-n_F) & = -n_{F'}, \ \forall n_F \in N_F \end{cases}$$

Step 3. Extend f to $f : \mathbb{Q}_F \to \mathbb{Q}_{F'}$ by

$$f\left(\frac{m_F}{n_F}\right) = \frac{m_{F'}}{n_{F'}}.$$

Actually, we could have started here, with this as our basic definition. Steps 1 and 2 were written down merely to help conceptualize the process.

Step 4. Prove that this $f : \mathbb{Q}_F \to \mathbb{Q}_{F'}$ satisfies properties (1), (2), and (9).

Step 5. Prove the following:

Lemma 1.7.2 *If F is a complete ordered field, then*

$$\forall x \in F,\ x = \sup\{r \in \mathbb{Q}_F : r < x\}.$$

That is, in a complete ordered field, every element is the supremum of the set of all *rational* elements less than it.

Step 6. Define $f : F \to F'$ by

$$f(x) = \sup\{f(r) : r \in \mathbb{Q}_F \text{ and } r < x\}.$$

(Before you can accept this as a valid definition, you must first prove that $\sup\{f(r) : r \in \mathbb{Q}_F \text{ and } r < x\}$ *exists*.)

Step 7. Prove that the function $f : F \to F'$ just defined satisfies properties (1), (2), and (9). To do this, you will need results from Exercise Set 1.6-B.

Warning: Completing Step 7 may be quite challenging! ■

SUMMARY

Since there is essentially only one complete ordered field, we give it a special name and symbol. The complete ordered field is called **"the Real Number System"** and is denoted with the special symbol, $\mathbb{R}$.

Definition 1.7.3 (The Real Number System)

$\mathbb{R}$ = the complete ordered field; its members are called *real numbers*.

For further insight on the uniqueness of the complete ordered field, the reader may consult references [10], [15], [37], [63], and [93].

Chapter 2

Sequences

Sections 2.1–2.7, through Theorem 2.7.4, contain essential core material. Indeed, the concepts and methodology introduced here lie at the very heart of the subject, and will be used throughout the remainder of the book. This chapter also contains some material labeled "$*$" that may be covered in courses lasting more than one semester. Section 2.8 should be learned sometime in a student's career, but not necessarily here. Upper and lower limits of sequences are covered in (optional) Section 2.9.

In this chapter we experience our first encounter with the notion of "limit." This notion is a central concept in the subject of analysis. It will appear in several guises throughout the course. I believe, as do many mathematicians, that the concept and techniques of limits are most easily learned in the context of (infinite) sequences. Later forms and properties of limits are then readily seen as extensions of those learned in this context.

We shall develop the theory of sequences to a level where it can be seen as a powerful tool in analysis. Much of the power of analysis is anticipated in the theory of sequences, and many of its deepest results can be formulated in the language and conceptual framework of sequences. In fact, the concepts and methods presented in this chapter will be used in every remaining chapter of the book.

2.1 Basic Concepts: Convergence and Limits

A commonly-held, intuitive understanding of an (infinite) sequence is that it is an "infinite succession of numbers," not necessarily different:

$$x_1,\, x_2,\, x_3,\, \cdots, x_n,\, x_{n+1},\, \cdots.$$

Of course, this is merely a suggestive description. It cannot serve as a rigorous definition, since it does not define what is meant by an "infinite succession" of numbers. Another intuitive way of describing an infinite sequence is as a vector with infinitely many components:

$$(x_1,\, x_2,\, x_3,\, \cdots, x_n,\, x_{n+1},\, \cdots).$$

Again, this description may help us feel more comfortable with the notion of a sequence, but it fails to be a rigorous definition since it leaves "vector" and "component" undefined.

Finally, we give a precise and rigorous definition.

Definition 2.1.1 A **sequence** of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$.

That is, given any natural number n, there is a corresponding real number $x(n)$.

Comments and some conventions:

(1) We shall call $x(n)$ the **nth term** of the sequence.

(2) We shall hereafter always write the nth term as x_n, using subscript notation rather than the functional notation $x(n)$.

(3) The sequence itself will be denoted $\{x_n\}$, or occasionally $\{x_n\}_{n=1}^{\infty}$.

(4) Since by Definition 2.1.1, all sequences contain infinitely many[1] terms, it will not be necessary to call them *infinite* sequences. We merely call them sequences.

(5) Conventions (2) and (3) together make rigorous the intuitive view of a sequence $\{x_n\}$ as an infinite succession of numbers,

$$\{x_n\} = x_1,\, x_2,\, x_3,\, \cdots, x_n,\, x_{n+1},\, \cdots.$$

1. Strictly speaking, a sequence need not have infinitely many *different* terms, since some (or all) of them may be the same.

or as a vector with infinitely many components,

$$\{x_n\} = (x_1, x_2, x_3, \cdots, x_n, x_{n+1}, \cdots).$$

(6) We must be careful not to let the braces in the notation mislead us into thinking that we are talking about a *set* of numbers. For example, the *sequence* $\{1\}$ consisting of infinitely many terms, each of which equals 1, is different from the *set* $\{1\}$, which contains only one element, 1. We use the same notation $\{1\}$ in both cases; the context will determine which interpretation we mean.

Example 2.1.2 The first six terms of the sequence $\left\{2 + \frac{(-1)^n}{n}\right\}$ are

$$x_1 = 1, \quad x_2 = 2\tfrac{1}{2}, \quad x_3 = 1\tfrac{2}{3},$$

$$x_4 = 2\tfrac{1}{4}, \quad x_5 = 1\tfrac{4}{5}, \quad x_6 = 2\tfrac{1}{6}.$$

CONVERGENCE OF A SEQUENCE

The most important concept associated with sequences is that of *convergence to a limit.* Intuitively, when we say that a sequence $\{x_n\}$ converges to limit L, we mean that as n gets larger and larger, without bound, the terms x_n of the sequence get "close to" the number L; equivalently, the distance between x_n and L, which we measure by $|x_n - L|$, gets smaller than any positive real number. If we plot the function $x : \mathbb{N} \to \mathbb{R}$ in a coordinate system with a horizontal n-axis and vertical y-axis (where $y = x_n$), then the statement that $\{x_n\}$ converges to limit L is equivalent to saying that this graph has the horizontal line $y = L$ as an asymptote. See Figure 2.1(a).

Example 2.1.3 We graph the sequence $\{x_n\} = \left\{2 + \frac{(-1)^n}{n}\right\}$ by plotting the function $y = x_n$ in a two-dimensional coordinate system. Observe in Figure 2.1(a) that the horizontal line $y = 2$ is an asymptote. We thus say that the sequence $\{x_n\}$ converges to the number 2 as its limit.

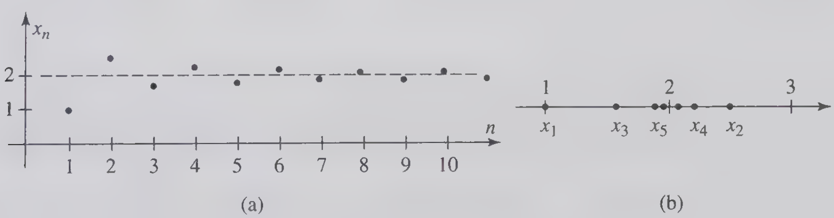


Figure 2.1

We can also plot the terms of the sequence as points on a number line, as in Figure 2.1(b), and notice that the successive terms of the sequence "cluster" around the limit, 2. We will find this notion of clustering to be a useful one later in the chapter.

We are now ready for the official definition of convergence and limit.

Definition 2.1.4 Let $\{x_n\}$ be a sequence and L be a real number. Then

$$\boxed{\lim_{n\to\infty} x_n = L \text{ if } \forall\varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni \forall n \in \mathbb{N},\, n \geq n_0 \Rightarrow |x_n - L| < \varepsilon.}$$ [2]

If $\lim_{n\to\infty} x_n = L$, we say that $\{x_n\}$ **converges** to L; and write

$$\boxed{x_n \to L.}$$

If there is no real number to which $\{x_n\}$ converges, we say that $\{x_n\}$ **diverges**.

USING DEFINITION 2.1.4 TO PROVE THAT $\lim_{n\to\infty} x_n = L$

Using Definition 2.1.4 will require a mental change of gears. Ignoring the quantifiers for the moment, notice that implementing the definition will require us to prove that one inequality implies another. Specifically, we must show that the inequality $n \geq n_0$ implies the inequality $|x_n - L| < \varepsilon$. When we throw in the quantifiers $\forall\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$, and $\forall n \in \mathbb{N}$, it can look pretty confusing! The secret to avoiding confusion is to understand what the definition *means*. Intuitively, it means we must show that corresponding to an arbitrary positive real number ε, after a certain number n_0 of terms, all the remaining terms of the sequence will be within a distance ε from L.

Verbal paraphrases of Definition 2.1.4:[3]

$\lim_{n\to\infty} x_n = L$ means:

- x_n can be made arbitrarily close to L by making n sufficiently large.
- $|x_n - L|$ can be made arbitrarily small by making n sufficiently large.
- For every positive ε there is some n_0 such that $|x_n - L| < \varepsilon$ whenever $n \geq n_0$.

Notice the important role played by *inequalities* in Definition 2.1.4. Why inequalities? The simplest explanation for this is that analysis must deal with *infinity*; both the infinitely large and the infinitely small. Since no real number

2. In practice, we usually leave out the third quantifier ($\forall n \in \mathbb{N}$) in the interest of simplicity. It is understood to be present even when not written.

3. Although Definition 2.1.4 is officially correct, and should be memorized, it is equally important (for the sake of understanding) to be able to paraphrase it in words.

is infinitely large or infinitely small, we must find a way to express the concept of infinity using only finite quantities. I call this the problem of "**finitizing the infinite**." It was a problem of critical importance in the development of analysis as a rigorous subject, and was solved in the nineteenth century by Cauchy, Weierstrass, and others. Their remarkable discovery was that inequalities and quantifiers provide the perfect mathematical tools for "finitizing the infinite."

Coming into this course you may not feel very comfortable with either inequalities or quantifiers. You haven't had to use them nearly as often as you have used equations. That is why the mental change of gears is necessary. In order to succeed in analysis, you will have to become quite skilled in handling both inequalities and quantifiers.

Strategy for using Definition 2.1.4 to prove that $\lim_{n\to\infty} x_n = L$:

1. Start by letting ε denote an arbitrary positive real number. That means, simply assume $\varepsilon > 0$; you know nothing about ε other than it is positive.

2. Examine the inequality $|x_n - L| < \varepsilon$. Try to find out how large n must be in order to guarantee that $|x_n - L| < \varepsilon$. This amounts to "playing with inequalities."

3. Once you think you have found a value for n_0 that will guarantee that $n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$, you must *prove* that this implication is true.

The following examples will illustrate important methods to be used in implementing the definition of $\lim_{n\to\infty} x_n = L$. Pay careful attention to them.

Example 2.1.5 Consider the limit statement $\lim_{n\to\infty} \left(\dfrac{2n+3}{3n-7}\right) = \dfrac{2}{3}$.

(a) After how many terms are we guaranteed that $\dfrac{2n+3}{3n-7}$ is within a distance of .01 of 2/3?

(b) After how many terms are we guaranteed that the n^{th} term of this sequence is an accurate approximation of the limit, to within 3 decimal places?

(c) For arbitrary $\varepsilon > 0$, after how many terms are we guaranteed that $\dfrac{2n+3}{3n-7}$ is within a distance ε of 2/3?

Solution: (a) We want an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < .01$. Now,

$$\begin{aligned}\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| &= \left|\frac{3(2n+3) - 2\,(3n-7)}{3(3n-7)}\right| \\ &= \left|\frac{6n+9-6n+14}{3(3n-7)}\right| \\ &= \frac{23}{|9n-21|}.\end{aligned}$$

We can eliminate the absolute value bars in the denominator if $9n - 21 > 0$. This inequality is true if $9n > 21$, which is true when $n \geq 3$. Thus, when $n \geq 3$, $|9n-21| = 9n-21$. Thus, our objective now is to find an $n_0 \in \mathbb{N} \ni$ both $n_0 \geq 3$ and

$$n \geq n_0 \Rightarrow \frac{23}{9n-21} < .01.$$

The latter inequality will be true if

$$\begin{aligned}\frac{9n-21}{23} &> 100 \\ 9n - 21 &> 2300 \\ 9n &> 2321 \\ n &> 257.88\cdots.\end{aligned}$$

Take $n_0 = 258$. We have shown that $n \geq 258 \Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < .01$.

(b) To make sure that the n^{th} term of the sequence approximates the limit accurately to three decimal places, we want to guarantee that rounding off to three decimal places does not cause a change in the third decimal digit. That is, we want to guarantee that the n^{th} term is within .0005 of the limit. So, we want to find an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < .0005$. As shown above, if $n \geq 3$,

$$\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \frac{23}{9n-21}.$$

Thus, our objective is to find an $n_0 \in \mathbb{N} \ni n_0 \geq 3$ and

$$n \geq n_0 \Rightarrow \frac{23}{9n-21} < .0005 = \frac{1}{2,000}.$$

The latter inequality will be true if

$$\frac{9n-21}{23} > 2,000$$
$$9n - 21 > 46,000$$
$$9n > 46,021$$
$$n > 5,113.444\cdots.$$

Take $n_0 = 5,114$. We have shown that $n \geq 5,114 \Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < .001$. That is, when $n \geq 5,114$, x_n will approximate $\lim\limits_{n\to\infty} x_n$ to three decimal places.

(c) Let $\varepsilon > 0$ be a fixed but arbitrary positive number. We want to find an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < \varepsilon$. As shown above, if $n \geq 3$,

$$\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| = \frac{23}{9n-21}.$$

Thus, our objective is to find an $n_0 \in \mathbb{N} \ni n_0 \geq 3$ and

$$n \geq n_0 \Rightarrow \frac{23}{9n-21} < \varepsilon.$$

Notice that when $n > 3$,

$$\begin{aligned}\frac{23}{9n-21} &< \frac{27}{9n-21} \quad (\text{since } 23 < 27 \text{ and } 9n-21 > 0)\\ &< \frac{27}{9n-27} \quad \Bigl(\text{since } -21 > -27 \Rightarrow 9n-21 > 9n-27\\ &\qquad\qquad\qquad \Rightarrow \frac{1}{9n-21} < \frac{1}{9n-27}\Bigr)\\ &< \frac{3}{n-3}.\end{aligned}$$

Thus, when $n > 3$, $\frac{23}{9n-21} < \varepsilon$ if $\frac{3}{n-3} < \varepsilon$. But the last inequality will be guaranteed if

$$\frac{n-3}{3} > \frac{1}{\varepsilon}$$
$$\text{i.e., } n - 3 > \frac{3}{\varepsilon}$$
$$\text{i.e., } n > \frac{3}{\varepsilon} + 3.$$

By the Archimedean property (remember that?), $\exists\, n_0 \in \mathbb{N} \ni n_0 > \dfrac{3}{\varepsilon} + 3$. Take $n_0 =$ such a natural number.[4] The above analysis shows that

$$n \geq n_0 \Rightarrow \left| \frac{2n+3}{3n-7} - \frac{2}{3} \right| < \varepsilon. \quad \square$$

Example 2.1.6 Prove that $\displaystyle\lim_{n\to\infty} \left(\frac{2n+3}{3n-7} \right) = \frac{2}{3}$.

Note: To prove this limit statement, we essentially work Part (c) of Example 2.1.5 backwards. The work we did there is best regarded as "scratchwork" for the proof we are about to give. We may include it as needed in our proof. But keep in mind that our proof must stand alone. For that reason, whatever we need from Example 2.1.5 (c) must be redone here.

Pay careful attention to the proof given here, as it will serve as a paradigm for the proofs you will be required to give.

Let $\varepsilon > 0$. By the Archimedean property, $\exists\, n_0 \in \mathbb{N} \ni n_0 > \dfrac{3}{\varepsilon} + 3$. Then,

$$n \geq n_0 \Rightarrow n > \frac{3}{\varepsilon} + 3 \text{ and } n > 3$$ to make $|x|$ go away

$$\Rightarrow n - 3 > \frac{3}{\varepsilon} \text{ and } n - 3 > 0$$

$$\Rightarrow \frac{n-3}{3} > \frac{1}{\varepsilon} \text{ and } n - 3 > 0$$

$$\Rightarrow \frac{3}{n-3} < \varepsilon \quad \text{(by Theorem 1.2.10, and since } n-3>0\text{)}$$

$$\Rightarrow \frac{27}{9n-27} < \varepsilon$$

$$\Rightarrow \frac{23}{9n-21} < \varepsilon.$$

$$\left(\begin{array}{l} \text{Since } -21 > -27 \Rightarrow 9n-21 > 9n-27 \Rightarrow \dfrac{1}{9n-21} < \dfrac{1}{9n-27} \\ \text{and since } 23 < 27. \end{array} \right)$$

---

4. Note that n_0 depends on ε. There is *no* n_0 such that $n_0 > \frac{\varepsilon}{3} + 3$ for *all* $\varepsilon > 0$.

Now, $n > 3 \Rightarrow \left|\dfrac{23}{9n-21}\right| = \dfrac{23}{9n-21}$. Thus,

$$n \geq n_0 \Rightarrow \left|\frac{23}{9n-21}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{6n+9-6n+14}{3(3n-7)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3(2n+3)-2\,(3n-7)}{3(3n-7)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| < \varepsilon.$$

That is, $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\dfrac{2n+3}{3n-7} - \dfrac{2}{3}\right| < \varepsilon.$

Therefore, $\lim\limits_{n\to\infty} \left(\dfrac{2n+3}{3n-7}\right) = \dfrac{2}{3}$ by Definition 2.1.1. □

The type of reasoning used in Examples 2.1.5 and 2.1.6 is so important in analysis that we shall give another pair of examples. Compare these new examples with 2.1.5 and 2.1.6 to see where they are similar. To facilitate the comparison, we will ask the same three questions we asked in Example 2.1.5, and ask for a "proof" just as we did in Example 2.1.6.

Example 2.1.7 Consider the limit statement $\lim\limits_{n\to\infty} \left(\dfrac{3n^2-4n}{n^2+5}\right) = 3.$

(a) After how many terms are we guaranteed that $\dfrac{3n^2-4n}{n^2+5}$ is within a distance of .01 of 3?

(b) After how many terms are we guaranteed that the n^{th} term of this sequence is an accurate approximation of the limit, to within 3 decimal places?

(c) For arbitrary $\varepsilon > 0$, after how many terms are we guaranteed that $\dfrac{3n^2-4n}{n^2+5}$ is within a distance ε of 3?

Solution: (a) We want an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\frac{3n^2 - 4n}{n^2+5} - 3\right| < .01$. Now,

$$\begin{aligned}
\left|\frac{3n^2-4n}{n^2+5} - 3\right| &= \left|\frac{(3n^2-4n) - 3\left(n^2+5\right)}{n^2+5}\right| \\
&= \left|\frac{-4n-15}{n^2+5}\right| \\
&= \frac{4n+15}{n^2+5}.
\end{aligned}$$

Thus, our objective is to find an $n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow \frac{4n+15}{n^2+5} < .01.$$

The latter inequality will be true if

$$\begin{gathered}
\frac{n^2+5}{4n+15} > 100, \text{ i.e.,} \\
n^2 + 5 > 400n + 1500 \\
n^2 - 400n > 1495 \\
n(n-400) > 1495.
\end{gathered}$$

Note that $404(404-400) = 404 \cdot 4 = 1616 > 1495$. Thus,

$$n \geq 404 \Rightarrow n(n-400) \geq 404(4) > 1495.$$

Thus, we take $n_0 = 404$. The above analysis shows that

$$n \geq 404 \Rightarrow \left|\frac{3n^2-4n}{n^2+5} - 3\right| < .01.$$

(b) To make sure that the n^{th} term of the sequence approximates the limit accurately to three decimal places, we want to guarantee that rounding off to three decimal places does not cause a change in the third decimal digit. That is, we want to guarantee that the n^{th} term is within .0005 of the limit. Thus, we want to find an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\frac{3n^2-4n}{n^2+5} - 3\right| < .0005$. As shown above,

$$\left|\frac{3n^2-4n}{n^2+5} - 3\right| = \frac{4n+15}{n^2+5}.$$

Thus, our objective is to find an $n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow \frac{4n+15}{n^2+5} < .0005.$$

The latter inequality will be true if

$$\begin{aligned} \frac{n^2+5}{4n+15} &> 2,000; \text{ i.e.,} \\ n^2+5 &> 8000n + 30,000 \\ n^2 - 8000n &> 29,995 \\ n(n-8000) &> 29,995. \end{aligned}$$

Note that $8004(8004 - 8000) = 8004 \cdot 4 = 32,016 > 29,995$. In fact

$$n \geq 8004 \Rightarrow n(n-8000) \geq 8004(4) > 29,995.$$

Thus, we take $n_0 = 8004$. The above analysis shows that

$$n \geq 8004 \Rightarrow \left| \frac{3n^2 - 4n}{n^2+5} - 3 \right| < .0005.$$

(c) Let $\varepsilon > 0$ be a fixed but arbitrary positive number. We want to find an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left| \dfrac{3n^2-4n}{n^2+5} - 3 \right| < \varepsilon$. As shown above,

$$\left| \frac{3n^2-4n}{n^2+5} - 3 \right| = \frac{4n+15}{n^2+5}.$$

$\frac{3n^2-4n}{n^2+5} - \frac{3n^2-15}{n^2+5}$

$\left| \frac{-4n-15}{n^2+5} \right.$

Thus, our objective is to find an $n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow \frac{4n+15}{n^2+5} < \varepsilon.$$

$\frac{4n+15}{n^2+5}$

Notice that

$$\begin{aligned} \frac{4n+15}{n^2+5} &< \frac{4n+15}{n^2} \\ &< \frac{4n+n}{n^2} \quad \text{if } n > 15 \\ &= \frac{5}{n}. \end{aligned}$$

$\frac{4n+15}{n^2+\cancel{5}}$

$\frac{5}{\varepsilon} < n$

$n_0 > \frac{5}{\varepsilon} - 15$

Thus, $\dfrac{4n+15}{n^2+5} < \varepsilon$ if $n > 15$ and $\dfrac{5}{n} < \varepsilon$. But the last inequality will be guaranteed if $\dfrac{n}{5} > \dfrac{1}{\varepsilon}$; i.e., $n > \dfrac{5}{\varepsilon}$.

By the Archimedean property, $\exists\, n_0 \in \mathbb{N} \ni n_0 > \dfrac{5}{\varepsilon} + 15$.

$\left(\text{We want to be sure that both } n > 15 \text{ and } n > \dfrac{5}{\varepsilon}. \right)$

Let n_0 be such a natural number. The above analysis shows that

$$n \geq n_0 \Rightarrow \left| \frac{3n^2 - 4n}{n^2 + 5} - 3 \right| < \frac{5}{n} < \varepsilon. \quad \square$$

Example 2.1.8 Prove that $\lim\limits_{n \to \infty} \left(\dfrac{3n^2 - 4n}{n^2 + 5} \right) = 3.$

Note: As in Example 2.1.6, to prove this limit statement, we essentially work Part (c) of Example 2.1.7 backwards. But remember, our proof must stand alone, independently of Example 2.1.7.

Proof. Let $\varepsilon > 0$. By the Archimedean property,[5] $\exists\, n_0 \in \mathbb{N} \ni n_0 > \dfrac{5}{\varepsilon} + 15$. Then,

$$\begin{aligned}
n \geq n_0 &\Rightarrow n > \frac{5}{\varepsilon} + 15 \Rightarrow n > \frac{5}{\varepsilon} \text{ and } n > 15 \\
&\Rightarrow \frac{n}{5} > \frac{1}{\varepsilon} \text{ and } n > 15 \\
&\Rightarrow \frac{5}{n} < \varepsilon \text{ and } n > 15 \\
&\Rightarrow \frac{4n + n}{n^2} < \varepsilon \text{ and } n > 15 \\
&\Rightarrow \frac{4n + 15}{n^2 + 5} < \frac{4n + n}{n^2} < \varepsilon \\
&\Rightarrow \left| \frac{-4n - 15}{n^2 + 5} \right| < \varepsilon \\
&\Rightarrow \left| \frac{(3n^2 - 4n) - 3(n^2 + 5)}{n^2 + 5} \right| < \varepsilon \\
&\Rightarrow \left| \frac{3n^2 - 4n}{n^2 + 5} - 3 \right| < \varepsilon.
\end{aligned}$$

Thus, $n \geq n_0 \Rightarrow \left| \dfrac{3n^2 - 4n}{n^2 + 5} - 3 \right| < \varepsilon$. Therefore, $\lim\limits_{n \to \infty} \left(\dfrac{3n^2 - 4n}{n^2 + 5} \right) = 3.$ $\square$

5. This choice of n_0 came from Example 2.1.7. We know that, but there is no need to say that here. It would not help the proof in any way.

SUMMARY: HOW TO PROVE $\lim_{n\to\infty} x_n = L$

1. Let $\varepsilon > 0$.
2. Find a real number r such that $|x_n - L| < \varepsilon$ for all $n \geq r$.
   (This is what we did in Part (c) of Examples 2.1.5 and 2.1.7.)
3. Let n_0 denote any natural number $\geq r$ (found in Step 2).
   (The Archimedean property guarantees the existence of this n_0.)
4. Prove directly that for this value of n_0, $n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$.
   (This is what we did in Examples 2.1.6 and 2.1.8.)

Note: Step 2 above, although of critical importance in finding n_0, is not considered part of the **proof** of $\lim_{n\to\infty} x_n = L$. It is never included when the proof is written up. It may be discarded once Step 4 is completed. In fact, step 4 is usually done by working Step 2 backwards, as demonstrated in Examples 2.1.6 and 2.1.8.

EXERCISE SET 2.1

1. Write out the first eight terms of each of the following sequences:

   (a) $\left\{\dfrac{1}{n^2}\right\}$ (b) $\{(-1)^n\}$

   (c) $\left\{n^{\frac{1}{n}}\right\}$ (d) $\left\{\left(1+\frac{1}{n}\right)^n\right\}$

   (e) $\{\sin(n\pi)\}$ (f) $\{\cos(n\pi)\}$

   (g) $\left\{\cos\left(\frac{n\pi}{3}\right)\right\}$ (h) $\left\{\dfrac{n^2-2n}{3n}\right\}$

2-15 $-3(n^2+5)$ / n^2+5

2. In each of the following exercises, a limit statement $\lim_{n\to\infty} x_n = L$ is given. In each case, answer the following questions:

   (1) After how many terms are we guaranteed that x_n is within .01 of L?

   (2) After how many terms are we guaranteed that x_n is an accurate approximation of L to within 3 decimal places?

   (3) For arbitrary but unknown $\varepsilon > 0$, after how many terms are we guaranteed that x_n is within ε of L?

(a) $\lim_{n\to\infty} \frac{5}{n^2} = 0$ (b) $\lim_{n\to\infty} \frac{3}{n+4} = 0$

(c) $\lim_{n\to\infty} \frac{7n}{n^2+3} = 0$ (d) $\lim_{n\to\infty} \frac{11}{1-n^2} = 0$

(e) $\lim_{n\to\infty} \frac{3n}{n+4} = 3$ (f) $\lim_{n\to\infty} \frac{2n-5}{n-6} = 2$

(g) $\lim_{n\to\infty} \frac{3n+4}{7n-1} = \frac{3}{7}$ (h) $\lim_{n\to\infty} \frac{2n}{1-5n} = -\frac{2}{5}$

(i) $\lim_{n\to\infty} \frac{5n}{11+n^2} = 0$ (j) $\lim_{n\to\infty} \frac{n}{1+8n^2} = 0$

(k) $\lim_{n\to\infty} \frac{n^2-2}{n^2+n} = 1$ (l) $\lim_{n\to\infty} \frac{8n^2+3}{5n^2-2n} = \frac{8}{5}$

(m) $\lim_{n\to\infty} \frac{n-2n^2}{3n^2+1} = -\frac{2}{3}$ (n) $\lim_{n\to\infty} \frac{2n^2-n}{n^2-5n-7} = 2$

(o) $\lim_{n\to\infty} \frac{n^2+3n}{10-n^2} = -1$ (p) $\lim_{n\to\infty} \frac{n^2+6n}{n^3-5n+1} = 0$

3. Use the methods of this section to **prove** each of the limit statements (a)–(p) given in Exercise 2 above.

2.2 Algebra of Limits

In this section we establish some basic rules that allow us to evaluate limits algebraically, without resorting to ε-n_0 arguments.

Theorem 2.2.1 (***Absolute Value and Limits***) *Suppose $\{x_n\}$ is a sequence. Then*

(a) $x_n \to 0 \Leftrightarrow |x_n| \to 0$;

(b) $x_n \to L \Leftrightarrow |x_n - L| \to 0$;

(c) $x_n \to L \Rightarrow |x_n| \to |L|$. *(Note: we do not claim "$\Leftrightarrow$.")*

Proof. Exercise 1. ■

Definition 2.2.2 A sequence $\{x_n\}$ is called a **constant sequence** if $\exists\, c \in \mathbb{R} \ni \forall n \in \mathbb{N}, x_n = c$.

Theorem 2.2.3 *A constant sequence $\{x_n\} = \{c\}$ converges (to c).*

Proof. Exercise 3. ■

Definition 2.2.4 A sequence $\{x_n\}$ is said to be **eventually constant** if $\exists\, c \in \mathbb{R}$ and $\exists\, n_0 \in \mathbb{N} \ni \forall n \geq n_0, x_n = c$.

Theorem 2.2.5 *An eventually constant sequence converges (to that constant).*

Proof. Exercise 4. ■

Theorem 2.2.6 (***Fundamental Limit***) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$.

Proof. Let $\varepsilon > 0$. Then $\dfrac{1}{\varepsilon} > 0$. By the Archimedean property, $\exists\, n_0 \in \mathbb{N} \ni n_0 > \dfrac{1}{\varepsilon}$. Then

$$\begin{aligned} n \geq n_0 &\Rightarrow n > \frac{1}{\varepsilon} \\ &\Rightarrow 0 < \frac{1}{n} < \varepsilon \\ &\Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. ■

Theorem 2.2.7 (***Uniqueness of Limits***) *A sequence cannot converge to more than one real number.*

Proof. Suppose $\{x_n\}$ is a sequence, with $\lim\limits_{n\to\infty} x_n = L$ and $\lim\limits_{n\to\infty} x_n = M$. We want to prove that $L = M$. We shall use the "forcing principle" [Theorem 1.5.9 (d)]. Let $\varepsilon > 0$.

Since $x_n \to L$, $\exists\, n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |x_n - L| < \dfrac{\varepsilon}{2}$.

Since $x_n \to M$, $\exists\, n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |x_n - M| < \dfrac{\varepsilon}{2}$.

Let $n_0 = \max\{n_1, n_2\}$. Then $n_0 \geq n_1$ and $n_0 \geq n_2$, and so

$$|x_{n_0} - L| < \frac{\varepsilon}{2} \text{ and } |x_{n_0} - M| < \frac{\varepsilon}{2}$$
$$|x_{n_0} - L| + |x_{n_0} - M| < \varepsilon$$
$$|L - x_{n_0}| + |x_{n_0} - M| < \varepsilon$$
$$|L - x_{n_0} + x_{n_0} - M| \leq |L - x_{n_0}| + |x_{n_0} - M| < \varepsilon \text{ (Why?)}$$
$$|L - x_{n_0} + x_{n_0} - M| < \varepsilon$$
$$|L - M| < \varepsilon.$$

Thus, $\forall \varepsilon > 0$, $|L - M| < \varepsilon$. Therefore, by the "forcing principle," $L = M$.
■

Theorem 2.2.8 (***Alternate Definition of Limit***) *$x_n \to L$ iff $\forall \varepsilon > 0$, all but finitely many terms of the sequence $\{x_n\}$ are in the interval $(L - \varepsilon, L + \varepsilon)$.*

Proof. (a) (The $\Rightarrow$ part): Suppose $x_n \to L$. Let $\varepsilon > 0$. By Definition 2.1.4, $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$. That is, all terms of the sequence, from the $n_0 th$ term on, belong to the interval $(L - \varepsilon, L + \varepsilon)$. That means that all terms but the first $n_0 - 1$ of them are guaranteed to belong to $(L - \varepsilon, L + \varepsilon)$. In other words, all but finitely many terms of $\{x_n\}$ are in the interval $(L - \varepsilon, L + \varepsilon)$.

(b) (The $\Leftarrow$ part): Suppose that all but finitely many terms of the sequence $\{x_n\}$ are in the interval $(L - \varepsilon, L + \varepsilon)$. Say all but the first n_1 terms of $\{x_n\}$ are in the interval $(L - \varepsilon, L + \varepsilon)$. That is, $n > n_1 \Rightarrow x_n$ is in $(L - \varepsilon, L + \varepsilon)$. Take $n_0 = n_1 + 1$. Then, $n \geq n_0 \Rightarrow x_n \in (L - \varepsilon, L + \varepsilon)$. That is, $n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$. Therefore, $x_n \to L$. ■

Definition 2.2.9 A sequence $\{x_n\}$ is **bounded** if the set $\{x_n : n \in \mathbb{N}\}$ is a bounded set. There are two equivalent ways of saying that $\{x_n\}$ is **bounded**:

(1) $\exists\, a, b \in \mathbb{R} \ni \forall n \in \mathbb{N}, a \leq x_n \leq b$.

(2) $\exists\, M > 0 \ni \forall n \in \mathbb{N}$, $|x_n| \leq M$.

Theorem 2.2.10 (***Boundedness***) *Every convergent sequence is bounded.*

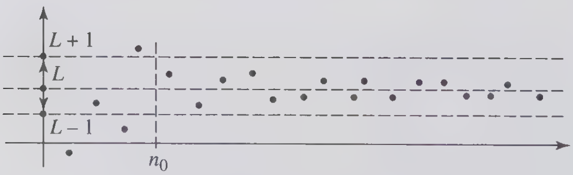


Figure 2.2

Proof. Let $\{x_n\}$ denote a convergent sequence, say $x_n \to L$. Taking $\varepsilon = 1$ in Definition 2.1.4, $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n - L| < 1$. Then,

$$n \geq n_0 \Rightarrow -1 < x_n - L < 1 \Rightarrow L - 1 < x_n < L + 1.$$

Let $a = \min\{x_1, x_2, \cdots, x_{n_0}, L-1\}$ and $b = \max\{x_1, \cdots, x_{n_0}, L+1\}$. Then, $\forall n \in \mathbb{N}$, $a \leq x_n \leq b$. Therefore, $\{x_n\}$ is bounded. ■

Definition 2.2.11 A sequence $\{x_n\}$ is **bounded away from 0** (by C) if

$$\exists\, C > 0 \text{ and } \exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n| \geq C.$$

Theorem 2.2.12 (*Boundedness Away from 0*) *If a sequence converges to a nonzero number, then it is bounded away from 0. More precisely, if $x_n \to L \neq 0$ and C is any number between 0 and $|L|$, then $\{x_n\}$ is bounded away from 0 by C. In fact,*

(a) *If $0 < C < L$, then $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n > C$.*

(b) *If $L < C < 0$, then $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n < C$.*

Proof. Suppose $x_n \to L \neq 0$. Then $|x_n| \to |L| \neq 0$.

Suppose $0 < C < |L|$. Then $|L| - C > 0$. Using Definition 2.1.4, with $\varepsilon = |L| - C$, $\exists\, n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \geq n_0 &\Rightarrow ||x_n| - |L|| < |L| - C \\ &\Rightarrow C - |L| < |x_n| - |L| < |L| - C \\ &\Rightarrow C < |x_n| < 2|L| - C \\ &\Rightarrow |x_n| > C \\ &\Rightarrow x_n > C \text{ if } 0 < C < L, \text{ and } x_n < C \text{ if } L < C < 0. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded away from 0 (by C).[6] ■

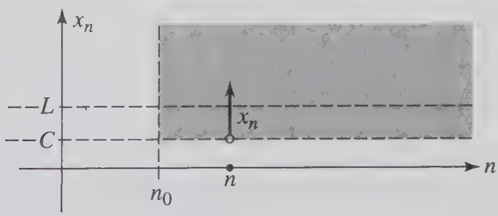


Figure 2.3

6. In practice, C is often taken to be $\frac{L}{2}$ or $\frac{|L|}{2}$.

Main Theorem

Theorem 2.2.13 (***Algebra of Limits***) *Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and $c \in \mathbb{R}$.*

Then

(a) $\lim\limits_{n\to\infty} (cx_n) = c \cdot \lim\limits_{n\to\infty} x_n$;

(b) $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$;

(c) $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} y_n$;

(d) $\lim\limits_{n\to\infty} (x_n y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$;

(e) $\lim\limits_{n\to\infty} \left(\dfrac{1}{y_n}\right) = \dfrac{1}{\lim\limits_{n\to\infty} y_n}$ $\quad$ (*if* $\lim\limits_{n\to\infty} y_n \neq 0$);

(f) $\lim\limits_{n\to\infty} \left(\dfrac{x_n}{y_n}\right) = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ $\quad$ (*if* $\lim\limits_{n\to\infty} y_n \neq 0$);

(g) $\lim\limits_{n\to\infty} (\sqrt{x_n}) = \sqrt{\lim\limits_{n\to\infty} x_n}$ (*if* $\lim\limits_{n\to\infty} x_n \geq 0$, *and* $\exists n_1 \ni n \geq n_1 \Rightarrow x_n \geq 0$).

Proof. Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences, and $c \in \mathbb{R}$. In fact, suppose $x_n \to L$ and $y_n \to M$. Then

(a) Case 1 ($c \neq 0$): Let $\varepsilon > 0$. Since $x_n \to L$, $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n - L| < \dfrac{\varepsilon}{|c|}$.

$$\begin{aligned} \text{Then,} \quad n \geq n_0 \quad &\Rightarrow |cx_n - cL| = |c||x_n - L| < |c| \cdot \frac{\varepsilon}{|c|} \\ &\Rightarrow |cx_n - cL| < \varepsilon. \end{aligned}$$

Therefore, $cx_n \to cL$.

Case 2 ($c = 0$): Exercise 9.

(b) Let $\varepsilon > 0$.

Since $x_n \to L$, $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |x_n - L| < \dfrac{\varepsilon}{2}$.

Since $y_n \to M$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |y_n - M| < \dfrac{\varepsilon}{2}$.

Let $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned}
n \geq n_0 &\Rightarrow n \geq n_1 \text{ and } n \geq n_2 \\
&\Rightarrow |x_n - L| < \frac{\varepsilon}{2} \text{ and } |y_n - M| < \frac{\varepsilon}{2} \\
&\Rightarrow |x_n - L| + |y_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\
&\Rightarrow |(x_n - L) + (y_n - M)| < \varepsilon \text{ (Why?)} \\
&\Rightarrow |(x_n + y_n) - (L + M)| < \varepsilon
\end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}(x_n + y_n) = L + M = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$.

(c) Exercise 11.

(d) Let $\varepsilon > 0$. Since $\{x_n\}$ converges, it is bounded; say that $\forall n \in \mathbb{N}$, $|x_n| \leq B$, where $B > 0$.

Since $x_n \to L$, $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |x_n - L| < \dfrac{\varepsilon}{2(|M| + 1)}$.

Since $y_n \to M$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |y_n - M| < \dfrac{\varepsilon}{2B}$.

Let $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned}
n \geq n_0 &\Rightarrow n \geq n_1 \text{ and } n \geq n_2 \\
&\Rightarrow |x_n - L| < \frac{\varepsilon}{2(|M| + 1)} \text{ and } |y_n - M| < \frac{\varepsilon}{2B} \\
&\Rightarrow (|M| + 1)|x_n - L| < \frac{\varepsilon}{2} \text{ and } B|y_n - M| < \frac{\varepsilon}{2} \\
&\Rightarrow |M||x_n - L| < \frac{\varepsilon}{2} \text{ and } B|y_n - M| < \frac{\varepsilon}{2} \\
&\Rightarrow |M||x_n - L| < \frac{\varepsilon}{2} \text{ and } |x_n||y_n - M| < \frac{\varepsilon}{2} \\
&\Rightarrow |M||x_n - L| + |x_n||y_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\
&\Rightarrow |M(x_n - L)| + |x_n(y_n - M)| < \varepsilon \\
&\Rightarrow |M(x_n - L) + x_n(y_n - M)| < \varepsilon \text{ (Why?)}
\end{aligned}$$

$$\Rightarrow |x_n y_n - ML + Mx_n - Mx_n| < \varepsilon$$

$$\Rightarrow |x_n y_n - LM| < \varepsilon.$$

Therefore, $\lim\limits_{n\to\infty}(x_n y_n) = LM = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$.

(e) Let $\varepsilon > 0$. Since $y_n \to M \neq 0$, $\{y_n\}$ is bounded away from 0. In fact, by Theorem 2.2.12, $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |y_n| > \dfrac{|M|}{2}$. Also, since $y_n \to M$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |y_n - M| < \dfrac{\varepsilon|M|^2}{2}$. Let $n_0 = \max\{n_1, n_2\}$. Then,

$$\begin{aligned} n \geq n_0 \quad &\Rightarrow n \geq n_1 \text{ and } n \geq n_2 \\ &\Rightarrow |y_n| > \frac{|M|}{2} \text{ and } |y_n - M| < \frac{\varepsilon|M|^2}{2} \\ &\Rightarrow \frac{1}{|y_n|} < \frac{2}{|M|} \text{ and } |y_n - M| < \frac{\varepsilon|M|^2}{2} \\ &\Rightarrow \frac{|y_n - M|}{|y_n||M|} = \frac{1}{|y_n|}\frac{|y_n - M|}{|M|} < \frac{2}{|M|}\cdot\frac{\varepsilon|M|^2}{2|M|} \\ &\Rightarrow \left|\frac{1}{y_n} - \frac{1}{M}\right| = \left|\frac{M - y_n}{y_n M}\right| = \frac{|y_n - M|}{|y_n||M|} < \varepsilon \\ &\Rightarrow \left|\frac{1}{y_n} - \frac{1}{M}\right| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}\left(\dfrac{1}{y_n}\right) = \dfrac{1}{M} = \dfrac{1}{\lim\limits_{n\to\infty} y_n}$.

(f) Exercise 14.

(g) Suppose $\lim\limits_{n\to\infty} x_n = L \geq 0$, and $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow x_n \geq 0$. Let $\varepsilon > 0$.

Case 1 ($L = 0$): Then $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |x_n - 0| < \varepsilon^2$. Choose $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned} n \geq n_0 &\Rightarrow 0 \leq x_n < \varepsilon^2 \\ &\Rightarrow \sqrt{x_n} < \varepsilon \\ &\Rightarrow \left|\sqrt{x_n} - 0\right| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty}\sqrt{x_n} = 0 = \sqrt{L}$.

Case 2 ($L > 0$): Since $x_n \to L$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |x_n - L| < \varepsilon\sqrt{L}$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$n \geq n_0 \Rightarrow |\sqrt{x_n} - \sqrt{L}| = \frac{|x_n - L|}{\sqrt{x_n} + \sqrt{L}} < \frac{|x_n - L|}{\sqrt{L}} < \frac{\varepsilon\sqrt{L}}{\sqrt{L}} = \varepsilon.$$ That is,

$n \geq n_0 \Rightarrow |\sqrt{x_n} - \sqrt{L}| < \varepsilon$. Therefore, $\sqrt{x_n} \to \sqrt{L}$. ■

Example 2.2.14 Use the "Algebra of Limits" to prove $\lim\limits_{n\to\infty} \left(\dfrac{2n+3}{3n-7}\right) = \dfrac{2}{3}$.

Solution: $\forall n \in \mathbb{N}$, $\dfrac{2n+3}{3n-7} = \dfrac{2+\frac{3}{n}}{3-\frac{7}{n}}$. Thus, by Theorems 2.2.13 and 2.2.6,[7]

$$\lim_{n\to\infty}\left(\frac{2n+3}{3n-7}\right) = \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}}{3-\frac{7}{n}}\right) = \frac{\lim\limits_{n\to\infty}\left(2+\frac{3}{n}\right)}{\lim\limits_{n\to\infty}\left(3-\frac{7}{n}\right)} = \frac{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty}\frac{3}{n}}{\lim\limits_{n\to\infty} 3 - \lim\limits_{n\to\infty}\frac{7}{n}}$$
$$= \frac{2 + 3\lim\limits_{n\to\infty}\frac{1}{n}}{3 - 7\lim\limits_{n\to\infty}\frac{1}{n}} = \frac{2+0}{3-0} = \frac{2}{3}. \qquad \square$$

ONLY THE "TAIL" MATTERS!

Definition 2.2.15 For a fixed $m \in \mathbb{N}$, the ***m*-tail** of a sequence $\{x_n\}$ is the sequence

$$\begin{aligned} T_m &= \{x_m, x_{m+1}, \cdots, x_{m+n}, \cdots\} \\ &= \{x_k\}_{k=m}^{\infty} \\ &= \{x_{m+n}\}_{n=0}^{\infty} \end{aligned}$$

That is, the m-tail of $\{x_n\}$ is the sequence that results when the first $m-1$ terms of $\{x_n\}$ are deleted.

Theorem 2.2.16 *A sequence $\{x_n\}$ converges to $L \Leftrightarrow$ all of its m-tails T_m converge to $L \Leftrightarrow$ one of its m-tails T_m converges to L. (That is, for fixed $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} x_{m+n}$.)*

Proof. Exercise 18. ■

In this sense, only the "tail" of a sequence matters in considerations of convergence and limit. The first few (finite number) of terms of a sequence do not matter.

---

7. To be strictly correct, the equations below must be read from right to left, to avoid writing limits before we are sure that they exist.

We have already had occasion to make use of this fact. An "eventually constant" sequence (Definition 2.2.4) is one that has a constant tail; in Theorem 2.2.5 we saw that the constant tail determines its convergence. Again, in stating and proving Theorem 2.2.13 (g), we ignore all terms of $\{x_n\}$ that come before the n_1th one (that is, all terms where $\sqrt{x_n}$ might not exist) and deal only with the n_1-tail. Now we know why we can do that.

EXERCISE SET 2.2

1. Prove Theorem 2.2.1.

2. Show by example that the converse of Theorem 2.2.1(c) is not true.

3. Prove Theorem 2.2.3.

4. Prove Theorem 2.2.5.

5. Give an example of a bounded sequence that does not converge.

6. Prove that if $\{a_n\}$ is a bounded sequence (not necessarily convergent) and $\{b_n\}$ converges to 0, then $a_n b_n \to 0$. Use this result to prove that $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = \lim\limits_{n\to\infty} \dfrac{\cos n}{n} = 0$.

7. Give an example of sequences $\{a_n\}$ and $\{b_n\}$ for which $b_n \to 0$, but $a_n b_n \not\to 0$.

8. Prove Parts (a) and (b) of Theorem 2.2.12 as easy consequences of (c).

9. Prove Theorem 2.2.13 (a), Case 2.

10. Give an example of sequences $\{a_n\}$ and $\{b_n\}$ for which $\{a_n+b_n\}$ converges but one or both of $\{a_n\}$ and $\{b_n\}$ does (do) not. In such a case, we cannot say $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$. Does this contradict Theorem 2.2.13 (b)? Explain.

11. Prove Part (c) of Theorem 2.2.13 as an easy consequence of (a) and (b).

12. Give an example of sequences $\{a_n\}$ and $\{b_n\}$ for which $\{a_n b_n\}$ converges but one or both of $\{a_n\}$ and $\{b_n\}$ does (do) not. In such a case, we cannot say $\lim\limits_{n\to\infty} (x_n y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$. Does this contradict Theorem 2.2.13 (d)? Explain.

13. Suppose $\lim_{n\to\infty} x_n = 5$ and $\lim_{n\to\infty} y_n = -3$. Find each of the following:

(a) $\lim_{n\to\infty}(2x_n - 3y_n)$ (b) $\lim_{n\to\infty}\frac{x_n^2 - 2y_n^2}{4x_n y_n}$

(c) $\lim_{n\to\infty}\frac{|x_n + 2y_n|}{x_n - y_n}$ (d) $\lim_{n\to\infty}\frac{1+\sqrt{x_n}}{\sqrt{1-y_n}}$

14. Prove Theorem 2.2.13 (f).

15. In each of the following, find the limit and use the "Algebra of Limits" and other theorems of this section to justify your answer:

(a) $\lim_{n\to\infty}\frac{6}{2+7n} =$ (b) $\lim_{n\to\infty}\frac{1}{4-5n} =$

(c) $\lim_{n\to\infty}\frac{n}{3n+8} =$ (d) $\lim_{n\to\infty}\frac{2n}{3-4n} =$

(e) $\lim_{n\to\infty}\left(\frac{10n-11}{7-2n}\right)^2 =$ (f) $\lim_{n\to\infty}\left(\frac{5-2n}{1+6n}\right)^3 =$

(g) $\lim_{n\to\infty}\frac{100}{n^2} =$ (h) $\lim_{n\to\infty}\frac{6n}{n^4} =$

(i) $\lim_{n\to\infty}\sqrt{\frac{n}{n^2+5}} =$ (j) $\lim_{n\to\infty}\frac{8n+9}{8-3n^2} =$

(k) $\lim_{n\to\infty}\frac{3n^2+n-5}{n^2+6n} =$ (l) $\lim_{n\to\infty}\frac{4n^2-5n}{8n^2+3n-1} =$

(m) $\lim_{n\to\infty}\frac{5n}{2n^3-7} =$ (n) $\lim_{n\to\infty}\frac{3n^2+7n-4}{n^3-4} =$

(o) $\lim_{n\to\infty}\frac{n^3-2n^2}{6-3n+2n^3} =$ (p) $\lim_{n\to\infty}\frac{2n^3+8n^2-23}{5n^3-6n} =$

(q) $\lim_{n\to\infty}\frac{1+\sqrt{n}}{3-\sqrt{n}} =$ (r) $\lim_{n\to\infty}\frac{\sqrt{1+n^2}}{1-n} =$

16. Use mathematical induction to extend Theorem 2.2.13 (d) to prove that if $\{x_n\}$ converges, then $\forall p \in \mathbb{N}$, $\{x_n^p\}$ converges, and $\lim_{n\to\infty} x_n^p = \left(\lim_{n\to\infty} x_n\right)^p$.

17. Prove that each of the following sequences diverges:

(a) $\{4n-5\}$ (b) $\{\sqrt{3n+1}\}$

(c) $\left\{\dfrac{2n+3}{\sqrt{n}}\right\}$ (d) $\left\{\dfrac{n^2+7}{8n+3}\right\}$

(Hint: in each case, show that the sequence is unbounded.)

18. Prove Theorem 2.2.16.

19. Justify the following assertion: $\forall n_0 \in \mathbb{N}$, deleting or altering any or all terms of a sequence $\{x_n\}$ before the $n_0 th$ one will affect neither its convergence/divergence, nor its limit in case of convergence.

20. Prove or disprove (if false, give a counterexample):

(a) If $\{x_n\}$ and $\{y_n\}$ are both bounded above, then so is their sum $\{x_n + y_n\}$.

(b) If $\{x_n\}$ and $\{y_n\}$ are both bounded above, then so is their difference $\{x_n - y_n\}$.

(c) If $\{x_n\}$ and $\{y_n\}$ are sequences of nonnegative real numbers that are bounded above, then so is their product $\{x_n y_n\}$.

(d) If $\{x_n\}$ and $\{y_n\}$ are sequences of positive real numbers that are bounded above, then so is their quotient $\left\{\dfrac{x_n}{y_n}\right\}$.

21. Prove that if $a_n \to u$ and $b_n \to v$, then $\max\{a_n, b_n\} \to \max\{u, v\}$ and $\min\{a_n, b_n\} \to \min\{u, v\}$. [See Exercise 1.2-B.6.]

22. Suppose $a_n + b_n \to u \in \mathbb{R}$ and $a_n - b_n \to v \in \mathbb{R}$. Prove that $\{a_n\}$, $\{b_n\}$, and $\{a_n b_n\}$ all converge, and find their limits.

23. **Rearrangements:** We say that a sequence $\{y_n\}$ is a **rearrangement** of a sequence $\{x_n\}$ if there is a 1-1 correspondence $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}$, $y_n = x_{f(n)}$. Suppose $\{y_n\}$ is a rearrangement of $\{x_n\}$. Prove that $\{y_n\}$ converges iff $\{x_n\}$ converges, and when they converge they have the same limit.

24. **(Project) Arithmetic Means:** For a given $\{x_n\}$ we define its sequence of arithmetic means to be $\{\sigma_n\}$, where $\sigma_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

(a) Prove that if $x_n \to L$, then $\sigma_n \to L$.

(b) Prove that the converse of (a) is false, by finding a sequence $\{x_n\}$ such that $\{\sigma_n\}$ converges but $\{x_n\}$ does not.

Outline of a proof of (a) [fill in missing steps, and justify]:

Case 1 ($L = 0$): Then $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |x_n| < \frac{\varepsilon}{2}$. Holding n_1 fixed, $\frac{x_1 + x_2 + \cdots + x_{n_1}}{n} \to 0$, so $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow \frac{|x_1 + x_2 + \cdots + x_{n_1}|}{n} < \frac{\varepsilon}{2}$. For $n \geq n_0 = \max\{n_1, n_2\}$, $|\sigma_n| = \frac{|x_1 + x_2 + \cdots + x_{n_1}|}{n} + \frac{|x_{n_1+1} + x_{n_1+2} + \cdots + x_n|}{n} < \frac{\varepsilon}{2} + \frac{1}{n}(n - n_1)\frac{\varepsilon}{2} < \varepsilon$.

Case 2 ($L \neq 0$): Apply Case 1 to the sequence $\{x_n - L\}$.

25. **(Project) Geometric Means:** For a given sequence $\{x_n\}$ we define its sequence of geometric means to be $\{\tau_n\}$, where $\tau_n = \sqrt[n]{x_1 x_2 \cdots x_n}$.

    (a) Prove that if $x_n \to L$, then $\tau_n \to L$.

    (b) Prove that the converse of (a) is false, by finding a sequence $\{x_n\}$ such that $\{\tau_n\}$ converges but $\{x_n\}$ does not.

2.3 Inequalities and Limits

The following theorem and its corollary provide two very useful tools for proving that sequences converge to a limit L. You will find yourself using these tools, especially the second, quite often.

Theorem 2.3.1 (***The First "Squeeze" Principle***) *If $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are sequences such that $a_n \to L$, $c_n \to L$, and $\forall n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$, then $b_n \to L$.*

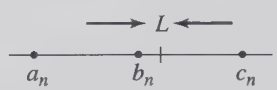


Figure 2.4

Proof. Suppose $\{a_n\}, \{b_n\}$, and $\{c_n\}$ are convergent sequences such that $a_n \to L$, $c_n \to L$, and $\forall n \in \mathbb{N}$, $a_n \leq b_n \leq c_n$. Let $\varepsilon > 0$.
Since $a_n \to L$, $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow |a_n - L| < \varepsilon$.
Since $c_n \to L$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |c_n - L| < \varepsilon$.

Let $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned}
n \geq n_0 &\Rightarrow n \geq n_1 \text{ and } n \geq n_2 \\
&\Rightarrow |a_n - L| < \varepsilon \text{ and } |c_n - L| < \varepsilon \\
&\Rightarrow -\varepsilon < a_n - L < \varepsilon \text{ and } -\varepsilon < c_n - L < \varepsilon \\
&\Rightarrow L - \varepsilon < a_n < L + \varepsilon \text{ and } L - \varepsilon < c_n < L + \varepsilon \\
&\Rightarrow L - \varepsilon < a_n \text{ and } c_n < L + \varepsilon \\
&\Rightarrow L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon \\
&\Rightarrow L - \varepsilon < b_n < L + \varepsilon \\
&\Rightarrow |b_n - L| < \varepsilon.
\end{aligned}$$

Therefore, $b_n \to L$. ■

Corollary 2.3.2 (***The Second "Squeeze" Principle***) *If $\{a_n\}$ and $\{b_n\}$ are sequences such that $b_n \to 0$ and $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |a_n - L| \leq b_n$, then $a_n \to L$.*

Proof. Exercise 1. ■

USING THE SQUEEZE PRINCIPLES TO PROVE CONVERGENCE

The second squeeze principle provides an easy way to convert the "scratch-work" done in Section 2.1 (as in Part (c) of Examples 2.1.5 and 2.1.7) into proofs, without having to rewrite the work into a proof like we did in that section (as in Examples 2.1.6 and 2.1.8). The following two examples illustrate this procedure.

Example 2.3.3 Use the second squeeze principle to prove that $\lim_{n\to\infty} \left(\dfrac{2n+3}{3n-7}\right) = \dfrac{2}{3}$. (See Example 2.1.6.)

Solution: $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\left|\frac{2n+3}{3n-7} - \frac{2}{3}\right| &= \left|\frac{3(2n+3) - 2\,(3n-7)}{3(3n-7)}\right| \\
&= \left|\frac{6n+9-6n+14}{3(3n-7)}\right| \\
&= \frac{23}{|9n-21|} \\
&= \frac{23}{9n-21} \quad \text{if } n \geq 3 \\
&< \frac{23}{9n-n} \quad \text{if } n \geq 22 \\
&< \frac{24}{8n} = \frac{3}{n} \quad \text{if } n \geq 22.
\end{aligned}$$

Now $\frac{3}{n} \to 0$. Therefore, by the second squeeze principle, with $a_n = \frac{2n+3}{3n-7}$, $L = \frac{2}{3}$, $b_n = \frac{3}{n}$, and $n_0 = 22$, we have proved that $\frac{2n+3}{3n-7} \to \frac{2}{3}$. □

Example 2.3.4 Use the second squeeze principle to prove that $\lim_{n\to\infty} \left(\frac{3n^2-4n}{n^2+5}\right) = 3$. (See Example 2.1.8.)

Solution: $\forall n \in \mathbb{N}$,

$$\begin{aligned}
\left|\frac{3n^2-4n}{n^2+5} - 3\right| &= \left|\frac{(3n^2-4n) - 3\left(n^2+5\right)}{n^2+5}\right| \\
&= \left|\frac{-4n-15}{n^2+5}\right| \\
&= \frac{4n+15}{n^2+5} \\
&< \frac{4n+n}{n^2} \text{ if } n \geq 15 \\
&= \frac{5n}{n^2} = \frac{5}{n}.
\end{aligned}$$

That is, $n \geq 15 \Rightarrow \left|\frac{3n^2-4n}{n^2+5} - 3\right| < \frac{5}{n} \to 0$. Therefore, by the second squeeze principle, $\frac{3n^2-4n}{n^2+5} \to 3$. □

MORE APPLICATIONS OF THE SQUEEZE PRINCIPLE

Theorem 2.3.5 *Let $A \subseteq \mathbb{R}$.*

(a) *If $u = \inf A$, then $\exists$ sequence $\{a_n\}$ of elements of A such that $a_n \to u$.*

(b) *If $u = \sup A$, then $\exists$ sequence $\{a_n\}$ of elements of A such that $a_n \to u$.*

Proof. (a) Suppose $u = \inf A$. Let $n \in \mathbb{N}$. Then $u + \frac{1}{n}$ is not a lower bound for A, so $\exists a_n \in A \ni a_n < u + \frac{1}{n}$ (see Theorem 1.6.7, ε-criterion for infimum).

Moreover, $\forall n \in \mathbb{N}$, $a_n \geq u = \inf A$. Thus, $\{a_n\}$ is a sequence of elements of A such that $\forall n \in \mathbb{N}$,

$$u - \frac{1}{n} < u \leq a_n < u + \frac{1}{n}, \text{ so}$$
$$|a_n - u| < \frac{1}{n}.$$

Then, by the second squeeze principle, $a_n \to u$.

(b) Exercise 3. ■

The following theorem is of fundamental importance, and will be used frequently throughout the remainder of the course. However, its proof is easy and is left as an exercise.

Theorem 2.3.6 (***Denseness of the Rationals and Irrationals in*** $\mathbb{R}$) *Let x be any real number. Then*

(a) *$\exists$ sequence $\{r_n\}$ of rational numbers different from $x \ni r_n \to x$.*

(b) *$\exists$ sequence $\{z_n\}$ of irrational numbers different from $x \ni z_n \to x$.*

Proof. Exercise 4. ■

Theorem 2.3.7 *If $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.*

Proof.[8] Suppose $|a| < 1$. Then $\frac{1}{|a|} > 1$. Let $\delta = \frac{1}{|a|} - 1$. Then $\delta > 0$, and

$$\frac{1}{|a|} = \delta + 1, \text{ so}$$
$$|a| = \frac{1}{1+\delta}.$$

Then, by Bernoulli's inequality (see Exercise 1.3.14),

$$(1+\delta)^n \geq 1 + n\delta > n\delta, \text{ so}$$
$$|a|^n = \frac{1}{(1+\delta)^n} < \frac{1}{n\delta} \to 0.$$

Therefore, by the second squeeze principle, $|a|^n \to 0$. ■

8. For an easier proof of Theorem 2.3.7, See Exercise 2.5.16.

Example 2.3.8 $\lim_{n\to\infty} \sqrt[n]{n} = 1$. (Assume the existence[9] of $\sqrt[n]{n}$.)

Proof. $\forall n \in \mathbb{N}$, let $a_n = \sqrt[n]{n} - 1$; i.e., $\sqrt[n]{n} = a_n + 1$.

By the Algebra of Limits theorem, it suffices to prove that $a_n \to 0$. Now, $\forall n \in \mathbb{N}$, $a_n \geq 0$ (prove). Then, from the equation above, we have

$$n = (1 + a_n)^n, \text{ so by Exercise 1.3.15,}$$

$$n \geq 1 + na_n + \tfrac{1}{2}n(n-1)a_n^2, \text{ so}$$

$$n > \tfrac{1}{2}n(n-1)a_n^2; \text{ that is,}$$

$$\begin{aligned} 2 &> (n-1)a_n^2, \text{ or} \\ \frac{2}{n-1} &> a_n^2 \quad (\text{if } n \geq 2). \end{aligned}$$

Hence, $\quad 0 \leq a_n^2 < \dfrac{2}{n-1} \to 0.$

Therefore, by the second squeeze principle, $a_n^2 \to 0$. Then, by the Algebra of Limits theorem, $\sqrt{a_n^2} \to 0$. That is, $a_n \to 0$. $\square$

Example 2.3.9 For any fixed $c > 0$, $\lim_{n\to\infty} \sqrt[n]{c} = 1$.

Proof. Case 1 ($c > 1$): By the Archimedean property, $\exists n_0 \in \mathbb{N} \ni n_0 > c$. Then $n \geq n_0 \Rightarrow \sqrt[n]{n} > \sqrt[n]{c}$. Thus, $\forall n \in \mathbb{N}$,

$$n \geq n_0 \Rightarrow 1 < \sqrt[n]{c} < \sqrt[n]{n} \to 1 \text{ (by Example 2.3.8).}$$

Therefore, by the first squeeze principle, $\sqrt[n]{c} \to 1$.

Case 2 ($c = 1$): Trivial.

Case 3 ($0 < c < 1$): Then, $\dfrac{1}{c} > 1$, so by Case 1, $\lim_{n\to\infty} \sqrt[n]{\dfrac{1}{c}} \to 1$; that is,

$$\frac{1}{\sqrt[n]{c}} \to 1.$$

By the Algebra of Limits theorem, this implies that $\sqrt[n]{c} \to \dfrac{1}{1} = 1$. $\square$

---

9. Examples 2.3.8 and 2.3.9 make use of the n^{th} root function $f(x) = \sqrt[n]{x}$. This function will be defined rigorously in Section 5.5, specifically, in Exercise 14 of that section. In the meantime, we shall assume its familiar algebraic properties.

*__Theorem 2.3.10__ *If $\{x_n\}$ is a sequence of nonzero numbers such that* $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = L < 1$, *then,* $x_n \to 0$.

Proof. Suppose $\{x_n\}$ is a sequence such that $\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = L < 1$. Then $1 - L > 0$. Let $\delta = \frac{1-L}{2}$. Then $\delta > 0$. Since $\left|\frac{x_{n+1}}{x_n}\right| \to L$, $\exists n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \geq n_0 &\Rightarrow \left|\left|\frac{x_{n+1}}{x_n}\right| - L\right| < \delta \\ &\Rightarrow L - \delta < \left|\frac{x_{n+1}}{x_n}\right| < L + \delta \\ &\Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < L + \frac{1-L}{2} = \frac{L+1}{2} < 1. \quad \text{(Why?)} \\ &\Rightarrow \left|\frac{x_{n+1}}{x_n}\right| < \frac{L+1}{2} < 1. \end{aligned}$$

Taking $r = \frac{L+1}{2}$, we have $n \geq n_0 \Rightarrow |x_{n+1}| < r|x_n|$. Thus,

$$\begin{aligned} &|x_{n_0+1}| < r|x_{n_0}| \\ &|x_{n_0+2}| < r|x_{n_0+1}| < r^2|x_{n_0}| \\ &\qquad\vdots \\ &|x_{n_0+k}| < r^k|x_{n_0}|. \end{aligned} \tag{1}$$

By Theorem 2.3.7, $\lim_{k\to\infty} r^k = 0$, since $0 < r < 1$. Then (recall that n_0 is fixed),

$$\lim_{k\to\infty} r^k|x_{n_0}| = |x_{n_0}| \cdot \lim_{k\to\infty} r^k = 0.$$

Putting this together with Inequality (1), we have

$$|x_{n_0+k}| < r^k|x_{n_0}| \to 0. \quad (\text{as } k \to \infty)$$

Therefore, by the second squeeze principle, $\lim_{k\to\infty} |x_{n_0+k}| = 0$. Therefore, $\lim_{n\to\infty} |x_n| = 0$ (Exercise 10), so $\lim_{n\to\infty} x_n = 0$. ■

Corollary 2.3.11 *For any fixed real number c,* $\lim_{n\to\infty}\left(\frac{c^n}{n!}\right) = 0$.

Proof. Exercise 11. ■

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*An asterisk before a theorem, proof, or other item in this chapter indicates that the item is challenging or can be omitted, especially in a one-semester course.

LIMITS PRESERVE INEQUALITIES

In our future work we shall have many occasions to use the following two theorems.

Theorem 2.3.12 (***Limits Preserve Inequalities, I***)

(a) *If* $\{a_n\}$ *converges and* $\forall n \in \mathbb{N}$, $a_n \leq K$, *then* $\lim\limits_{n\to\infty} a_n \leq K$.

(b) *If* $\{a_n\}$ *converges and* $\forall n \in \mathbb{N}$, $a_n \geq K$, *then* $\lim\limits_{n\to\infty} a_n \geq K$.

Proof. (a) Suppose $\{a_n\}$ converges and $\forall n \in \mathbb{N}$, $a_n \leq K$. Say $a_n \to L$. We must prove that $L \leq K$. For contradiction, suppose $L > K$. Let $\varepsilon = L - K$. Then $\varepsilon > 0$. Since $a_n \to L$, $\exists\, n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \geq n_0 &\Rightarrow |a_n - L| < \varepsilon \\ &\Rightarrow -\varepsilon < a_n - L < \varepsilon \\ &\Rightarrow L - \varepsilon < a_n < L + \varepsilon \\ &\Rightarrow L - (L - K) < a_n \\ &\Rightarrow K < a_n; \text{ contradiction!} \end{aligned}$$

Therefore, $L \leq K$.

(b) Exercise 12. ■

Theorem 2.3.13 (***Limits Preserve Inequalities, II***) *If* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences, and* $\forall n \in \mathbb{N}$, $a_n \leq b_n$, *then* $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$.

Proof. Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and $\forall n \in \mathbb{N}$, $a_n \leq b_n$. Define the sequence $\{c_n\}$ by $c_n = b_n - a_n$. Then, by the algebra of limits, $\{c_n\}$ converges and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n.$$

On the other hand, $\forall n \in \mathbb{N}$, $c_n \geq 0$, so by Theorem 2.3.12, $\lim\limits_{n\to\infty} c_n \geq 0$. That is, $\lim\limits_{n\to\infty} b_n - \lim\limits_{n\to\infty} a_n \geq 0$; i.e., $\lim\limits_{n\to\infty} a_n \leq \lim\limits_{n\to\infty} b_n$. ■

Theorem 2.3.14 (***Partial Converse of 2.3.13***) *If* $\{a_n\}$ *and* $\{b_n\}$ *are convergent sequences such that* $\lim\limits_{n\to\infty} a_n < \lim\limits_{n\to\infty} b_n$, *then* $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow a_n < b_n$.

Proof. Exercise 16. ■

EXERCISE SET 2.3

1. Prove Corollary 2.3.2.

2. In each of the following, find the limit and use the second squeeze principle to prove that answer. (See Exercise 2.2.15.)

(a) $\lim_{n\to\infty} \dfrac{6}{2+7n} =$ (b) $\lim_{n\to\infty} \dfrac{1}{4-5n} =$

(c) $\lim_{n\to\infty} \dfrac{n}{3n+8} =$ (d) $\lim_{n\to\infty} \dfrac{2n}{3-4n} =$

(e) $\lim_{n\to\infty} \dfrac{10n-11}{7-2n} =$ (f) $\lim_{n\to\infty} \dfrac{5-2n}{1+6n} =$

(g) $\lim_{n\to\infty} \dfrac{100}{n^2} =$ (h) $\lim_{n\to\infty} \dfrac{6n}{n^4} =$

(i) $\lim_{n\to\infty} \dfrac{n}{n^2+5} =$ (j) $\lim_{n\to\infty} \dfrac{8n+9}{8-3n^2} =$

(k) $\lim_{n\to\infty} \dfrac{3n^2+n-5}{n^2+6n} =$ (l) $\lim_{n\to\infty} \dfrac{4n^2-5n}{8n^2+3n-1} =$

(m) $\lim_{n\to\infty} \dfrac{5n}{2n^3-7} =$ (n) $\lim_{n\to\infty} \dfrac{3n^2+7n-4}{n^3-4} =$

(o) $\lim_{n\to\infty} \dfrac{n^3-2n^2}{6-3n+2n^3} =$ (p) $\lim_{n\to\infty} \dfrac{2n^3+8n^2-23}{5n^3-6n} =$

3. Prove Theorem 2.3.5 (b).

4. Prove Theorem 2.3.6 using Theorem 1.5.7 and the squeeze principle.

5. Prove each of the following:

(a) $\lim_{n\to\infty} \dfrac{1+2+3+\cdots+n}{n^2} = \dfrac{1}{2}$

(b) $\lim_{n\to\infty} \dfrac{1^2+2^2+3^2+\cdots+n^2}{n^3} = \dfrac{1}{3}$

(c) $\lim_{n\to\infty} \dfrac{1^3+2^3+3^3+\cdots+n^3}{n^4} = \dfrac{1}{4}$

[Hint: See Exercises 1.3.3–1.3.5.]

6. **Geometric Series:** (a) Given that $|r| < 1$, prove that

$\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \left(a + ar + ar^2 + \cdots + ar^n\right) = \dfrac{a}{1-r}$. [See the formula for finite geometric sums, Exercise 1.3.12.]

(b) Calculate $\lim_{n\to\infty} \left(\dfrac{1}{3} + \dfrac{1}{9} + \dfrac{1}{27} + \cdots + \dfrac{1}{3^n}\right)$.

(c) Calculate $\lim_{n\to\infty}\left(\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots+\frac{9}{10^n}\right)$. What does this tell you about the decimal $0.9999999\cdots$?

7. Let c be a fixed real number. Prove that $\forall p \in \mathbb{N}$,

(a) $\lim_{n\to\infty}\frac{c^n}{n^p}=0$ if $|c|\le 1$, and (b) $\lim_{n\to\infty}\frac{n^p}{c^n}=0$ if $|c|>1$.

8. Prove that if $0\le a\le b$, then $\lim_{n\to\infty}\sqrt[n]{a^n+b^n}=b$.

9. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n^2+b_n^2\to 0$. Prove that $a_n\to 0$ and $b_n\to 0$. Does this conclusion remain true if $a_n^2+b_n^2$ is replaced by $a_n^3+b_n^3$? Justify your answer.

10. Finish the proof of Theorem 2.3.10, by explaining why $\lim_{n\to\infty}|x_n|=0$ follows from $\lim_{k\to\infty}|x_{n_1+k}|=0$.

11. Prove Corollary 2.3.11. (Consider $c=0$ as a separate case.) For $c\ne 0$, use Theorem 2.3.10 and the first squeeze principle.

12. Prove Theorem 2.3.12 (b).

13. Prove that if $x>1$, then $\lim_{n\to\infty}\frac{n}{x^n}=0$.

14. Prove that if $k\in\mathbb{N}\ni k\ge 2$, then $\lim_{n\to\infty}\frac{n^k}{k^n}=0$.

15. Find each of the following limits, using the appropriate limit theorems to justify your answer.

(a) $\lim_{n\to\infty}\frac{1+2^n}{3^n}$ (b) $\lim_{n\to\infty}\frac{2^n+3^n}{5^n}$

(c) $\lim_{n\to\infty}\frac{1+2^n}{1-2^n}$ (d) $\lim_{n\to\infty}\frac{1-\sqrt{n}}{1-n}$

(e) $\lim_{n\to\infty}\left(\sqrt{n^2+2}-n\right)$ (f) $\lim_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$

(g) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$ (h) $\lim_{n\to\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

(i) $\lim_{n\to\infty}\frac{n^2}{n!}$ (j) $\lim_{n\to\infty}\frac{n^2}{2^n}$

16. Prove Theorem 2.3.14.

17. Show by example that Theorems 2.3.12–2.3.14 do not remain true if **all** the inequalities are changed from $\le$ to $<$, or $\ge$ to $>$. Which inequalities **can** be changed without jeopardizing the truth of the theorems?

18. Prove that if all terms of a convergent sequence $\{x_n\}$ lie in a closed interval $[a, b]$, then its limit is also in $[a, b]$. What if $[a, b]$ is changed to (a, b)?

19. Let $c \in \mathbb{R}$ be fixed. Prove that $\forall p \in \mathbb{N}$, if $|c| < 1$, $\lim\limits_{n\to\infty} n^p c^n = 0$.

20. $\forall x \in \mathbb{R}$, let $[x]$ = the greatest integer less than or equal to x, and let $c \in \mathbb{R}$. Prove that

(a) $\lim\limits_{n\to\infty} \dfrac{[cn]}{n} = c.$ (b) If $c > 0$, $\lim\limits_{n\to\infty} \left[\dfrac{1}{cn}\right] n = 0.$

2.4 Divergence to Infinity

Definition 2.4.1 Suppose $\{x_n\}$ is a sequence of real numbers. We say that

(a) $\{x_n\}$ **diverges to** $+\infty$ $\left(\lim\limits_{n\to\infty} x_n = +\infty\right)$ if

$$\forall M > 0, \exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n > M;$$

(b) $\{x_n\}$ **diverges to** $-\infty$ $\left(\lim\limits_{n\to\infty} x_n = -\infty\right)$ if

$$\forall M > 0, \exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n < -M.$$

Note that this definition implies that if $\lim\limits_{n\to\infty} x_n = +\infty$ (or $-\infty$) then $\{x_n\}$ is *unbounded*, hence $\{x_n\}$ cannot converge (why?). So we will *not* say that $\{x_n\}$ converges to $+\infty$ or $-\infty$, or that $\lim\limits_{n\to\infty} x_n$ exists in these cases, even though we use the notation $\lim\limits_{n\to\infty} x_n$.

Example 2.4.2 Consider the limit statement $\lim\limits_{n\to\infty} \left(\dfrac{3n^2 - 2n}{5n + 23}\right) = +\infty.$

(a) After how many terms are we guaranteed that $\dfrac{3n^2 - 2n}{5n + 23} > 100$?

(b) For arbitrary $M > 0$, after how many terms are we guaranteed that $\dfrac{3n^2 - 2n}{5n + 23} > M$?

Solution: (a) We want an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \dfrac{3n^2 - 2n}{5n + 23} > 100$. Now,

$$\begin{aligned}
\frac{3n^2 - 2n}{5n + 23} &> \frac{3n^2 - 2n}{5n + n}, \text{ if } n \geq 24 \\
&= \frac{3n^2 - 2n}{6n}, \text{ if } n \geq 24 \\
&= \frac{3n - 2}{6}, \text{ if } n \geq 24.
\end{aligned}$$

Thus, we want to find an $n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } \frac{3n-2}{6} > 100; \text{ i.e.,}$$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } 3n - 2 > 600; \text{ i.e.,}$$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } 3n > 602.$$

This will be satisfied if $n \geq 201$. Thus, take $n_0 = 201$.

(b) We want an $n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \dfrac{3n^2 - 2n}{5n + 23} > M$. As shown above,

$$\frac{3n^2 - 2n}{5n + 23} > \frac{3n-2}{6}, \text{ if } n \geq 24.$$

Thus, we want to find an $n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } \frac{3n-2}{6} > M; \text{ i.e.,}$$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } 3n - 2 > 6M; \text{ i.e.,}$$

$$n \geq n_0 \Rightarrow n \geq 24 \text{ and } 3n > 6M + 2.$$

This will be satisfied if $n \geq 24$ and $n \geq \dfrac{6M+2}{3}$. Thus, we take $n_0 \geq \max\left\{24, \dfrac{6M+2}{3}\right\}$. □

Example 2.4.3 Prove that $\lim\limits_{n\to\infty} \left(\dfrac{3n^2 - 2n}{5n + 23}\right) = +\infty$.

Solution: Let $M > 0$. By the Archimedean property, $\exists\, n_0 \in \mathbb{N} \ni n_0 > \max\left\{24, \dfrac{6M+2}{3}\right\}$. Then,

$$n \geq n_0 \Rightarrow n > \frac{6M+2}{3} \text{ and } n \geq 24$$

$$\Rightarrow 3n > 6M + 2 \text{ and } n \geq 24$$

$$\Rightarrow \frac{3n-2}{6} > M \text{ and } n \geq 24$$

$$\Rightarrow \frac{3n^2 - 2n}{6n} > M \text{ and } n \geq 24$$

$$\Rightarrow \frac{3n^2 - 2n}{5n + 23} > \frac{3n^2 - 2n}{6n} > M$$

Therefore, $\lim_{n\to\infty}\left(\dfrac{3n^2-2n}{5n+23}\right)=+\infty$. $\square$

Theorem 2.4.4 *If $\{x_n\}$ is a sequence of positive real numbers, then*

(a) $\lim_{n\to\infty} x_n = +\infty$ *if and only if* $\lim_{n\to\infty}\left(\dfrac{1}{x_n}\right)=0$;

(b) $\lim_{n\to\infty} x_n = 0$ *if and only if* $\lim_{n\to\infty}\left(\dfrac{1}{x_n}\right)=+\infty$.

Proof. Exercise 4. ■

Example 2.4.5 If $a>1$, then $\lim_{n\to\infty} a^n = +\infty$.

Proof. Suppose $a>1$. Let $b=\dfrac{1}{a}$. Then $0<b<1$, so by Theorem 2.3.4, $b^n \to 0$. Thus, by Theorem 2.4.4 (b), $\dfrac{1}{b^n}\to+\infty$. That is, $\lim_{n\to\infty} a^n=+\infty$. $\square$

Theorem 2.4.6 (*Summary of* $\lim_{n\to\infty} a^n$) *Let $a\in\mathbb{R}$. Then*

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a|<1 \\ 1 & \text{if } a=1 \\ +\infty & \text{if } a>1. \end{cases}$$

If $a\le -1$, then $\{a^n\}$ diverges. ■

Theorem 2.4.7 (***Comparison Test***) *Suppose $\{a_n\}$ and $\{b_n\}$ are sequences such that $\forall n\in\mathbb{N}$, $a_n\le b_n$.*

(a) *If* $\lim_{n\to\infty} a_n=+\infty$, *then* $\lim_{n\to\infty} b_n=+\infty$.

(b) *If* $\lim_{n\to\infty} b_n=-\infty$, *then* $\lim_{n\to\infty} a_n=-\infty$.

Proof. (a) Suppose $\lim_{n\to\infty} a_n=+\infty$. Let $M>0$. Then, by Definition 2.4.1, $\exists n_0\in\mathbb{N}\ni n>n_0\Rightarrow a_n>M$. Since $\forall n\in\mathbb{N}$, $a_n\le b_n$, it follows that $n>n_0\Rightarrow b_n>M$. Therefore, $\lim_{n\to\infty} b_n=+\infty$.

(b) Exercise 5. ■

Example 2.4.8 $\lim_{n\to\infty}\dfrac{2^n+4^n}{3^n}=+\infty$.

Proof. By Example 2.4.5 above, $\lim_{n\to\infty}\left(\frac{4}{3}\right)^n = +\infty$. Now, $\forall n \in \mathbb{N}$,

$$\frac{2^n + 4^n}{3^n} > \frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n \to +\infty.$$

Therefore, by Theorem 2.4.7, $\lim_{n\to\infty} \frac{2^n + 4^n}{3^n} = +\infty$. $\square$

ALGEBRA OF INFINITE LIMITS

Theorem 2.4.9 *Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ are sequences such that $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = +\infty$, $\lim_{n\to\infty} c_n = -\infty$, and $\lim_{n\to\infty} d_n = -\infty$. Then*

(a) $\lim_{n\to\infty} (a_n + b_n) = +\infty$;

(b) $\lim_{n\to\infty} (c_n + d_n) = -\infty$;

(c) $\lim_{n\to\infty} (a_n b_n) = +\infty$;

(d) $\lim_{n\to\infty} (c_n d_n) = +\infty$;

(e) $\lim_{n\to\infty} (a_n c_n) = -\infty$.

***Proof.** (a) Suppose $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = +\infty$. Let $M > 0$. Since $\lim_{n\to\infty} a_n = +\infty$, $\exists n_1 \in \mathbb{N} \ni n > n_1 \Rightarrow a_n > M$. Since $\lim_{n\to\infty} b_n = +\infty$, $\exists n_2 \in \mathbb{N} \ni n > n_2 \Rightarrow b_n > M$. Let $n_0 = \max\{n_1, n_2\}$. Then,

$$\begin{aligned} n > n_0 &\Rightarrow a_n > M \text{ and } b_n > M \\ &\Rightarrow (a_n + b_n) > 2M > M. \end{aligned}$$

Therefore, $\lim_{n\to\infty} (a_n + b_n) = +\infty$.

(b) Exercise 6.
(c) Exercise 7.
(d) Exercise 8.
(e) Suppose $\lim_{n\to\infty} a_n = +\infty$, and $\lim_{n\to\infty} c_n = -\infty$. Let $M > 0$. Since $\lim_{n\to\infty} a_n = +\infty$, $\exists n_1 \in \mathbb{N} \ni n > n_1 \Rightarrow a_n > M$. Since $\lim_{n\to\infty} c_n = -\infty$, $\exists n_2 \in \mathbb{N} \ni n > n_2 \Rightarrow c_n < -M$. Let $n_0 = \max\{n_1, n_2\}$. Then,

$$\begin{aligned} n > n_0 &\Rightarrow a_n > M \text{ and } c_n < -1 \\ &\Rightarrow a_n > M \text{ and } -c_n > 1 \\ &\Rightarrow (a_n)(-c_n) > M \\ &\Rightarrow -a_n c_n > M \\ &\Rightarrow a_n c_n < -M. \end{aligned}$$

Therefore, $\lim_{n\to\infty} (a_n c_n) = -\infty$. ■

Symbolic Shorthand: The results of Theorem 2.4.9 are often expressed as a kind of "algebra" of $+\infty$ and $-\infty$, summarized in Table 2.1 as follows:

Table 2.1

Algebra of Infinite Limits	
(a)	$(+\infty) + (+\infty) = +\infty$
(b)	$(-\infty) + (-\infty) = -\infty$
(c)	$(+\infty) \cdot (+\infty) = +\infty$
(d)	$(-\infty) \cdot (-\infty) = +\infty$
(e)	$(+\infty) \cdot (-\infty) = -\infty$

However, the forms $(+\infty)+(-\infty)$ and $(+\infty)-(+\infty)$ are "**indeterminate**" in the sense that no answer can be given that is always true. That is, there are pairs of sequences $\{a_n\}$ and $\{b_n\}$ such that $\lim_{n\to\infty} a_n = +\infty$ and $\lim_{n\to\infty} b_n = -\infty$ for which $\lim_{n\to\infty} (a_n+b_n) = +\infty$, others for which $\lim_{n\to\infty} (a_n+b_n) = -\infty$, others for which $\lim_{n\to\infty} (a_n+b_n)$ is a finite number, and still others for which $\lim_{n\to\infty} (a_n+b_n)$ does not exist.

In the same sense (of limits of sequences), we can combine finite and infinite limits algebraically. Suppose $p > 0$ and $n < 0$ represent finite positive and negative limits of sequences. Table 2.2 summarizes the results:

Table 2.2

Algebra of Infinite Limits	
(a)	$(+\infty) + p(\text{or } n) = +\infty$
(b)	$(-\infty) + p(\text{or } n) = -\infty$
(c)	$(\pm\infty) \cdot p = \pm\infty$
(d)	$(\pm\infty) \cdot n = \mp\infty$
(e)	$(\pm\infty) \cdot 0$ is indeterminate
(f)	$\dfrac{1}{\pm\infty} = 0$
(g)	$\dfrac{1}{\lvert 0 \rvert} = +\infty$, but $\dfrac{1}{0}$ is indeterminate

We can improve our statement about the indeterminate $\frac{1}{0}$ by introducing a further bit of notation. Suppose $\{a_n\}$ is a sequence such that $a_n \to 0$. If $\{a_n\}$ has a tail consisting of all positive numbers, then we write

$$a_n \to 0^+.$$

If $\{a_n\}$ has a tail consisting of all negative numbers, then we write

$$a_n \to 0^-.$$

With this understanding, we have

Table 2.3

Algebra of Infinite Limits
(a) $\dfrac{1}{0^+} = +\infty$
(b) $\dfrac{1}{0^-} = -\infty$

FINAL CAUTION ABOUT $\pm\infty$:

Always remember that $+\infty$ and $-\infty$ are *not* real numbers. They should not be expected to obey the rules that pertain to real numbers. They are merely convenient symbols, which seem to obey *some* common algebraic rules. They are intended for use only in connection with limits.

EXERCISE SET 2.4

1. In each of the following, a limit statement is given. In each case, answer the following questions:

   (i) After how many terms are we guaranteed that $x_n > 100$ (or $x_n < -100$)?

   (ii) For arbitrary but unknown $M > 0$, after how many terms are we guaranteed that $x_n > M$ (or $x_n < -M$)?

   (a) $\lim_{n\to\infty} \sqrt{n} = +\infty$ (b) $\lim_{n\to\infty} \dfrac{n^2+1}{n+1} = +\infty$

   (c) $\lim_{n\to\infty} \dfrac{1-n}{\sqrt{n}} = -\infty$ (d) $\lim_{n\to\infty} \dfrac{1+4n-n^3}{3n} = -\infty$

2. Use Definition 2.4.1 to prove each of the limit statements in Exercise 1.

3. Prove each of the following limit statements, using the theorems of this and previous sections:

   (a) $\lim_{n\to\infty} \frac{3^n + 8^n}{7^n} = +\infty$ (b) $\lim_{n\to\infty} \frac{n^3 + \sin(3n)}{n-1} = +\infty$

   (c) $\lim_{n\to\infty} \frac{n!}{n^{100}} = +\infty$ (d) $\lim_{n\to\infty} \frac{3^n - 5^n}{4^n} = -\infty$

4. Prove Theorem 2.4.4.

5. Prove Theorem 2.4.7 (b).

6. Prove Theorem 2.4.9 (b).

7. Prove Theorem 2.4.9 (c).

8. Prove Theorem 2.4.9 (d).

9. In each of the following parts (a)–(d), find sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to +\infty$, $b_n \to +\infty$, and the given condition holds:

   (a) $a_n - b_n \to 0$ (b) $a_n - b_n \to +\infty$
   (c) $a_n - b_n \to -\infty$ (d) $a_n - b_n \to L \neq 0$

10. In each of the following parts (a)–(d), find sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to +\infty$, $b_n \to 0$, and the given condition holds:

    (a) $a_n b_n \to 0$ (b) $a_n b_n \to +\infty$
    (c) $a_n b_n \to -\infty$ (d) $a_n b_n \to L \neq 0$

11. Prove directly from the definitions that if $a_n \to +\infty$ and $\{b_n\}$ is a sequence of positive terms bounded away from 0, then $a_n b_n \to +\infty$.

12. Let $c \in \mathbb{R}$ and $p \in \mathbb{N}$ be fixed. Prove that

$$\lim_{n\to\infty} \frac{c^n}{n^p} = \begin{cases} 0 & \text{if } |c| \leq 1; \\ +\infty & \text{if } c > 1; \\ \text{does not exist (finite or infinite) if } c < -1. \end{cases}$$

    [See Exercise 2.3.7.]

13. Prove that if $\{x_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$, then, $x_n \to +\infty$.

14. **Limit Comparison Test:** Suppose $\{x_n\}$ and $\{y_n\}$ are sequences of positive real numbers such that $\dfrac{x_n}{y_n} \to L$, where $0 < L < \infty$. Prove that $x_n \to \infty \Leftrightarrow y_n \to \infty$.
strictly less than ∞

15. **Geometric Series:** Given that $r > 1$, find $\sum_{k=0}^{\infty} ar^k = \lim_{n\to\infty} \left(a + ar + ar^2 + \cdots + ar^n\right)$ and justify your answer. [See Exercise 2.3.6.]

16. Prove the relations (a) given in Table 2.2.

17. Prove the relations (b) given in Table 2.2.

18. Prove the relations (c) given in Table 2.2.

19. Prove the relations (d) given in Table 2.2.

20. Prove the relation (a) given in Table 2.3.

21. Prove the relation (b) given in Table 2.3.

2.5 Monotone Sequences

One of the most powerful tools in the theory and application of sequences is the notion of monotone increasing or monotone decreasing sequences. Such sequences have special convergence behavior that make them especially useful.

Definition 2.5.1 A sequence $\{a_n\}$ is said to be (see Figure 2.5)

(a) **monotone increasing** if $\forall n \in \mathbb{N}, a_n \leq a_{n+1}$; that is,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots.$$

(b) **monotone decreasing** if $\forall n \in \mathbb{N}, a_n \geq a_{n+1}$; that is,

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots.$$

(c) **strictly increasing** if $\forall n \in \mathbb{N}, a_n < a_{n+1}$; that is,

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots.$$

(d) **strictly decreasing** if $\forall n \in \mathbb{N}, a_n > a_{n+1}$; that is,

$$a_1 > a_2 > \cdots > a_n > a_{n+1} > \cdots.$$

(e) **monotone** if it is any one of (a) or (b) or (c) or (d).

(f) **strictly monotone** if it is either (c) or (d).

Figure 2.5

There are four methods commonly used to prove that a sequence is monotone. For example, any of the following methods will show that $\{a_n\}$ is monotone increasing:

(a) By subtracting successive terms, show that $\forall n \in \mathbb{N}$, $a_{n+1} - a_n \geq 0$.

(b) If all a_n are positive, divide successive terms and show that $\dfrac{a_{n+1}}{a_n} \geq 1$.

(c) If $f(x) = a_x$ is differentiable, show that $\forall x \geq 1$, $f'(x) \geq 0$. (We shall not use this method before Chapter 6 where derivatives are introduced.)

(d) Use mathematical induction to show that $\forall n \in \mathbb{N}$, $a_n \leq a_{n+1}$.

Examples 2.5.2 (a) The sequence $\left\{\dfrac{1}{n}\right\}$ is strictly decreasing, since $\forall n \in \mathbb{N}$,

$$\frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n(n+1)} = \frac{-1}{n(n+1)} < 0.$$

(b) The sequence $\left\{\dfrac{3n}{4n+5}\right\}$ is strictly increasing, since $\forall n \in \mathbb{N}$,

$$\frac{3(n+1)}{4(n+1)+5} \Big/ \frac{3n}{4n+5} = \frac{3n+3}{4n+9} \cdot \frac{4n+5}{3n} = \frac{12n^2+27n+15}{12n^2+27n} > 1.$$

(c) The sequence $\left\{\dfrac{3^n}{n^3}\right\}$ is strictly increasing after the first three terms, since
$\dfrac{3^{n+1}}{(n+1)^3} \cdot \dfrac{n^3}{3^n} = \dfrac{3n^3}{n^3+3n^2+3n+1} > 1$ whenever $3n^3 > n^3 + 3n^2 + 3n + 1$, or $n[n(2n-3)-3] - 1 > 0$, which is true when $n \geq 3$. □

Theorem 2.5.3 (*Monotone Convergence Theorem*) *Every bounded monotone sequence converges. More precisely,*

Main Theorem

(a) *if $\{a_n\}$ is a monotone increasing sequence that is bounded above, then* $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$;

(b) *if $\{a_n\}$ is a monotone decreasing sequence that is bounded below, then* $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Proof. (a) Suppose $\{a_n\}$ is bounded and monotone increasing. Since $\{a_n\}$ is bounded, the set $\{a_n : n \in \mathbb{N}\}$ has an upper bound. By the completeness of $\mathbb{R}$, $\exists\, u = \sup\{a_n : n \in \mathbb{N}\}$.

Let $\varepsilon > 0$. By Theorem 1.6.6 (ε-criterion for supremum), $\exists\, n_0 \in \mathbb{N} \ni a_{n_0} > u - \varepsilon$. But $\{a_n\}$ is monotone increasing; therefore, $n \geq n_0 \Rightarrow a_n \geq a_{n_0}$. Thus,

$$n \geq n_0 \Rightarrow a_n \geq a_{n_0} > u - \varepsilon. \qquad (2)$$

But since $u = \sup\{a_n : n \in \mathbb{N}\}$,

$$\forall n \in \mathbb{N}, a_n \leq u. \qquad (3)$$

Putting together (2) and (3), we have

$$\begin{aligned} n \geq n_0 &\Rightarrow u - \varepsilon < a_n \leq u < u + \varepsilon \\ &\Rightarrow u - \varepsilon < a_n < u + \varepsilon \\ &\Rightarrow |a_n - u| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = u = \sup\{a_n : n \in \mathbb{N}\}$.

(b) Exercise 3. ■

Corollary 2.5.4 (*Fundamental Theorem of Monotone Sequences*) *A monotone sequence converges if and only if it is bounded.*

Proof. Exercise 4. ■

*APPLICATION: DECIMAL REPRESENTATION OF REAL NUMBERS

In Chapter 1 we defined the "Real Number System." However, we did not say anything about the most familiar representation of real numbers—as decimals. We are now in a position to make the connection, clearly and precisely. A **decimal expansion** is an expression of the form

$$D = \pm\, K.d_1 d_2 d_3 \cdots d_n d_{n+1} \cdots \qquad (4)$$

where K is a nonnegative integer and $\forall n \in \mathbb{N}$, the "digit" d_n is an element of the set $\{0, 1, 2, 3, \cdots, 9\}$. But what do we actually mean by the expression (4)?

We can use sequences to make this meaning precise. To keep it simple, we shall consider only the case when $D \geq 0$. The case $D < 0$ is handled similarly.

First, notice that by definition the decimal (4) has infinitely many "digits." We are accustomed to thinking of some decimals, such as $\frac{1}{2} = 0.5$ as "terminating;" i.e., having only finitely many digits. Let us agree that such decimals really have infinitely many digits, by making all remaining digits zeros.

For each $n \in \mathbb{N}$, we define the *n-place truncation* of D to be the terminating decimal

$$D_n = K.d_1 d_2 d_3 \cdots d_n$$

by which we mean (the rational number)

$$D_n = K + \frac{d_1}{10} + \frac{d_2}{100} + \cdots + \frac{d_n}{10^n}.$$

Now, it is clear that $\{D_n\}$ is a monotone increasing sequence, bounded above, since

$$D_1 \leq D_2 \leq \cdots D_n \leq \cdots \leq K + 1.$$

Therefore, by the monotone convergence theorem, there is a (unique) real number $x \in \mathbb{R} \ni D_n \to x$. It is this real number, x, that is "represented" by the decimal expansion (4). We summarize these results in the following theorem.

***Theorem 2.5.5** *Every decimal expansion $D = K.d_1 d_2 d_3 \cdots d_n d_{n+1} \cdots$ defined above represents a unique nonnegative real number; namely, $x = \sup\{D_n : n \in \mathbb{N}\}$, or $x = \lim_{n \to \infty} D_n$, where D_n is as defined above.*

Theorem 2.5.5 establishes a one-way relationship between decimals and nonnegative real numbers,

$$\text{decimals} \to \text{nonnegative real numbers.}$$

We have not yet proved that the relationship goes the other way as well: that to each nonnegative real number there corresponds a decimal expansion of the form (4). We now turn our attention to that concern.

Let $x \geq 0$. (We consider here only the case $x \geq 0$. To represent $x < 0$ as a decimal, represent $|x|$ as a decimal and prefix a "$-$".) In Chapter 1 (Theorem 1.5.3) we showed that $\exists$ unique nonnegative integer $K \ni$

$$K \leq x < K + 1. \text{ Then}$$
$$0 \leq x - K < 1, \text{ so}$$
$$0 \leq 10x - 10K < 10.$$

Hence, $\exists$ unique integer $d_1 \in \{1, 2, \cdots, 9\}$ such that

$$d_1 \leq 10x - 10K < d_1 + 1$$
$$0 \leq 10x - 10K - d_1 < 1$$
$$0 \leq 10\left[x - \left(K + \frac{d_1}{10}\right)\right] < 1$$

$$0 \le 10\,[x - K.d_1] < 1.$$

Then, $0 \le 100\,[x - K.d_1] < 10$. We repeat the above process: $\exists$ unique integer $d_2 \in \{1, 2, \cdots, 9\}$ such that

$$\begin{gathered} d_2 \le 100(x - K.d_1) < d_2 + 1 \\ 0 \le 100x - 100K.d_1 - d_2 < 1 \\ 0 \le 100\left[x - \left(K + \frac{d_1}{10} + \frac{d_2}{100}\right)\right] < 1 \\ 0 \le 100\,[x - K.d_1 d_2] < 1. \end{gathered}$$

Continuing in this way, we obtain a sequence $\{d_n\}$ of "digits" $d_n \in \{1, 2, \cdots, 9\} \ni \forall n \in \mathbb{N}$,

$$0 \le 10^n\,[x - K.d_1 d_2 \cdots d_n] < 1;$$

$$\text{i.e., } 0 \le x - K.d_1 d_2 \cdots d_n < \frac{1}{10^n} \tag{5}$$

$$K.d_1 d_2 \cdots d_n \le x < K.d_1 d_2 \cdots d_n + \frac{1}{10^n}. \tag{6}$$

In words, inequality (6) tells us that corresponding to every given $n \in \mathbb{N}$, there is a unique n-place decimal $K.d_1 d_2 \cdots d_n \le x$ closest to x.

Inequality (5), along with the first squeeze principle, tells us that

$$\lim_{n\to\infty} K.d_1 d_2 \cdots d_n = x.$$

In that sense, we say that every nonnegative real number is "represented" by an infinite decimal:

$$x = K.d_1 d_2 \cdots d_n d_{n+1} \cdots.$$

Note: the decimal expansion of a real number is not necessarily unique. For example,

$$1 = \begin{cases} 1.0000 \cdots 0 \cdots \\ 0.9999 \cdots 9 \cdots \end{cases} \quad \text{and} \quad \frac{1}{2} = \begin{cases} 0.5000 \cdots 0 \cdots \\ 0.4999 \cdots 9 \cdots \end{cases}$$

***Example 2.5.6** Prove that $0.999 \cdots 9 \cdots = 1.000 \cdots 0 \cdots$.

Proof. By definition, $0.9999 \cdots 9 \cdots = \lim_{n\to\infty} 0.d_1 d_2 \cdots d_n$, where $d_i = 9$, for all i. Now, $\forall n \in \mathbb{N}$,

$$\begin{aligned} 0.d_1 d_2 \cdots d_n &= .9999 \cdots 9 \quad (n \text{ nines}) \\ &= 1 - 0.00 \cdots 01 \quad (n - 1 \text{ zeros}) \\ &= 1 - \frac{1}{10^n}. \end{aligned}$$

Thus, by Theorem 2.3.5, $\lim_{n\to\infty} 0.d_1d_2\cdots d_n = 1$. Since a sequence cannot have more then one limit, $0.999\cdots 9\cdots = 1.000\cdots 0\cdots$. □

We summarize this discussion (including the case $x < 0$) in the following theorem.

***Theorem 2.5.7** *Every real number x can be represented as an (infinite) decimal expansion, $x = K.d_1d_2\cdots d_nd_{n+1}\cdots$, where $d_n \in \{0,1,2,3,\cdots,9\}$. This decimal representation is unique except when one of them ends in all 0's and the other in all 9's.*

MORE APPLICATIONS

Example 2.5.8 Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \sqrt{x_n + 2}$. Prove that $\{x_n\}$ converges, and find its limit.

Solution: Part 1—Prove that $\{x_n\}$ converges.

(a) First, we prove that $\{x_n\}$ is monotone increasing. We use mathematical induction to prove the following: $\forall n \in \mathbb{N}$, $0 \le x_n \le x_{n+1}$.

(i) $x_1 = 1$ and $x_2 = \sqrt{1+2} = \sqrt{3}$. Thus, $0 \le x_1 \le x_2$.
(ii) Now, assume $0 \le x_k \le x_{k+1}$. Then,

$$\begin{aligned} 0 \le\ & x_k + 2 \le x_{k+1} + 2 \\ 0 \le\ & \sqrt{x_k + 2} \le \sqrt{x_{k+1} + 2} \\ 0 \le\ & x_{k+1} \le x_{k+2}. \end{aligned}$$

Therefore, by (i), (ii), and mathematical induction, $\forall n \in \mathbb{N}$, $0 \le x_n \le x_{n+1}$. That is, $\{x_n\}$ is monotone increasing.

(b) Now, we prove that $\{x_n\}$ is bounded above. We use mathematical induction to prove the following: $\forall n \in \mathbb{N}$, $x_n \le 3$.
(i) $x_1 = 1$ so $x_1 \le 3$.
(ii) Now, assume $x_k \le 3$. Then

$$\begin{aligned} x_k + 2 &\le 5 \\ \sqrt{x_k + 2} &\le \sqrt{5} \\ x_{k+1} &\le 3. \end{aligned}$$

Therefore, by (i), (ii), and mathematical induction, $\forall n \in \mathbb{N}$, $x_n \le 3$. That is, $\{x_n\}$ is bounded above by 3.

(c) Therefore, by (a), (b), and the monotone convergence theorem, $\{x_n\}$ converges.

Part 2—find $\lim_{n\to\infty} x_n$. By Part 1, we know that $\exists L = \lim_{n\to\infty} x_n \in \mathbb{R}$. We proceed to find L. Consider the defining equation: $x_{n+1} = \sqrt{x_n + 2}$. By squaring both sides, we have

$$(x_{n+1})^2 = x_n + 2. \text{ Thus,}$$
$$\lim_{n\to\infty} (x_{n+1})^2 = \lim_{n\to\infty} (x_n + 2).$$

Applying the algebra of limits to both sides of this equation,

$$\begin{aligned} &L^2 = L + 2 \\ &L^2 - L - 2 = 0 \\ &(L-2)(L+1) = 0 \\ &L = 2 \text{ or } L = -1. \end{aligned}$$

Now, $L \neq -1$ (see Theorem 2.3.12 (b) with $K = 0$). Thus, $L = 2$. $\square$

strictly positive? so $L \neq -1$

Example 2.5.9 (The Number e) The sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ converges.

Proof. (a) We first show that this sequence is strictly increasing. By the binomial theorem, $\left(1 + \frac{1}{n}\right)^n =$

$$1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{2\cdot 3}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n =$$

$$1 + n\left(\frac{1}{n}\right) + \frac{1}{2}\frac{n}{n}\frac{n-1}{n} + \frac{1}{2\cdot 3}\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n} + \cdots + \frac{1}{2\cdot 3\cdots n}\left[\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{1}{n}\right] =$$

$$1 + 1 + \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{1}{2\cdot 3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots \left(1 - \frac{n-1}{n}\right)\right].$$

Again, by the binomial theorem, $\left(1 + \frac{1}{n+1}\right)^{n+1} =$

$$1 + 1 + \frac{1}{2}\left(1 - \frac{1}{n+1}\right) + \frac{1}{2\cdot 3}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \cdots + \frac{1}{(n+1)!}\left[\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right)\right].$$

Notice that in the above expansion of $\left(1 + \frac{1}{n+1}\right)^{n+1}$,

(i) all terms are positive;

(ii) each of the first n terms is greater than the corresponding term in the expansion of $\left(1 + \frac{1}{n}\right)^n$;

(iii) there is one more term than in the expansion of $\left(1+\dfrac{1}{n}\right)^n$.

Putting (i)–(iii) together, we see that $\left(1+\dfrac{1}{n+1}\right)^{n+1} > \left(1+\dfrac{1}{n}\right)^n$. That is, the sequence $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is strictly increasing.

(b) Next, we show that this sequence is bounded above. From the proof of Part (a), we see that $\forall n \in \mathbb{N}$,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &\leq 1+1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n!}\\ &< 1+1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\cdots+\frac{1}{2^{n-1}}\\ &= 1+\left(2-\frac{1}{2^{n-1}}\right) \quad \text{(See Exercise 1.3.10.)}\\ &< 3.\end{aligned}$$

Hence, this sequence is bounded above by 3.

(c) Finally, by Parts (a) and (b) above, $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is a monotone increasing sequence with an upper bound. Hence, by the monotone convergence theorem, this sequence converges. $\square$

Definition 2.5.10 (The Number *e*) $e = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

EXISTENCE OF SQUARE ROOTS, AND HOW TO FIND THEM

In Chapter 1 we showed that in any complete ordered field there is a positive number whose square is 2. We can use sequences to extend that result, showing that every positive number a has a square root. The method we use is based on an ancient method of calculating square roots, called the "guess-and-average" method. We shall use it to find a sequence of real numbers converging to $\sqrt{a}$. The procedure goes as follows. Pick any $a > 0$. We want to find $\sqrt{a}$; that is, a positive number whose square is a. Let x_1 be any positive real number; it will serve as our "first guess." As our second guess, we take the average of x_1 and $\frac{a}{x_1}$. That is,

$$x_2 = \frac{x_1 + \dfrac{a}{x_1}}{2}.$$

(If we are lucky and our first guess is exactly $\sqrt{a}$, then $x_1 = \dfrac{a}{x_1}$, and $x_2 = x_1$.)

We repeat this process over and over to define a sequence $\{x_n\}$ inductively. The following theorem guarantees that this sequence converges to a positive number whose square is a.

Theorem 2.5.11 (*A Sequence Converging to* $\sqrt{a}$) *Let a be any positive real number. Define the sequence $\{x_n\}$ inductively by*

$$\left\{\begin{array}{l} x_1 = \text{ any positive real number} \\ \forall n \in \mathbb{N},\ x_{n+1} = \dfrac{x_n + \dfrac{a}{x_n}}{2}. \end{array}\right\}. \tag{7}$$

Then $\{x_n\}$ converges to a positive real number whose square is a. That is, $x_n \to \sqrt{a}$. Moreover, $\forall n \geq 2$,

$$\boxed{0 \leq x_n - \sqrt{a} \leq \frac{x_n^2 - a}{\sqrt{a}}.} \tag{8}$$

Proof. (a) Let $a > 0$, and define $\{x_n\}$ inductively by the above scheme. First, we prove that $\{x_n\}$ is bounded below. In fact, $\forall n \in \mathbb{N}$, $(x_{n+1})^2 \geq a$. To prove this, let $n \in \mathbb{N}$. From Equation (7) we have

$$2x_{n+1} = x_n + \frac{a}{x_n};$$

$$2x_n x_{n+1} = x_n^2 + a;$$

$$x_n^2 - 2x_n x_{n+1} + a = 0.$$

Consider this as a quadratic equation in the variable x_n. Since it has a real number solution, x_n, its discriminant must be ≥ 0. That is,

$$(-2x_{n+1})^2 - 4(1)(a) \geq 0;$$

$$4(x_{n+1})^2 - 4a \geq 0;$$

$$(x_{n+1})^2 \geq a.$$

(b) Next, we show that $\{x_n\}$ is monotone decreasing for $n \geq 2$. That is, $\forall n \geq 2$, $x_{n+1} \leq x_n$. To prove this, let $n \geq 2$. Then,

$$\begin{aligned} x_n - x_{n+1} &= x_n - \frac{x_n + \dfrac{a}{x_n}}{2} \\ &= \frac{1}{2}\left[2x_n - \left(x_n + \frac{a}{x_n}\right)\right] \\ &= \frac{1}{2}\frac{x_n^2 - a}{x_n} \\ &\geq 0, \text{ since } x_n^2 \geq a \text{ from Part (a) above} \\ &\qquad \text{(recall that } x_n > 0). \end{aligned}$$

That is, $x_n - x_{n+1} \geq 0$, from which it follows that $x_{n+1} \leq x_n$.

(c) By (a) and (b) together, $\{x_n\}_{n=2}^{\infty}$ is a monotone decreasing sequence that is bounded below. Therefore, by the monotone convergence theorem, $\{x_n\}$ converges.

(d) Let $L = \lim_{n\to\infty} x_n$. Then $L \geq 0$, since $x_n > 0$ for all n. By taking the limit of both sides of the equation $x_{n+1} = \dfrac{x_n + \dfrac{a}{x_n}}{2}$ we have

$$L = \frac{L + \dfrac{a}{L}}{2}, \text{ or } 2L = L + \frac{a}{L}.$$

$$\text{That is, } L = \frac{a}{L}, \quad \text{or} \quad L^2 = a$$

and thus, L is a positive number whose square is a; i.e., $L = \sqrt{a}$.

(e) Suppose $n \geq 2$. From the last line of (a) above, we see that $x_n \geq \sqrt{a}$. Thus,

$$\frac{a}{x_n} \leq \frac{a}{\sqrt{a}} = \sqrt{a} \leq x_n, \text{ so}$$

$$0 \leq x_n - \sqrt{a} \leq x_n - \frac{a}{x_n} = \frac{x_n^2 - a}{x_n} \leq \frac{x_n^2 - a}{\sqrt{a}}. \text{ Thus,}$$

$$\forall n \geq 2,\ 0 \leq x_n - \sqrt{a} \leq \frac{x_n^2 - a}{\sqrt{a}}. \qquad \blacksquare$$

*Example 2.5.12 (Calculating $\sqrt{a}$ to Any Specified Degree of Accuracy)

The inequality (8) in Theorem 2.5.11 is very useful in practice, to find $\sqrt{a}$ to a specified degree of accuracy. For example, suppose we wish to calculate $\sqrt{5}$ correctly to six decimal places, using the sequence $\{x_n\}$ defined in Theorem 2.5.11. We want n such that

$$0 \leq x_n - \sqrt{a} \leq .0000005 \quad (= 5 \times 10^{-7}).$$

By (2) it is sufficient to have

$$\frac{x_n^2 - 5}{\sqrt{5}} < 5 \times 10^{-7}, \text{ for which it suffices to have}$$

$$\frac{x_n^2 - 5}{2} < 5 \times 10^{-7}$$

$$x_n^2 - 5 < 10^{-6}$$

$$x_n^2 < 5 + 10^{-6}.$$

It is reasonable to take $x_1 = 2$ as our first guess. Using a calculator, we find

n	x_n	x_n^2
1	2	4
2	2.25	5.0625
3	2.2361111...	5.00019290...
4	2.2360679...	5.000000002...

$$\left(x_{n+1} = \frac{x_n + \dfrac{5}{x_n}}{2} \right)$$

In Part (b) of the proof of Theorem 2.5.11 we showed that $\{x_n\}$ is monotone decreasing, and this table shows that x_4^2 is already $< 5 + 10^{-6}$, so we can be sure that, as a decimal approximation to $\sqrt{5}$, $x_4 = 2.2360679\cdots$ is correct to six decimal places. Rounding off to six decimal places, we have $\sqrt{5} = x_5 = 2.236068$. □

*SUP AND INF AS LIMITS OF MONOTONE SEQUENCES

Theorem 2.5.13 *Let A be a nonempty set of real numbers. Then* $\inf A$ *and* $\sup A$ *(when they exist) are limits of monotone sequences of elements of A. More specifically,*

(a) *If $u = \inf A$, then $\exists$* ***monotone decreasing*** *sequence $\{a_n\}$ of elements of A such that $a_n \to u$. Moreover, if $\inf A \notin A$ then $\exists$* ***strictly*** *decreasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$.*

(b) *If $u = \sup A$, then $\exists$* ***monotone increasing*** *sequence $\{a_n\}$ of elements of A such that $a_n \to u$. Moreover, if $\sup A \notin A$ then $\exists$* ***strictly*** *increasing sequence $\{a_n\}$ of elements of A such that $a_n \to u$.*

***Proof.** Suppose A is a nonempty set of real numbers.

(a) Suppose $u = \inf A$.

(i) Case 1 ($u \in A$): In this case, merely take $\{a_n\}$ to be the constant sequence $a_n = u$.

(ii) Case 2 ($u \notin A$): Then $\forall a \in A$, $u < a$. We define the sequence $\{a_n\}$ by mathematical induction, as follows. Since $u + 1$ is not a lower bound for A, $\exists\, a_1 \in A \ni u < a_1 < u + 1$. Similarly, neither a_1 nor $u + \frac{1}{2}$ is a lower bound for A, so $\exists\, a_2 \in A \ni$

$$u < a_2 < \min\left\{a_1, u + \frac{1}{2}\right\}.$$

Now, suppose $n \in \mathbb{N}$ and a_n has been defined so that

$$u < a_n < \min\left\{a_{n-1}, u + \frac{1}{n}\right\}.$$

Then neither a_n nor $u+\frac{1}{n}$ is a lower bound for A, so $\exists\, a_{n+1} \in A \ni$

$$u < a_{n+1} < \min\left\{a_n, u + \frac{1}{n+1}\right\}.$$

Thus, $\forall n \in \mathbb{N}$, $a_{n+1} < a_n$. So, $\{a_n\}$ is a strictly decreasing sequence of elements of A. Since $u < a_n < u + \frac{1}{n}$, we conclude by the first squeeze principle that $a_n \to u$.

(b) Exercise 18. ■

UNBOUNDED MONOTONE SEQUENCES

Theorem 2.5.14 (*Unbounded Monotone Sequences Diverge to Infinity*)

(a) *If $\{x_n\}$ is a monotone increasing sequence that is unbounded above, then* $\lim\limits_{n\to\infty} x_n = +\infty$.

(b) *If $\{x_n\}$ is a monotone decreasing sequence that is unbounded below, then* $\lim\limits_{n\to\infty} x_n = -\infty$.

Proof. (a) Suppose $\{x_n\}$ is monotone increasing and unbounded above. Let $M > 0$. Since $\{x_n\}$ is unbounded above, M cannot be an upper bound for $\{x_n\}$, so $\exists\, n_0 \in \mathbb{N} \ni x_{n_0} > M$. Since $\{x_n\}$ is monotone increasing, $n \geq n_0 \Rightarrow x_n \geq x_{n_0} > M$. Therefore, by Definition 2.4.1, $\lim\limits_{n\to\infty} x_n = +\infty$.

(b) Exercise 19. ■

Corollary 2.5.15 (a) *A monotone increasing sequence either converges to a real number or diverges to $+\infty$.*

(b) *A monotone decreasing sequence either converges to a real number or diverges to $-\infty$.*

Proof. These are immediate consequences of Theorem 2.5.14. ■

A SLOWLY DIVERGENT SEQUENCE (WILL FOOL A COMPUTER OR CALCULATOR)

With the easy accessibility of computers and calculators, students in elementary calculus courses are being encouraged to explore sequences numerically, and "discover" by computation whether a sequence converges and to what

limit. To analyze a sequence $\{x_n\}$ for convergence, students are encouraged to calculate a number of terms of the sequence and observe the trend. For example, we could use a calculator to calculate terms of the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$, as shown in Table 2.4.

Table 2.4

n	1	5	10	20	100	1,000	5,000
$\left(1+\frac{1}{n}\right)^n$	2	2.488	2.594	2.653	2.705	2.717	2.718

As n gets larger and larger, the values of $\left(1+\frac{1}{n}\right)^n$ seem to be getting closer to each other; we could conclude from the computed values in the table that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ converges and that its limit is a number close to 2.718.

WARNING: Use of calculation to determine convergence and limits is frequently unreliable. Uncritical reliance upon calculation can lead to quite erroneous conclusions. The following example will demonstrate the dangers inherent in this approach.

Recall from your calculus course that an infinite series $\sum_{n=1}^{\infty} x_n$ is said to converge to a sum S if and only if the sequence $\{S_n\}$ of "partial sums" $S_n = \sum_{k=1}^{n} x_k$ converges to S. That is,

$$\sum_{n=1}^{\infty} x_n = S \Leftrightarrow \lim_{n\to\infty} \sum_{k=1}^{n} x_k = S.$$

Thus, convergence of a series $\sum x_n$ is determined by the convergence of an associated sequence $\{S_n\}$. The example we are going to give will be a series that diverges, although its divergence is extremely slow—so slow that any calculator or computer will be fooled into concluding that it converges and even produce a finite number as its sum.

Example 2.5.16 The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$; that is,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty.$$

$\frac{1}{n}$ $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

Proof. $\forall n \in \mathbb{N}$, let $S_n = \sum_{k=1}^{n} \frac{1}{k}$. We want to prove that $\lim_{n\to\infty} S_n = +\infty$. Now, $\forall n \in \mathbb{N}$, $S_{n+1} = S_n + \frac{1}{n+1} > S_n$. Thus, $\{S_n\}$ is monotone increasing. Hence, by Theorem 2.5.14 (a), we need only show that $\{S_n\}$ is unbounded above. Now, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
S_{2^n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right)\\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right)\\
&= 1+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} \qquad (n+1 \text{ terms; } n \ \tfrac{1}{2}\text{'s})\\
&= 1+\frac{n}{2}.
\end{aligned}$$

Thus, the sequence $\{S_n\}$ is unbounded above. Therefore, by Theorem 2.5.14 (a), $\lim_{n\to\infty} S_n = +\infty$. $\square$

Lessons Drawn from Example 2.5.16: This example provides an excellent demonstration of the folly of relying too heavily on calculators or computers in finding limits. Try this one on your favorite calculator or computer. We have just proved that the sequence $\{S_n\}$ diverges to $+\infty$. It would be expected, therefore, that after not too many terms, S_n will exceed a small number, say 10. Try computing terms S_n for yourself. How many terms does it take to reach 10? If you actually try this by directly calculating S_n on your calculator, you will probably give up in frustration. But please, read on.

By the "integral test" for infinite series used in elementary calculus, we know that

$$\ln(n+1) < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n.$$

Thus, using a calculator to calculate $\ln(n+1)$ and $\ln n$ we can see that after 1,000 terms S_n has not yet reached 8. It takes about 10,000 terms for S_n to reach 10. After 100,000 terms, S_n is only about 12. After a million terms, S_n is only about 14, after a billion terms, S_n is about 20, and after a trillion terms, S_n is not yet 30! The sequence $\{S_n\}$ is what we call a "slowly diverging" sequence.

But even worse: eventually, say when $n \geq n_0$, $\frac{1}{n}$ is so small that the calculator cannot distinguish it from 0. That is, $\forall n \geq n_0$, the calculator uses 0 instead of $\frac{1}{n}$. Then $\forall n \geq n_0$, the calculator thinks $S_n = S_{n_0}$. Said another way, the calculator thinks that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges, because it thinks the sequence $\{S_n\}$ is eventually constant.

SUMMARY:

(a) To the calculator, every convergent sequence is eventually constant!

(b) To the calculator, many divergent sequences are eventually constant, hence convergent, as far as you can tell from the calculator.

(c) Evidence gathered about convergence by direct calculation may cause you to conclude, mistakenly, that a divergent sequence converges.

CANTOR'S NESTED INTERVALS THEOREM

We come now to one of the most famous theorems of the early twentieth century in elementary real analysis. It is important because of the pivotal role it plays in establishing the powerful Bolzano-Weierstrass Theorem in Section 2.6—which in turn plays a key role in establishing deep results about Cauchy sequences in Section 2.7, the topology of the real number system in Chapter 3, and continuous functions in Chapter 4 and Section 5.7. It is included here because it can easily be proved using the monotone convergence theorem.

Theorem 2.5.17 (*Cantor's Nested Intervals Theorem*) *Let $\{I_n\}$ be a sequence of nonempty closed intervals $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, and $\lim\limits_{n\to\infty} (b_n - a_n) = 0$. Then $\bigcap\limits_{n=1}^{\infty} I_n$ consists of exactly one point.*

In the interest of clarity, we shall state and prove a slightly more general version of this theorem. The reader will see that Cantor's theorem follows directly from this alternate version.

Alternate Theorem 2.5.17: *Let $\{I_n\}$ be a sequence of nonempty closed intervals $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$. Then*

(a) $\bigcap\limits_{n=1}^{\infty} I_n$ *is a nonempty closed interval, and*

(b) *if* $\lim\limits_{n\to\infty} (b_n - a_n) = 0$, *then* $\bigcap\limits_{n=1}^{\infty} I_n$ *consists of exactly one point.*

Proof. (a) Let $\{I_n\}$ be a sequence of nonempty closed intervals $I_n = [a_n, b_n]$ that are "nested"; i.e., $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$. Consider the sequences

$\{a_n\}$ and $\{b_n\}$ of left and right endpoints of the intervals I_n. Since the intervals are nested, we have $\forall n \in \mathbb{N}$,

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \cdots \leq b_2 \leq b_1.$$

Thus, the sequence $\{a_n\}$ is monotone increasing and bounded above by b_1. Likewise, the sequence $\{b_n\}$ is monotone decreasing and bounded below by a_1. By the monotone convergence theorem,

$$\exists\, a = \lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}, \text{ and } \exists\, b = \lim_{n\to\infty} b_n = \inf\{b_n : n \in \mathbb{N}\}.$$

Now, $\forall n \in \mathbb{N}$, $a_n \leq b_n$. Thus, by Theorem 2.3.12, $a \leq b$. We shall prove that $\bigcap_{n=1}^{\infty} I_n = [a, b]$.

Let $x \in [a, b]$. Then $\forall n \in \mathbb{N}$, $a_n \leq a \leq x \leq b \leq b_n$, so $x \in I_n$. Thus, $\forall n \in \mathbb{N}$, $[a, b] \subseteq I_n$. Therefore,

$$[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n. \tag{9}$$

We now show that the set containment goes the other way also. Suppose $y \in \bigcap_{n=1}^{\infty} I_n$. Then $\forall n \in \mathbb{N}$, $a_n \leq y \leq b_n$. Then, by Theorem 2.3.12 (limits preserve inequalities), $\lim_{n\to\infty} a_n \leq y \leq \lim_{n\to\infty} b_n$. That is, $a \leq y \leq b$. Therefore,

$$\bigcap_{n=1}^{\infty} I_n \subseteq [a, b]. \tag{10}$$

Putting (9) and (10) together, we have

$$\bigcap_{n=1}^{\infty} I_n = [a, b]. \tag{11}$$

Since $a \leq b$, this interval is nonempty.

(b) Suppose further, that $\lim_{n\to\infty} (b_n - a_n) = 0$. Then, by the algebra of limits, $\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$. But $\lim_{n\to\infty} b_n = b$ and $\lim_{n\to\infty} a_n = a$. Thus, $b - a = 0$; that is, $b = a$.

Therefore, by (11), $\bigcap_{n=1}^{\infty} I_n = \{a\}$. ■

EXERCISE SET 2.5

1. A sequence $\{x_n\}$ is said to be **eventually monotone (increasing or decreasing)** if it has a tail that is monotone (increasing or decreasing). Restate Theorem 2.5.3 for eventually monotone sequences, and prove either (a) or (b) of the resulting theorem.

2. Which of the following sequences are eventually monotone (or strictly) increasing (or decreasing)? Justify your answers, assuming the usual properties of trigonometric functions where necessary.

   (a) $\left\{\frac{(-1)^n}{n}\right\}$ (b) $\left\{n - \frac{1}{n}\right\}$

   (c) $\left\{n + \frac{(-1)^n}{n}\right\}$ (d) $\{n^2 - 10n + 100\}$

   (e) $\{2 + (-1)^n\}$ (f) $\left\{\frac{3n^2 + (-1)^n}{n}\right\}$

   (g) $\left\{\sin \frac{n\pi}{2}\right\}$ (h) $\{\sin n\pi\}$

   (i) $\left\{\frac{n}{2^n}\right\}$ (j) $\left\{\frac{5^n}{n!}\right\}$

   (k) $\left\{\cos \frac{\pi}{2n}\right\}$ (l) $\left\{\sin \frac{\pi}{3n}\right\}$

   (m) $\left\{\frac{3n + 5}{n^2 - n - 2}\right\}$ (n) $\left\{\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right\}$

3. Prove Theorem 2.5.3 (b).

4. Prove Corollary 2.5.4.

5. The proof of Theorem 2.5.3 (monotone convergence theorem) depends heavily on the completeness property of $\mathbb{R}$. Find a bounded, monotone increasing sequence in the (incomplete) field $\mathbb{Q}$ that fails to converge in $\mathbb{Q}$.

6. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \sqrt{6 + x_n}$. Prove that $\{x_n\}$ converges, and find its limit. [Hint: See Example 2.5.8.]

7. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \sqrt{4x_n + 5}$. Prove that $\{x_n\}$ converges, and find its limit. [Hint: See Example 2.5.8.]

8. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \dfrac{2x_n + 3}{7}$. Prove that $\{x_n\}$ converges, and find its limit.

9. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \dfrac{nx_n^2}{n+1}$. Prove that $\{x_n\}$ converges, and find its limit.

10. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = c \in \mathbb{R}$, and $\forall n \in \mathbb{N}$, $x_{n+1} = x_n^2$. For what values of c does $\{x_n\}$ converge, and to what?

11. Consider the sequence $\{x_n\}$ defined inductively by $x_1 = 1$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \sqrt[3]{x_n + 6}$. Prove that $\{x_n\}$ converges, and find its limit.

12. Define $\{x_n\}$ inductively by $x_1 = 1$ and $x_{n+1} = x_n + \dfrac{1}{x_n}$. Prove that $\{x_n\}$ is monotone increasing, but diverges.

13. Define $\{x_n\}$ inductively by $x_1 = 2$ and $x_{n+1} = 2 - \dfrac{1}{x_n}$. Write out the first few terms of this sequence and conjecture a formula for x_n. Prove this formula by mathematical induction, and use it to find $\lim\limits_{n\to\infty} x_n$.

14. Prove that $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2n)}\right\}$ converges.[10]

15. Use the method of Example 2.5.12 to calculate $\sqrt{10}$ to four decimal places.

16. Use the method of Example 2.5.12 to calculate $\sqrt{45}$ to five decimal places.

17. Use the methods of this section to give an easy proof of Theorem 2.3.7.

18. Prove Theorem 2.5.13 (b).

19. Prove Theorem 2.5.14 (b).

20. About how large is $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ when $n = 500,000$?

21. $\forall n \in \mathbb{N}$, let $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$. Prove that $\{S_n\}$ converges, and find its limit. Hint: Find an easy formula for S_n by writing it out as a sum and using the "partial fraction decomposition" of $\dfrac{1}{k(k+1)}$.

22. $\forall n \in \mathbb{N}$, let $T_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2}$. Prove that $\{T_n\}$ converges. [Hint: use the monotone convergence theorem combined with Exercise 21.]

23. Prove or disprove (if false, give a counterexample.):
    (a) If $\{x_n\}$ and $\{y_n\}$ are both monotone increasing, then so is their sum $\{x_n + y_n\}$.
    (b) If $\{x_n\}$ and $\{y_n\}$ are both monotone increasing, then so is their difference $\{x_n - y_n\}$.

10. In fact, the limit is 0, but the proof is postponed until Exercise 8.2.44.

(c) If $\{x_n\}$ and $\{y_n\}$ are sequences of nonnegative real numbers that are monotone increasing, then so is their product $\{x_n y_n\}$.

(d) If $\{x_n\}$ and $\{y_n\}$ are arbitrary sequences of real numbers that are monotone increasing, then so is their product $\{x_n y_n\}$.

(e) If $\{x_n\}$ and $\{y_n\}$ are sequences of positive real numbers that are monotone increasing, then so is their quotient $\left\{\dfrac{x_n}{y_n}\right\}$.

24. Prove that if $\{x_n\}$ is monotone, then $\{x_n\}$ converges iff $\{x_n^2\}$ converges. What if $\{x_n\}$ is not monotone?

25. Prove that the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$, $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \cdots$ converges and find its limit. What happens when all the 2's are replaced by 3's?

26. Let a be a fixed positive real number. Prove that the sequence $\sqrt{a}, \sqrt{a+\sqrt{a}}$, $\sqrt{a+\sqrt{a+\sqrt{a}}}, \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}, \cdots$ converges, and find its limit. [Suggestion: When proving the sequence is bounded, consider two separate cases: $a \geq 2$ and $0 < a < 2$.] Find the first five integer values of a for which the limit L is an integer. What do you notice about these limits? Conjecture the next integer value of a for which L is an integer. Write L as a function of a. How does this help you generate more integer values of a and L that "work?"

2.6 Subsequences and Cluster Points

Intuitively speaking, when some of the terms of a sequence are deleted from it, a new sequence results, which we call a **subsequence** of the original sequence. We make this idea rigorous in the following definition.

Definition 2.6.1 Suppose $\{x_n\}$ is a sequence. If $\{n_k\}$ is a strictly increasing sequence of natural numbers; (i.e., $n_1 < n_2 < \cdots < n_k < \cdots$) then the sequence $\{x_{n_k}\}$ is said to be a **subsequence** of $\{x_n\}$. Thus, $\{x_{n_k}\}$ is the sequence

$$x_{n_1}, x_{n_2}, \cdots, x_{n_k}, x_{n_{k+1}}, \cdots.$$

Examples 2.6.2 The following are examples of subsequences of $\{x_n\}$:

$$x_1, x_3, x_5, x_7, \cdots x_{2n-1}, \cdots$$
$$x_2, x_4, x_6, x_8, \cdots x_{2n}, \cdots$$
$$x_5, x_{10}, x_{15}, x_{20}, \cdots x_{5n}, \cdots$$
$$x_8, x_9, x_{10}, x_{11}, \cdots x_{n+7}, \cdots.$$

The following lemma is simply a technical observation that frequently proves useful in what is to come.

Lemma 2.6.3 *If $\{n_k\}$ is a strictly increasing sequence of natural numbers, then $\forall k \in \mathbb{N}$, $n_k \geq k$.*

Proof. Exercise 1. ■

In the following definition and subsequent results, we develop some useful terminology.

Definition 2.6.4 (a) A sequence $\{x_n\}$ is said to be **eventually in** a set A, if $\exists\, n_o \in \mathbb{N} \ni \forall n \geq n_0$, $x_n \in A$.

(b) A sequence $\{x_n\}$ is said to be **frequently**[11] **in** a set A, if $\forall n_o \in \mathbb{N}$, $\exists\, n \geq n_0 \ni x_n \in A$.

In words, a sequence is "eventually in" a set A if some tail of the sequence is in A; a sequence is "frequently in" A if every tail of the sequence contains a member of A.

Example 2.6.5 The sequence $\left\{2 + \dfrac{1}{n}\right\}$ is eventually in the interval $(1.99, 2.01)$. [That interval contains the 101-tail of the sequence.] The sequence $\left\{(-1)^n\left(2 + \dfrac{1}{n}\right)\right\}$ is frequently in, but is not eventually in, the interval $(1.99, 2.01)$. [No tail of the sequence is in that interval, but every tail does contain some member of the interval.]

convergence

Lemma 2.6.6 (a) *A sequence $\{x_n\}$ is* **eventually in** *a set $A \Leftrightarrow A$ contains all but a finite number of terms of $\{x_n\}$; that is, A contains x_n for all but finitely many $n \in \mathbb{N}$.*

cluster pt

(b) *A sequence $\{x_n\}$ is* **frequently in** *a set $A \Leftrightarrow A$ contains infinitely many terms of $\{x_n\}$; that is, A contains x_n for infinitely many $n \in \mathbb{N}$.*

Proof. Exercise 2. ■

We put this new terminology to work in the following theorem.

Theorem 2.6.7 *Let $\{x_n\}$ be a sequence and let L be a real number. Then*

(a) *x_n converges to $L \Leftrightarrow \forall \varepsilon > 0$, $\{x_n\}$ is eventually in $(L - \varepsilon, L + \varepsilon)$.*

(b) *$\{x_n\}$ has a subsequence converging to $L \Leftrightarrow \forall \varepsilon > 0$, $\{x_n\}$ is frequently in $(L - \varepsilon, L + \varepsilon)$.*

---

11. Although "frequently" is in common usage, "infinitely often" might be a better way to describe this situation.

Proof. (a) This is merely a restatement of the definition of $x_n \to L$.

(b) Part 1 ($\Rightarrow$): Suppose $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to L. Let $\varepsilon > 0$. Since $x_{n_k} \to L$, $\exists\, k_0 \in \mathbb{N} \ni k \geq k_0 \Rightarrow |x_{n_k} - L| < \varepsilon$. Then infinitely many terms of $\{x_{n_k}\}$ are in $(L-\varepsilon, L+\varepsilon)$; hence, infinitely many terms of $\{x_n\}$ are in $(L-\varepsilon, L+\varepsilon)$.

Part 2 ($\Leftarrow$): Suppose $\{x_n\}$ is a sequence $\ni \forall \varepsilon > 0$, $\{x_n\}$ is frequently in $(L-\varepsilon, L+\varepsilon)$. We define a subsequence $\{x_{n_k}\}$ by the alternate principle of mathematical induction on k (Theorem 1.3.9):

> (1) Since $\{x_n\}$ is frequently in $(L-1, L+1)$, $\exists\, n_1 \in \mathbb{N} \ni x_{n_1} \in (L-1, L+1)$. This defines x_{n_1}.
>
> (2) Suppose we have found $x_{n_1}, x_{n_2}, \cdots, x_{n_k} \ni \forall i = 1, 2, \cdots, k$, $x_{n_i} \in (L - \frac{1}{i}, L + \frac{1}{i})$ and $n_1 < n_2 < \cdots < n_k$. We shall find $x_{n_{k+1}}$. Since $\{x_n\}$ is frequently in $\left(L - \frac{1}{k+1}, L + \frac{1}{k+1}\right)$, $\exists$ infinitely many natural numbers n such that $x_n \in \left(L - \frac{1}{k+1}, L + \frac{1}{k+1}\right)$. Hence, $\exists\, n_{k+1} > n_k \ni x_{n_{k+1}} \in \left(L - \frac{1}{k+1}, L + \frac{1}{k+1}\right)$.

Hence, by the alternate principle of mathematical induction, $\forall k \in \mathbb{N}$, $\exists\, x_{n_k} \in \left(L - \frac{1}{k}, L + \frac{1}{k}\right)$, and $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. Hence, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and by the squeeze principle,

$$\lim_{n\to\infty} x_{n_k} = L.$$

Therefore, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to L. ■

The following theorem is frequently useful.

Theorem 2.6.8 *A sequence $\{x_n\}$ converges to a real number $L \Leftrightarrow$ every subsequence of $\{x_n\}$ converges to L.*

Proof. Part 1 ($\Rightarrow$): Suppose $x_n \to L$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$. We shall prove that $x_{n_k} \to L$.

Let $\varepsilon > 0$. Since $x_n \to L$, $\exists\, n_o \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n - L| < \varepsilon$. Then,

$$\begin{aligned} k \geq n_0 &\Rightarrow n_k \geq k \geq n_0 \quad \text{(see Lemma 2.6.3);} \\ &\Rightarrow |x_{n_k} - L| < \varepsilon. \end{aligned}$$

$k \geq n_0 \Rightarrow \geq n_k \geq k \geq \;\; \geq n$

Therefore, $x_{n_k} \to L$.

Part 2 ($\Leftarrow$): Exercise 3. ■

Example 2.6.9 Find $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n$.

Solution: By writing out the first few terms of the sequence, we have

$$\left(1+\frac{1}{2}\right)^1, \left(1+\frac{1}{4}\right)^2, \left(1+\frac{1}{6}\right)^3, \left(1+\frac{1}{8}\right)^4, \left(1+\frac{1}{10}\right)^5, \cdots.$$

We can rewrite these terms as

$$\sqrt{\left(1+\frac{1}{2}\right)^2}, \sqrt{\left(1+\frac{1}{4}\right)^4}, \sqrt{\left(1+\frac{1}{6}\right)^6}, \sqrt{\left(1+\frac{1}{8}\right)^8}, \sqrt{\left(1+\frac{1}{10}\right)^{10}} \cdots.$$

Thus, $\left\{\left(1+\frac{1}{2n}\right)^n\right\}$ is a subsequence of $\left\{\sqrt{\left(1+\frac{1}{n}\right)^n}\right\}$. Hence,

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\sqrt{\left(1+\frac{1}{n}\right)^n} = \sqrt{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \sqrt{e}. \quad \square$$

Corollary 2.6.10 (*Application to Proving Divergence*)

(a) *If a sequence has two subsequences that converge to different limits, then it diverges.*

(b) *If a sequence has a divergent subsequence, then it diverges.*

Examples 2.6.11 The following sequences diverge:

(a) The sequence $\left\{\frac{1+(-1)^n}{2}\right\} = \{0,1,0,1,0,1,0,1,0,1,\cdots\}$ diverges. It has a subsequence $\{0,0,0,\cdots\}$ converging to 0, and a subsequence $\{1,1,1,\cdots\}$ converging to 1.

(b) The sequence $\left\{1,1,2,\frac{1}{2},3,\frac{1}{3},4,\frac{1}{4},5,\frac{1}{5},\cdots,n,\frac{1}{n},\cdots\right\}$ diverges. It has a subsequence $\{1,2,3,4,5,\cdots\}$ that diverges. $\square$

Theorem 2.6.12 (a) *A sequence diverges to $+\infty$ $\Leftrightarrow$ every subsequence diverges to $+\infty$.*

(b) *A sequence diverges to $-\infty$ $\Leftrightarrow$ every subsequence diverges to $-\infty$.*

Proof. Exercise 7. ■

Theorem 2.6.13 *Let $\{x_n\}$ be a sequence. Then*

(a) *$\{x_n\}$ diverges to $+\infty \Leftrightarrow \forall M > 0$, $\{x_n\}$ is eventually in $(M, +\infty)$.*

(b) *$\{x_n\}$ has a subsequence diverging to $+\infty \Leftrightarrow \forall M > 0$, $\{x_n\}$ is frequently in $(M, +\infty)$.*

(c) *$\{x_n\}$ diverges to $-\infty \Leftrightarrow \forall M > 0$, $\{x_n\}$ is eventually in $(-\infty, -M)$.*

(d) *$\{x_n\}$ has a subsequence diverging to $-\infty \Leftrightarrow \forall M > 0$, $\{x_n\}$ is frequently in $(-\infty, -M)$.*

Proof. Exercises 8 and 9. ■

CLUSTER POINTS

Definition 2.6.14 A real number x is a **cluster point of a sequence** $\{x_n\}$ if the sequence $\{x_n\}$ has a subsequence converging to x. [Equivalently, $\forall \varepsilon > 0$, $x_n \in (x - \varepsilon, x + \varepsilon)$ for infinitely many n.]

We say that $+\infty$ is a cluster point of $\{x_n\}$ if $\{x_n\}$ has a subsequence diverging to $+\infty$. [Equivalently, $\forall M > 0$, $x_n > M$ for infinitely many n.]

We say that $-\infty$ is a cluster point of $\{x_n\}$ if $\{x_n\}$ has a subsequence diverging to $-\infty$. [Equivalently, $\forall M > 0$, $x_n < -M$ for infinitely many n.]

Examples 2.6.15 (a) Find all the cluster points of the sequence $\left\{\sin \frac{n\pi}{4}\right\}$.

Solution. Writing out this sequence, we see that
$\left\{\sin \frac{n\pi}{4}\right\} = \left\{\frac{1}{\sqrt{2}},\ 1,\ \frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}},\ -1,\ -\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}},\ 1,\ \cdots\right\}$.
Thus, this sequence has five cluster points: $0, 1, -1, \frac{1}{\sqrt{2}}$, and $-\frac{1}{\sqrt{2}}$.

(b) Find all the cluster points of the sequence $\{x_n\}$, where

$$x_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \dfrac{1 + (-1)^{n/2}}{2} & \text{if } n \text{ is even.} \end{cases}$$

Solution. Writing out this sequence, we have

$$\{x_n\} = \{1, 0, 3, 1, 5, 0, 7, 1, 9, 0, 11, 1, 13, 0, \cdots\}$$

Thus, we see that the cluster points are 0, 1, and $+\infty$. $\square$

BOLZANO-WEIERSTRASS THEOREM

The following theorem has profound consequences, which we shall see in later sections and chapters. It should be regarded as of major importance.

Theorem 2.6.16 (*Bolzano-Weierstrass Theorem for Sequences*) *Every bounded sequence has a convergent subsequence.*

Proof. Suppose $\{x_n\}$ is a bounded sequence. Then $\exists\, A, B \in \mathbb{R} \ni \forall n \in \mathbb{N}$, $A \le x_n \le B$. We construct a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ as follows.

Let $x_{n_1} = x_1$.

Consider the closed interval $I_1 = [a_1, b_1] = [A, B]$. One of the two halves of this interval, $\left[a_1, \dfrac{a_1+b_1}{2}\right]$ or $\left[\dfrac{a_1+b_1}{2}, b_1\right]$ must contain x_n for infinitely many n, and hence must contain some x_{n_2} where $n_2 > n_1$. Pick one of the half-intervals that does and call it $I_2 = [a_2, b_2]$. Note: $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$.

Consider the two halves of this interval, $\left[a_2, \dfrac{a_2+b_2}{2}\right]$ and $\left[\dfrac{a_2+b_2}{2}, b_2\right]$. One of them must contain x_n for infinitely many n, hence must contain some x_{n_3} where $n_3 > n_2$. Pick one of the half-intervals that does and call it $I_3 = [a_3, b_3]$. Note: $b_3 - a_3 = \frac{1}{2}(b_2 - a_2) = \frac{1}{4}(b_1 - a_1)$.

Continuing in this way, we develop a sequence of closed intervals $\{I_k\}$ where $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$, and each interval $I_k = [a_k, b_k]$ contains x_{n_k}, with $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$. Thus, $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Moreover, each interval I_k has length

$$b_k - a_k = \frac{1}{2^{k-1}}(b_1 - a_1).$$

Thus, $\lim\limits_{k\to\infty}(b_k - a_k) = \lim\limits_{k\to\infty}\left[\dfrac{1}{2^{k-1}}(b_1 - a_1)\right] = 0.$

Therefore, we can apply Cantor's Nested Intervals Theorem to conclude that there is a unique real number L such that

$$\bigcap_{k=1}^{\infty} I_k = \{L\}.$$

Claim: $\lim\limits_{k\to\infty} x_{n_k} = L$.

Proof: $\forall k \in \mathbb{N}$, $L \in I_k$. But also, $x_{n_k} \in I_k$. Since x_{n_k} and L both belong to the interval I_k, $|x_{n_k} - L| \le (b_k - a_k)$, the "length" of I_k. But, as observed above, $(b_k - a_k) \to 0$. Therefore, by the squeeze principle,

$$\lim_{k\to\infty} x_{n_k} = L.$$

Thus, $\{x_n\}$ has a convergent subsequence, $\{x_{n_k}\}$. $\blacksquare$

Theorem 2.6.17 *A bounded sequence converges* $\Leftrightarrow$ *it has one and only one cluster point.*

Proof. Part 1 ($\Rightarrow$): Exercise 11.

Part 2 ($\Leftarrow$): Suppose $\{x_n\}$ is a bounded sequence with one and only one cluster point, say L. We shall prove that $x_n \to L$. For contradiction, suppose $x_n \not\to L$. Then

$$\exists \varepsilon > 0 \ni \forall n_0 \in \mathbb{N}, \exists n \geq n_0 \ni |x_n - L| \geq \varepsilon.$$

Thus, there exists a strictly increasing sequence $\{\alpha_n\}$ of natural numbers such that

$$\forall n \in \mathbb{N}, |x_{\alpha_n} - L| \geq \varepsilon. \qquad (12)$$

2.6.8 x_{β_n} is subsequence of x_{α_n}

Since $\{x_{\alpha_n}\}$ is a subsequence of $\{x_n\}$ it is also bounded, so by the Bolzano-Weierstrass Theorem, it has a convergent subsequence $\{x_{\beta_n}\}$; say $x_{\beta_n} \to M$. Then M is a cluster point of $\{x_n\}$ since $\{x_{\beta_n}\}$ is a subsequence of $\{x_n\}$. But $\{x_n\}$ has only one cluster point, so $M = L$. That is, $x_{\beta_n} \to L$. Then,

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_{\beta_n} - L| < \varepsilon.$$

This contradicts (12), since $\{x_{\beta_n}\}$ is a subsequence of $\{x_{\alpha_n}\}$. Therefore, $x_n \to L$. ■

***Theorem 2.6.18** *Every convergent sequence has a monotone subsequence (converging to the same limit). In fact, if a convergent sequence is not eventually constant, then it has a strictly monotone subsequence.*

***Proof.** Let $\{x_n\}$ be a convergent sequence, $x_n \to L$. If $\{x_n\}$ is eventually constant, it has a constant tail, which serves as a monotone subsequence of $\{x_n\}$. Henceforth, we assume that $\{x_n\}$ is not eventually constant. Then infinitely many terms of $\{x_n\}$ are different from L. So, one of the two intervals $(-\infty, L)$ or $(L, +\infty)$ must contain infinitely many terms of $\{x_n\}$.

Case 1, $(-\infty, L)$ contains infinitely many terms of $\{x_n\}$: Let $\varepsilon > 0$. Since $x_n \to L$, all but finitely many terms of $\{x_n\}$ are in $(L - \varepsilon, L + \varepsilon)$. Thus, by our Case 1 hypothesis, infinitely many terms of $\{x_n\}$ must be in $(L - \varepsilon, L)$. Thus,

$$\forall \varepsilon > 0, \text{ infinitely many terms of } \{x_n\} \text{ are in } (L - \varepsilon, L). \qquad (13)$$

Let $\varepsilon_1 = 1$. Then, by (13), $\exists n_1 \in \mathbb{N} \ni x_{n_1} \in (L - \varepsilon_1, L)$. This defines the first term x_{n_1} of our subsequence.

Let $\varepsilon_2 = L - x_{n_1}$. Then $\varepsilon_2 > 0$, so by (1), $\exists$ infinitely many $n \in \mathbb{N} \ni x_n \in (L - \varepsilon_2, L)$. Hence, $\exists n_2 > n_1 \ni x_{n_2} \in (L - \varepsilon_2, L)$. Note that $x_{n_2} > L - \varepsilon_2 = L - (L - x_{n_1}) = x_{n_1}$. That is, $x_{n_1} < x_{n_2}$. This defines the second term, x_{n_2}.

We proceed by the principle of mathematical induction. In the general step, we assume that we have found $n_k \in \mathbb{N}$ $(k \geq 2) \ni$

$$n_{k-1} < n_k \text{ and } x_{n_{k-1}} < x_{n_k} < L.$$

Define $\varepsilon_{k+1} = L - x_{n_k}$. Then $\varepsilon_{k+1} > 0$, so by (13), $\exists$ infinitely many $n \in \mathbb{N} \ni x_n \in (L - \varepsilon_{k+1}, L)$. Hence, $\exists n_{k+1} > n_k \ni x_{n_{k+1}} \in (L - \varepsilon_{k+1}, L)$. Note that $x_{n_{k+1}} > L - \varepsilon_{k+1} = L - (L - x_{n_k}) = x_{n_k}$. That is, $x_{n_k} < x_{n_{k+1}}$.

By mathematical induction, we have defined a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that is (strictly) monotone increasing.

Case 2, $(L, +\infty)$ contains infinitely many terms of $\{x_n\}$: Exercise 19. ■

Corollary 2.6.19 *Every bounded sequence has a monotone subsequence.*

Proof. An immediate consequence of Theorems 2.6.16 and 2.6.18. ■

EXERCISE SET 2.6

1. Prove Lemma 2.6.3. [Hint: Use mathematical induction.]

2. Prove Lemma 2.6.6.

3. Complete Part (2) of the Proof of Theorem 2.6.8.

4. Find each of the following limits:

   (a) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n}$ (b) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{n}$

   (c) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n^2}$ (d) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}$

   (e) $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$

5. In each of (a) and (b), prove or disprove that for all disjoint nonempty sets A and B, it is possible for a sequence $\{x_n\}$ of real numbers to satisfy the given condition.

   (a) $\{x_n\}$ is eventually in A and eventually in B.

   (b) $\{x_n\}$ is frequently in A and frequently in B.

6. Suppose that the subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of even-numbered and odd-numbered terms, respectively, of $\{x_n\}$ both have limit L. Prove that $x_n \to L$. (L may be finite or infinite.)

7. Prove Theorem 2.6.12.

8. Prove Theorem 2.6.13 (a) and (b).

9. Prove Theorem 2.6.13 (c) and (d).

10. Find all cluster points of the following sequences. Then use Corollary 2.6.10 or Theorem 2.6.17 to tell whether the sequence converges.
    (a) $\left\{2, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{4}, 5, \frac{1}{5}, \cdots\right\}$ (b) $\left\{1, \frac{1}{2}, 2, 1, \frac{1}{3}, 3, 1, \frac{1}{4}, 4, 1, \frac{1}{5}, 5, \cdots\right\}$
    (c) $\left\{1, -2, \frac{1}{3}, 4, -5, \frac{1}{6}, 7, -8, \frac{1}{9}, 10, -11, \frac{1}{12}, \cdots\right\}$
    (d) $\left\{(-1)^n 5 + \frac{1}{n}\right\}$ (e) $\left\{(-1)^n \left(5 + \frac{1}{n}\right)\right\}$
    (f) $\left\{5 + \frac{(-1)^n}{n}\right\}$ (g) $\left\{\sin \frac{n\pi}{6}\right\}$
    (h) $\left\{\cos \frac{n\pi}{3}\right\}$ (i) $\left\{\tan \frac{\pi}{4n}\right\}$
    (j) $\left\{\cos\left(\pi + \frac{1}{n}\right)\right\}$ (k) $\left\{\sin\left(\frac{(-1)^n\pi}{2} + \frac{1}{n}\right)\right\}$
11. Prove Theorem 2.6.17 (a).
12. Give an example of a sequence that has exactly 100 cluster points.
13. Show that a sequence can have infinitely many cluster points, by giving an example of a sequence that has every natural number as a cluster point.
14. Prove that if a monotone sequence $\{x_n\}$ has a cluster point x (finite or infinite), then $x_n \to x$.
15. Prove that a sequence that is not bounded above has a subsequence diverging to $+\infty$, and a sequence that is not bounded below has a subsequence diverging to $-\infty$.
16. Prove that a sequence diverges if and only if it either has more than one cluster point or is unbounded.
17. Prove that a sequence $\{x_n\}$ converges to a real number L if and only if every subsequence of $\{x_n\}$ has a subsequence converging to L.
18. Prove that a sequence $\{x_n\}$ has no convergent subsequence $\Leftrightarrow |x_n| \to \infty$.
19. Finish the proof of Theorem 2.6.18 by proving Case 2.
20. Use subsequences to prove that $\forall x \in \mathbb{R}$, $\{\sin nx\}$ diverges unless $x = k\pi$ for some $k \in \mathbb{N}$. [Hint: Assume $\sin nx \to L$. Case 1: $L \neq 0$. Show $\cos nx \to \frac{1}{2}$ by using $2 \cos nx \sin nx = \sin 2nx \to L$. Take limit of both sides of $\cos 2nx = \cos^2 nx - \sin^2 nx$ to get $L^2 < 0$. Case 2: $L = 0$. Then $\cos^2 nx = 1 - \sin^2 nx \to 1$. Take the limit of both sides of $\sin(n+1)x = \sin n \cos x + \cos nx \sin x$ to get a contradiction.] Finally, use the above results to prove that $\{\cos nx\}$ diverges unless x is an even multiple of π.

21. Suppose $\{x_n\}$ is a bounded sequence. Prove that if all its *convergent* subsequences have the same limit, L, then $\{x_n\}$ also converges, and has limit L.

22. Use Exercise 6 to prove that $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$. Hint: first show that $n^n < (2n)!$, from which you can get $\sqrt[2n]{(2n)!} > \sqrt{n}$. Then show that $(2n+1)! > n^{n+1}$, from which you can get[12] $\sqrt[2n+1]{(2n+1)!} > \sqrt{n}$.

23. **(Project) Recursive Arithmetic Means:**[13] Let $a, b > 0$. Define $\{x_n\}$ by $x_1 = a$, $x_2 = b$, and $\forall n \in \mathbb{N}$, $x_{n+2} = \dfrac{x_{n+1} + x_n}{2}$. That is, every term beginning with the third is the arithmetic mean of the preceding two terms.

    (a) By writing out eight or ten terms, conjecture a formula for the odd-numbered terms, and a formula for the even-numbered terms. Specifically, find formulas for r_n and s_n such that $\forall n \in \mathbb{N}$,
    $$x_{2n-1} = b + r_{2n-1}(a-b) \text{ and } x_{2n} = b + s_{2n}(a-b).$$

    (b) Use mathematical induction to prove the formulas for r_n and s_n conjectured in (a). [Hint: prove both at the same time.]

    (c) Show that one of the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ is monotone increasing and the other is monotone decreasing.

    (d) Prove that both $\{x_{2n+1}\}$ and $\{x_{2n}\}$ converge, and to the same limit.

    (e) Use Exercise 6 to determine $\lim_{n\to\infty} x_n$.

24. **(Project) Recursive Geometric Means:** Let $a, b > 0$. Define $\{x_n\}$ by $x_1 = a$, $x_2 = b$, and $\forall n \in \mathbb{N}$, $x_{n+2} = \sqrt{x_{n+1}x_n}$. That is, every term beginning with the third is the geometric mean of the preceding two terms. Repeat the instructions (a)–(e) of Exercise 23, but this time show that $\forall n \in \mathbb{N}$,
$$x_{2n+1} = b\left(\frac{a}{b}\right)^{r_{2n-1}} \text{ and } x_{2n} = b\left(\frac{a}{b}\right)^{s_{2n}}$$
where r_n and s_n are the same as used in Exercise 23.

25. **(Project) The Arithmetic-Geometric Mean of Two Positive Numbers:** Let $0 < a < b$. Define two sequences $\{x_n\}$ and $\{y_n\}$ inductively by $x_1 = \dfrac{a+b}{2}$, $y_1 = \sqrt{ab}$, and $\forall n \in \mathbb{N}$, $x_{n+1} = \dfrac{x_x + y_n}{2}$, and $y_{n+1} = \sqrt{x_n y_n}$.

12. You may assume that $\forall m < n$ in $\mathbb{N}$, and $x > 1$, $\sqrt[m]{x} > \sqrt[n]{x}$.

13. For an easier derivation of the limit of this sequence, see Exercise 2.7.9.

First show[14] that $\forall u, v \in \mathbb{R}$, $\sqrt{uv} \le \dfrac{u+v}{2}$ and equality holds $\Leftrightarrow u = v$. Use this to show that $a < y_1 < y_2 < \cdots < y_n < x_n < \cdots < x_2 < x_1 < b$. Also, show that $\forall n \in \mathbb{N}$, $|x_n - y_n| < \dfrac{a+b}{2^{n-1}}$. Use these results to conclude that $\{x_n\}$ and $\{y_n\}$ converge and have the same limit. This common limit is called the **arithmetic-geometric mean** of a and b.

2.7 Cauchy Sequences

It is often possible to prove that a sequence converges without knowing its actual limit. The monotone convergence theorem provides one tool for doing that. The "Cauchy criterion" we are about to study is another.

Definition 2.7.1 A sequence $\{x_n\}$ is a **Cauchy sequence** if it satisfies the "Cauchy criterion:"

$$\boxed{\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni m, n \ge n_0 \Rightarrow |x_m - x_n| < \varepsilon.}$$

The significance of the Cauchy criterion is that, unlike Definition 2.1.4, this criterion makes no reference to a limit "L." Thus, it can prove useful when we do not know in advance whether a sequence has a limit, or whether it is monotone. We shall prove shortly that every Cauchy sequence converges.

Theorem 2.7.2 *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose $\{x_n\}$ converges, say $x_n \to L$. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow |x_n - L| < \dfrac{\varepsilon}{2}$. Then

$$\begin{aligned} m, n \ge n_0 &\Rightarrow |x_m - L| < \frac{\varepsilon}{2} \text{ and } |x_n - L| < \frac{\varepsilon}{2} \\ &\Rightarrow |x_m - L| + |x_n - L| < \varepsilon \\ &\Rightarrow |x_m - L| + |L - x_n| < \varepsilon \\ &\Rightarrow |x_m - L + L - x_n| < \varepsilon \quad \text{by the triangle inequality} \\ &\Rightarrow |x_m - x_n| < \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. ■

The next theorem should remind you of Theorem 2.2.10 for convergent sequences. The proofs are almost the same.

Theorem 2.7.3 *Every Cauchy sequence is bounded.*

14. See Exercise 1.2-B.10.

Proof. Suppose $\{x_n\}$ is a Cauchy sequence. Then, taking $\varepsilon = 1$ in Definition 2.7.1, $\exists\, n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow |x_m - x_n| < 1$. Then

$$\begin{aligned} n \geq n_0 &\Rightarrow |x_n - x_{n_0}| < 1 \\ &\Rightarrow -1 < x_n - x_{n_0} < 1 \\ &\Rightarrow x_{n_0} - 1 < x_n < x_{n_0} + 1. \end{aligned}$$

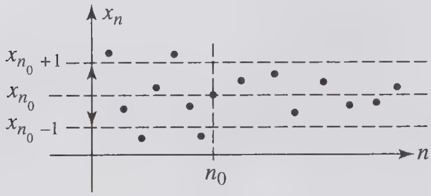


Figure 2.6

As in the proof of Theorem 2.2.10, we define

$A = \min\{x_1, x_2, \cdots, x_{n_0}, x_{n_0} - 1\}$ and $B = \max\{x_1, x_2, \cdots, x_{n_0}, x_{n_0} + 1\}$.

Then $\forall n \in \mathbb{N}$, $A \leq x_n \leq B$. That is, $\{x_n\}$ is bounded. ■

Main Theorem

Theorem 2.7.4 ***Every Cauchy sequence converges.***

Proof. Suppose $\{x_n\}$ is a Cauchy sequence. By Theorem 2.7.3, $\{x_n\}$ is bounded. Hence, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence, say $\{x_{n_k}\}$, with $x_{n_k} \to L$.

To prove: $x_n \to L$. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence, $\exists\, n_1 \in \mathbb{N} \ni m, n \geq n_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2}$. Since $x_{n_k} \to L$, $\exists\, n_2 \in \mathbb{N} \ni k \geq n_2 \Rightarrow |x_{n_k} - L| < \frac{\varepsilon}{2}$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned} m \geq n_0 &\Rightarrow m \geq n_1 \text{ and } n_m \geq m \geq n_1 \text{ and } n_m \geq m \geq n_2 \\ &\Rightarrow |x_m - x_{n_m}| < \frac{\varepsilon}{2} \text{ and } |x_{n_m} - L| < \frac{\varepsilon}{2} \\ &\Rightarrow |x_m - x_{n_m}| + |x_{n_m} - L| < \varepsilon \\ &\Rightarrow |x_m - x_{n_m} + x_{n_m} - L| < \varepsilon \quad \text{by the triangle inequality} \\ &\Rightarrow |x_m - L| < \varepsilon. \end{aligned}$$

Therefore, $x_n \to L$. ■

Theorem 2.7.5 *Suppose that $\{x_n\}$ is a sequence for which there is a constant C such that $\forall n \in \mathbb{N}$, $|x_{n+1} - x_n| < \frac{C}{2^n}$. Then $\{x_n\}$ is a Cauchy sequence; hence it converges.*

Proof. Suppose that $\{x_n\}$ is a sequence $\ni \forall n \in \mathbb{N}$, $|x_{n+1} - x_n| < \dfrac{C}{2^n}$. Then, whenever $m > n$ in $\mathbb{N}$,

$$\begin{aligned}
|x_m - x_n| &= |(x_{n+1} - x_n) + (x_{n+2} - x_{n+1}) + \cdots + (x_m - x_{m-1})| \\
&\leq |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \cdots + |x_m - x_{m-1})| \\
&\leq \frac{C}{2^n} + \frac{C}{2^{n+1}} + \cdots + \frac{C}{2^{m-1}} \\
&= \frac{C}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) \\
&< \frac{C}{2^n}(2) = \frac{C}{2^{n-1}}. \qquad (14)
\end{aligned}$$

Let $\varepsilon > 0$. Since $\dfrac{C}{2^{n-1}} \to 0$, $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \dfrac{C}{2^{n-1}} < \varepsilon$. Thus, from (14), $m, n \geq n_0 \Rightarrow |x_m - x_n| < \varepsilon$. This means $\{x_n\}$ is a Cauchy sequence. ■

***Example 2.7.6** (An Application) The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ converges. That is, the sequence $\displaystyle\left\{\sum_{k=1}^{n} \frac{1}{k!}\right\}_{n=1}^{\infty}$ converges.

Proof. $\forall n \in \mathbb{N}$, let $S_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k!}$. Then, $\forall n \in \mathbb{N}$,

$$|S_{n+1} - S_n| = \left|\sum_{k=1}^{n+1} \frac{1}{k!} - \sum_{k=1}^{n} \frac{1}{k!}\right| = \frac{1}{(n+1)!} \leq \frac{1}{2^n}.$$

(By Exercise 1.3.13, $2^n \leq (n+1)!$) Hence, by Theorem 2.7.5, $\{S_n\}$ converges. □

EXERCISE SET 2.7

1. Prove directly from Definition 2.7.1 that each of the following is a Cauchy sequence.

   (a) $\left\{\dfrac{1}{n}\right\}$ (b) $\left\{\dfrac{2n+1}{n}\right\}$ (c) $\left\{\dfrac{n}{n^2+1}\right\}$ (d) $\left\{\dfrac{n-2}{3n+4}\right\}$.

2. Prove directly from Definition 2.7.1 that each of the following is *not* a Cauchy sequence.

   (a) $\{(-1)^n\}$ (b) $\left\{\dfrac{n^2+1}{n}\right\}$.

3. Suppose that $\{x_n\}$ is a sequence such that $\forall n \in \mathbb{N}$, $|x_{n+1} - x_n| < C^n$, for some constant $0 < C < 1$. Prove that $\{x_n\}$ is a Cauchy sequence, hence converges.

4. Find an example of a sequence $\{x_n\}$ such that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ and $\lim_{n\to\infty} x_n = +\infty$. In what sense is this a caution to those who would use a calculator or computer to conclude whether a sequence converges?

5. Without using Theorem 2.7.4, prove that if a Cauchy sequence has a (finite) cluster point L, then it must converge to L.

6. Prove (without using Theorem 2.7.4) that a Cauchy sequence of integers must be eventually constant.

7. Suppose $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, and $r \in \mathbb{R}$. Without using Theorem 2.7.4,
   (a) prove that $\{x_n + y_n\}$, $\{x_n - y_n\}$, $\{rx_n\}$, and $\{x_n y_n\}$ are Cauchy sequences, but $\left\{\dfrac{x_n}{y_n}\right\}$ is not necessarily a Cauchy sequence, even if $\forall n \in \mathbb{N}$, $y_n \neq 0$.
   (b) find a condition on $\{y_n\}$ that will guarantee that $\left\{\dfrac{x_n}{y_n}\right\}$ is a Cauchy sequence, and prove your claim.

8. A sequence $\{x_n\}$ is said to be a **contractive sequence** if $\exists$ some constant c, $0 < c < 1 \ni \forall n \in \mathbb{N}$, $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$. Prove that a contractive sequence must be a Cauchy sequence, and hence converges.

9. **(Project) Recursive Arithmetic Means:**[15] Let $a \neq b$ be arbitrary real numbers, and define the sequence $\{x_n\}$ by
$$x_1 = a,\ x_2 = b, \text{ and } \forall n \in \mathbb{N},\ x_{n+2} = \frac{x_{n+1} + x_n}{2}.$$
   That is, each new term beginning with the third is the average of the two previous terms.
   (a) Prove that $\{x_n\}$ converges by proving that it is a contractive sequence.
   (b) Prove that $\forall n \in \mathbb{N}$, $x_{n+1} + \frac{1}{2}x_n = b + \frac{1}{2}a$.
   (c) Use (b) and the algebra of limits to find $\lim_{n\to\infty} x_n$. Are you surprised by this answer? Notice that if you interchanged a and b the answer would be different.

15. See Exercise 2.6.23.

10. **(Project) Recursive Weighted Arithmetic Means:** Let $a \neq b$ be arbitrary real numbers, let $0 < t < 1$, and define the sequence $\{x_n\}$ by

$$x_1 = a,\ x_2 = b, \text{ and } \forall n \in \mathbb{N},\ x_{n+2} = tx_n + (1-t)x_{n+1}.$$

That is, each new term beginning with the third is a weighted average of the two previous terms. Geometrically, x_{n+2} is a point in the interval between x_n and x_{n+1} that cuts the interval into two segments whose lengths are in the ratio t to $1-t$. Prove that $\{x_n\}$ is contractive (defined in Exercise 8), and find its limit.

11. **Contraction Mappings:** Let $a < b$ and $I = [a, b]$. A function $f : I \to I$ is said to be a **contraction mapping** if $\exists c \ni 0 < c < 1$ and $\forall x, y \in I$, $|f(x) - f(y)| \leq c|x - y|$. Prove that a contraction mapping must have at least one "fixed point," $x \in I \ni f(x) = x$. [See Exercise 8.] Also prove that f cannot have more than one fixed point in I.

12. **(Project) Fibonacci Numbers:** The Fibonacci sequence consists of the Fibonacci numbers, $1, 1, 2, 3, 5, 8, 13, 21, \cdots$, and is defined recursively by $f_1 = 1$, $f_2 = 1$, and $\forall n \geq 2$, $f_{n+2} = f_{n+1} + f_n$. Each new term after the second is the sum of the previous two terms. Many interesting results have been proved about the Fibonacci numbers—enough to fill an entire book. We shall be concerned here with the sequence of ratios of successive Fibonacci numbers. We begin by defining the sequence $\{r_n\}$ by $r_n = \dfrac{f_{n+1}}{f_n}$.

    (a) Develop a table that shows the first 10 terms of $\{r_n\}$. On the basis of this table, conjecture answers to the questions: Does $\{r_n\}$ converge? Is it monotone? Eventually monotone? Can you find a strictly increasing subsequence? A strictly decreasing subsequence? (No proofs required.)

    (b) Prove that $\forall n \in \mathbb{N}$, $r_{n+1} = 1 + \dfrac{1}{r_n}$.

    (c) Prove that $\forall n \geq 2$, $\frac{3}{2} < r_n < 2$.

    (d) Prove that $\{r_n\}$ is "contractive," and hence is a Cauchy sequence.

    (e) Find $\lim\limits_{n\to\infty} r_n$. [Keep note of this limit; it will reappear.]

    (f) The quadratic equation $x^2 - x - 1 = 0$ has two solutions, $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. Show that $\alpha + \beta = 1$, $\alpha^2 = a + 1$, and $\beta^2 = \beta + 1$, and from these facts show that $\forall n \in \mathbb{N}$, $\alpha^{n+2} = a^{n+1} + \alpha^n$ and $\beta^{n+2} = \beta^{n+1} + \beta^n$.

(g) $\forall n \in \mathbb{N}$, define $u_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, where α and β are as defined in (f). Prove that $u_1 = 1$, $u_2 = 1$, and $\forall n \geq 2$, $u_{n+2} = u_{n+1} + u_n$. Thus, $\{u_n\}$ must be the Fibonacci sequence. We have found an explicit formula for the Fibonacci numbers: $f_n = u_n$.

(h) **Geometric significance** of α. Consider a rectangle whose width a and length $a + b$ are so proportioned that when a square of side a is removed, as shown here, the remaining rectangle has width and length in the same proportion. That is, $\dfrac{a+b}{a} = \dfrac{a}{b}$.

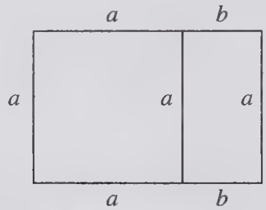


Figure 2.7

The classical Greek mathematicians called this ratio $R = \dfrac{a}{b}$ the "**Golden Ratio**," and any rectangle with sides in this ratio a "**Golden rectangle**." They considered it to be the most aesthetically pleasing of all rectangles, and used it frequently in their art and architecture. Prove algebraically that $R = \alpha$, defined in (f) above.

(i) Prove that $\forall n \geq 2$, $f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$.

(j) Prove that $\forall n \in \mathbb{N}$, $r_{n+1} - r_n = \dfrac{(-1)^{n+1}}{f_n f_{n+1}}$.

(k) Use (j) to prove that $\{r_{2n}\}$ is strictly decreasing and $\{r_{2n+1}\}$ is strictly increasing.

13. Let $a \geq 1$. Define the sequence $\{x_n\}$ by $x_1 = a$, and $x_{n+1} = a + \dfrac{1}{x_n}$. Prove that $\forall n \geq 2$, $a + \dfrac{1}{2a} \leq x_n \leq 2a$, and use this result to prove that $\{x_n\}$ is contractive. Find $\lim\limits_{n\to\infty} x_n$.

14. Let $a > 1$. Define the sequence $\{x_n\}$ by $x_1 = a$, and $x_{n+1} = \dfrac{1}{a + x_n}$. Prove that $\forall n \in \mathbb{N}$, $\dfrac{1}{2a} \leq x_n \leq a$, and use this result to prove that $\{x_n\}$ is contractive. Find $\lim\limits_{n\to\infty} x_n$. Compare this limit with that of Exercise 13.

*PROPERTIES EQUIVALENT TO COMPLETENESS

Several of the major theorems we have encountered so far are equivalent to the completeness property. We have derived them from the completeness property. We shall now show that in an Archimedean ordered field, any one of them will yield the completeness property as a consequence. The equivalences of these ideas is a result of pivotal significance in the foundations of real analysis.

Theorem 2.7.7** *Let F denote an Archimedean ordered field.* ***The following conditions are equivalent:

(a) *F is complete;*

(b) *Every bounded monotone sequence in F converges in F;*

(c) *Cantor's Nested Intervals Theorem;*

(d) *The Bolzano-Weierstrass Theorem (every bounded sequence has a convergent subsequence);*

(e) *Every Cauchy sequence in F converges in F.*

Proof.
$(a) \Rightarrow (b)$. Monotone convergence theorem (Theorem 2.5.3).
$(b) \Rightarrow (c)$. Cantor's nested intervals theorem (Theorem 2.5.17).
$(c) \Rightarrow (d)$. Bolzano-Weierstrass theorem (Theorem 2.6.16).
$(d) \Rightarrow (e)$. Theorem 2.7.4.
$(e) \Rightarrow (a)$. Proof: Suppose F is an Archimedean ordered field in which every Cauchy sequence converges. Let S be a nonempty subset of F with an upper bound. Then $\exists\, B \subseteq F \ni \forall x \in S,\ x \le B$. We shall prove that S has a least upper bound in F. We begin by constructing a sequence $\{x_n\}$ as follows:

Step 1. By the Archimedean property, the set

$$A = \{n \in \mathbb{N} : n \text{ is an upper bound for } S\}$$

is nonempty. Hence, by the well-ordering property, $\exists\, x_1 = \min A$. Then x_1 is an upper bound for S, but $x_1 - 1$ is not.

$$\text{Define } \ x_2 = \begin{cases} x_1 - \dfrac{1}{2} & \text{if } x_1 - \dfrac{1}{2} \text{is an upper bound for } S, \\ x_1 & \text{otherwise.} \end{cases}$$

Then,

- x_2 is an upper bound for S;
- $x_2 \le x_1$;
- $|x_2 - x_1| \le \frac{1}{2}$;
- $x_2 - \frac{1}{2}$ is not an upper bound for S, because, by definition of x_1 and x_2,

  if $x_2 = x_1 - \frac{1}{2}$, then $x_2 - \frac{1}{2} = x_1 - 1$, which is not an upper bound;

  if $x_2 = x_1$, then $x_1 - \frac{1}{2}$ is not an upper bound for S.

Define $x_3 = \begin{cases} x_2 - \frac{1}{4} & \text{if } x_2 - \frac{1}{4} \text{ is an upper bound for } S, \\ x_2 & \text{otherwise.} \end{cases}$

Then,

- x_3 is an upper bound for S;
- $x_3 \le x_2$;
- $|x_3 - x_2| \le \frac{1}{4}$;
- $x_3 - \frac{1}{4}$ is not an upper bound for S (reason as we did to show that $x_2 - \frac{1}{2}$ is not an upper bound for S).

Define

$$x_4 = \begin{cases} x_3 - \frac{1}{8} & \text{if } x_3 - \frac{1}{8} \text{ is an upper bound for } S, \\ x_3 & \text{otherwise.} \end{cases}$$

We continue by mathematical induction. We define

$$x_{k+1} = \begin{cases} x_k - \frac{1}{2^k} & \text{if } x_k - \frac{1}{2^k} \text{ is an upper bound for } S, \\ x_k & \text{otherwise.} \end{cases}$$

In this way, we arrive at a sequence $\{x_n\} \ni \forall n \in \mathbb{N}$,

$$\begin{cases} \bullet\ x_n \text{ is an upper bound for } S; \\ \bullet\ x_{n+1} \le x_n; \\ \bullet\ |x_{n+1} - x_n| \le \frac{1}{2^n}; \\ \bullet\ x_n - \frac{1}{2^{n-1}} \text{ is not an upper bound for } S. \end{cases} \tag{15}$$

Step 2. By (15) above, and by Theorem 2.7.5 above, $\{x_n\}$ is a Cauchy sequence. By our hypothesis, every Cauchy sequence converges. Thus, we may let $L = \lim_{n\to\infty} x_n$.

Step 3. Let $x \in S$. Since each x_n is an upper bound for S, $x_n \geq x$. Reviewing the proof of Theorem 2.3.12, we see that the proof does not depend on the completeness property. Thus, by Theorem 2.3.12, $L \geq x$. Hence, we conclude that L is an upper bound for S.

Step 4. Claim: $\forall n,\ x_n \geq L$.

Proof: In Step 3 we concluded that L is an upper bound for S. Now, suppose v is also an upper bound for S. To prove: $L \leq v$. For contradiction, suppose $v < L$.

Let $\varepsilon = L - v$. Since $\frac{1}{2^n} \to 0$, $\exists\, k \in \mathbb{N} \ni \frac{1}{2^k} < \varepsilon$. Now recall that in Step 1 we showed that $x_{k+1} - \frac{1}{2^k}$ is not an upper bound for S. Hence, $\exists\, x \in S \ni$

$$\begin{aligned} x &> x_{k+1} - \frac{1}{2^k} \\ &\geq L - \frac{1}{2^k} \quad \text{(See Step 4.)} \\ &> L - \varepsilon = v. \quad \left(\text{Since } \frac{1}{2^k} < \varepsilon.\right) \end{aligned}$$

Thus, $x > v$, contradicting the supposition that v is an upper bound for S. Therefore, by contradiction, $L \leq v$. Finally, we have proved that

$$L = \sup S. \quad \blacksquare$$

2.8 *Countable and Uncountable Sets

Georg Cantor (1845–1918) in his celebrated scheme for distinguishing between the relative sizes of infinite sets, introduced the notion of "countable" and "uncountable" sets. Briefly, he began by observing that two sets "have the same number of elements" if there is a 1-1 correspondence[16] between the elements of the two sets. Thus, infinite sets can have the same number of elements as proper subsets of themselves. For example the set of natural numbers $\mathbb{N}$ and the set of even numbers $\mathbf{E} = \{2, 4, 6, 8, \cdots\}$ have the same number of elements, since the function $f:\mathbb{N} \to \mathbf{E}$, given by $f(n) = 2n$, is a 1-1 correspondence.

Cantor showed that the set of natural numbers is the "smallest" infinite set in the sense that every infinite set must have a subset in 1-1 correspondence with $\mathbb{N}$. According to Cantor's definition, an infinite set is **countable** if it is

16. A 1-1 correspondence from a set A to a set B is a function $f : A \to B$ that is 1-1 and onto; see Appendix B.2.

in 1-1 correspondence with $\mathbb{N}$, and is **uncountable** otherwise. To make these ideas clear we must begin with precise definitions.

Definition 2.8.1 (Equivalent Sets) We say that two sets, A and B, are **equivalent** (in symbols, $A \cong B$) if there is a 1-1 correspondence $f : A \to B$. If $A \cong B$, we say that A and B **have the same number of elements**.

Definition 2.8.2 (a) A set S is **finite** if $\exists n \in \mathbb{N} \ni S \cong \{1, 2, \cdots, n\}$;
(b) A set S is **denumerable** if $S \cong \mathbb{N}$;
(b) A set is **countable** if it is finite or denumerable;
(c) A set is **uncountable** if it is not countable.

Example 2.8.3 The even numbers form a denumerable set; so do the odd numbers and sets like $\{1, 4, 9, 16, 25, \dots\} = \{n^2 : n \in \mathbb{N}\}$.

Theorem 2.8.4 *Every infinite set has a denumerable subset.*

Proof. Let S be an infinite set. Let $x_1 \in S$. Then[17] $S - \{x_1\}$ is still infinite. Choose any $x_2 \in S - \{x_1\}$. Then $x_2 \neq x_1$ and $S - \{x_1, x_2\}$ is still infinite. We proceed by mathematical induction. Assume $x_1, x_2, \cdots, x_n$ (all different) have been chosen in S; then $S - \{x_1, x_2, \cdots, x_n\}$ is still infinite. Choose $x_{n+1} \in S - \{x_1, x_2, \cdots, x_n\}$. By mathematical induction, we get a denumerable subset $\{x_1, x_2, \cdots, x_n, \cdots\}$ of S. ■

Thus, denumerable sets are the smallest infinite sets, while uncountable sets are a larger order of infinity in size. This topic fits nicely into this chapter, since **denumerable sets are those sets whose elements can be arranged in a sequence**. An uncountable set has so many elements that they cannot be arranged in a sequence; there are more elements in the set than there are natural numbers for use as subscripts in naming the elements of the set.

Cantor went on to classify many known infinite sets according to whether they were countable or uncountable. He made some surprising discoveries.

THE RATIONAL NUMBERS

One might guess that the rational numbers form an uncountable set, since they seem to be much more numerous than the natural numbers. After all, the natural numbers are discretely located at one-unit intervals along the number line, whereas the rational numbers are densely scattered everywhere along the line. In every interval of the number line, no matter how small, there are infinitely many rational numbers. Imagine the astonishment of the mathematical world when Cantor successfully proved that the rational numbers are countable. They are *not* more numerous than the natural numbers. Theorem 2.8.5 and its corollary are essentially his proof of this remarkable fact.

17. In Appendix B.1, we define $S - \{x_1\}$ to be $\{x \in S : x \neq x_1\}$.

Theorem 2.8.5 *There is a sequence whose range is* $\mathbb{Q}$. *That is, the set of rational numbers can be arranged as a subsequence of a sequence.*

Proof. Part 1: First, we show how to list all the *positive* rational numbers in a sequence. List in a (horizontal) row all the positive rational numbers with denominator 1, then in another row all those with denominator 2, then all those with denominator 3, and so on. Then construct a sequence by following the arrows in the pattern shown below.

Following the arrows will produce a sequence whose terms include all the positive rational numbers: $p_1, p_2, \cdots, p_n, p_{n+1}, \cdots$.

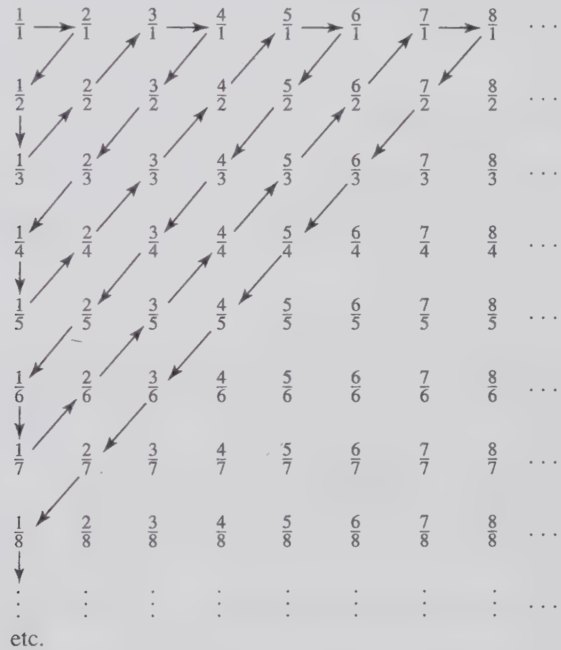


Figure 2.8

Part 2: The following sequence will include all the rational numbers, positive, negative, and zero: $0, p_1, -p_1, p_2, -p_2, \cdots, p_n, -p_n, p_{n+1}, -p_{n+1}, \cdots$.
■

Corollary 2.8.6 *The set* $\mathbb{Q}$ *of rational numbers is countable.*

Proof. In Theorem 2.8.5 we produced a sequence whose range is $\mathbb{Q}$; that is, a function f from $\mathbb{N}$ **onto** $\mathbb{Q}$. This sequence includes many repetitions; i.e.,

rational numbers are listed more than once in the sequence f. By eliminating the repetitions, we obtain a subsequence g of f:

$$g : \mathbb{N} \to \mathbb{Q} \quad \text{(1-1 and onto)}.$$

This g is a 1-1 correspondence, showing that $\mathbb{Q}$ is countable. ■

SOME UNCOUNTABLE SETS

Cantor further startled the mathematical world by proving the following remarkable result, demonstrating once and for all that there is more than one level of infinity.

Theorem 2.8.7 *The set* $\mathbb{R}$ *of real numbers is uncountable. (It is impossible to list the real numbers as a sequence.)*

Proof. If the real numbers *could* be listed as a sequence, then the open interval $(0, 1)$ would be a subsequence. Thus, it suffices to prove that it is impossible to list the elements of $(0, 1)$ as a sequence. For contradiction, suppose it is possible to list all the real numbers in $(0, 1)$ as a sequence,

$$\{x_n\}_{n=1}^{\infty}.$$

Each x_n has a decimal expansion. In the notation of Theorem 2.5.5, say

$$\left\{\begin{array}{c} x_1 = 0.d_{11}d_{12}d_{13}\cdots d_{1n}\cdots \\ x_2 = 0.d_{21}d_{22}d_{23}\cdots d_{2n}\cdots \\ x_3 = 0.d_{31}d_{32}d_{33}\cdots d_{3n}\cdots \\ \vdots \\ x_m = 0.d_{m1}d_{m2}d_{m3}\cdots d_{mn}\cdots \\ \vdots \end{array}\right\} \tag{16}$$

Define a new decimal $y = 0.e_1e_2e_3\cdots e_n\cdots$ by defining,

$$\forall k \in \mathbb{N},\ e_k \neq d_{kk},\ e_k \neq 0, \text{ and } e_k \neq 9.$$

That is, each e_k is different from d_{kk} and is neither 0 nor 9. Then,
$y \neq x_1$, since y and x_1 differ in the first decimal place and $e_1 \neq 0$ or 9;
$y \neq x_2$, since y and x_2 differ in the 2nd decimal place and $e_2 \neq 0$ or 9;

$$\vdots$$

$y \neq x_m$, since y and x_m differ in the mth decimal place and $e_m \neq 0, 9$;

$$\vdots$$

Thus, we have constructed a real number $y \in (0,1)$, which in not in the list (16). Contradiction! Therefore, it is impossible to list the elements of $(0,1)$ as a sequence. ■

The following is a very surprising result, which we can now prove. It says that the irrational numbers are more numerous than the rational numbers. This may come as a surprise, if you have always thought of irrational numbers as exceptional cases.

Corollary 2.8.8 *The set of irrational numbers is uncountable (hence, more numerous than the set of rationals).*

Proof. Exercise 2. ■

Significance of Theorems 2.8.5 and 2.8.7 and Their Corollaries: Taken together, these results give us profound insight into a fundamental contrast between the set of rational numbers and the sets of real numbers and irrational numbers. The set $\mathbb{Q}$ can be put into 1-1 correspondence with $\mathbb{N}$, whereas $\mathbb{R}$ and the set of irrational numbers cannot. The set of rational numbers is countable, whereas $\mathbb{R}$ and the set of irrational numbers are not. Said another way, **the rational numbers are *not* more numerous than the natural numbers, but the irrational numbers and the real numbers *are*.**

Further fascinating results of Cantor on infinite sets may be found in Section 3.4.

EXERCISE SET 2.8

1. Prove that the union of two denumerable sets is denumerable.

2. Prove Corollary 2.8.8. [Hint: Use Exercise 1.]

3. Prove that the union of a denumerable collection of denumerable sets is denumerable. [Hint: Use a diagonal procedure like the one used in proving Theorem 2.8.5. Double subscripts may help.]

4. Prove that if A and B are denumerable sets, then so is $A \times B$. [In fact, if $A_1, A_2, \cdots, A_n$ are denumerable, then so is $A_1 \times A_2 \times \cdots \times A_n$.]

5. Prove that the relation $\cong$ of Definition 2.8.1 has the following properties:

   (a) (Reflexivity) $\forall A,\ A \cong A$.

   (b) (Symmetry) $\forall A, B,\ A \cong B \Rightarrow B \cong A$.

   (c) (Transitivity) $\forall A, B, C$, if $A \cong B$ and $B \cong C$, then $A \cong C$.

6. Prove that if A is any infinite set and $x \in A$, then $A \cong A - \{x\}$. [Hint: Apply Theorem 2.8.4; make x the first element of a denumerable subset of A and consider the function $f(x_k) = x_{k+1}$ on this subset, while $f(x) = x$ otherwise.]

7. Prove that if A is an infinite set and B is any finite subset of A, then $A \cong A - B$.

8. Prove that if A is an infinite set, then there is some denumerable subset B of A, such that $A \cong A - B$.

9. Prove that if A is an uncountable set and B is any countable subset of A, then $A \cong A - B$.

10. Suppose $a < b$, and $c < d$. Prove that $(a, b) \cong (c, d)$ and $[a, b] \cong [c, d]$ by constructing 1-1 correspondences between the intervals.

11. Suppose $a < b$. Prove that $(a, b) \cong (a, +\infty)$ by constructing a 1-1 correspondence between the intervals.

12. Suppose $a < b$, and $c < d$. Prove that $(a, b) \cong [c, d]$.

13. Prove that $(0, 1) \cong \mathbb{R}$. [An interval is equivalent to the whole line!]

14. Prove that $(0, 1) \times (0, 1) \cong (0, 1)$. [Hint: Use decimal expansions.]

15. Prove that if $A \cong C$ and $B \cong D$, then $A \times B \cong C \times D$.

16. Prove that $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}$. [The plane is equivalent to a line!]

17. **(Project) Algebraic and Transcendental Numbers:** By definition, an **algebraic number** is any real number that is a solution of a polynomial equation $p(x) = 0$, where $p(x)$ has integer coefficients. A **transcendental number** is a real number that is not algebraic.

    Assume that every algebraic number x satisfies a unique polynomial equation with rational coefficients of the form $x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ of lowest degree. [You could prove this using the factor theorem and the unique factorization theorem of algebra.] The degree of this polynomial

is called the **degree** of x. For example, $\sqrt{3}$ is algebraic of degree 2, since it satisfies $x^2 - 3 = 0$.

(a) Prove that $\forall n \in \mathbb{N}$, there are only countably many algebraic numbers of degree n.

(b) Prove that the set of algebraic numbers is countable.

(c) Prove the existence of transcendental numbers by proving that the set of transcendental numbers is uncountable. [Recall how we proved that the set of irrational numbers is uncountable.]

(d) Search the literature and find proofs that π and e are transcendental. I suggest Niven, [100].

2.9 *Upper and Lower Limits

Not every sequence has a limit. But, after we make the appropriate definitions, we shall see that every sequence has a unique "upper limit" and a unique "lower limit," which may be a finite real number, or $+\infty$ or $-\infty$. These upper and lower limits will prove useful in Chapters 8 and 9, but will not really be needed before then.

Definition 2.9.1 (Upper Limit) Suppose $\{x_n\}$ is any sequence of real numbers.

Case 1: If $\{x_n\}$ is bounded above we define, $\forall n \in \mathbb{N}$,

$$\overline{x_n} = \sup\{x_k : k \geq n\}.$$

That is, $\overline{x_n}$ is the supremum of the n-tail[18] of $\{x_n\}$. Since $\{x_n\}$ is bounded above, each $\overline{x_n}$ is a real number. Note that

$$\overline{x_1} \geq \overline{x_2} \geq \cdots \geq \overline{x_n} \geq \cdots.$$

We define the **upper limit** of $\{x_n\}$ to be

$$\overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \overline{x_n}.$$

Since $\{\overline{x^n}\}$ is monotone decreasing, we know that $\overline{\lim\limits_{n\to\infty}}\, x_n$ is either a real number or $-\infty$.

Case 2: If $\{x_n\}$ is not bounded above then $\forall n \in \mathbb{N}$, $\overline{x_n} = +\infty$ and we define

$$\overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} (+\infty) = +\infty.$$

18. See Definition 2.2.15.

Definition 2.9.2 (Lower Limit) Suppose $\{x_n\}$ is any sequence of real numbers. Case 1: If $\{x_n\}$ is bounded below we define, $\forall n \in \mathbb{N}$,

$$\underline{x_n} = \inf\{x_k : k \geq n\}.$$

That is, $\underline{x_n}$ is the infimum of the n-tail[18] of $\{x_n\}$. Since $\{x_n\}$ is bounded below, each $\underline{x_n}$ is a real number. Note that

$$\underline{x_1} \leq \underline{x_2} \leq \cdots \leq \underline{x_n} \leq \cdots.$$

We define the **lower limit** of $\{x_n\}$ to be

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x_n}.$$

Since $\{\underline{x_n}\}$ is monotone increasing, we know that $\underline{\lim}_{n\to\infty} x_n$ is either a real number or $+\infty$.

Case 2: If $\{x_n\}$ is not bounded below then $\forall n \in \mathbb{N}$, $\underline{x_n} = -\infty$ and we define

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} (-\infty) = -\infty.$$

Remark 2.9.3 Although not every bounded sequence converges, every bounded sequence has both an upper limit and a lower limit, which are unique, finite real numbers. Even unbounded sequences have upper and lower limits; if a sequence is not bounded above, its upper limit is $+\infty$, and if a sequence is not bounded below, its lower limit is $-\infty$.

Examples 2.9.4 Find the upper and lower limits of each of the following sequences:

(a) $\{x_n\} = \{1, 0, 1, 0, 1, 0, 1, \cdots\}$

(b) $\{y_n\} = \{1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \cdots\}$

(c) $\{z_n\} = \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \cdots\}$

(d) $\{w_n\} = \{-n\}$.

Solutions:

(a) $\forall n \in \mathbb{N}$, $\underline{x_n} = 0$, so $\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \underline{x_n} = \lim_{n\to\infty} 0 = 0$. Similarly, $\overline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x_n} = \lim_{n\to\infty} 1 = 1$.

(b) $\forall n \in \mathbb{N}$, $\underline{y_n} = 0$, so $\underline{\lim}_{n\to\infty} y_n = \lim_{n\to\infty} \underline{y_n} = \lim_{n\to\infty} 0 = 0$. Also, $\overline{\lim}_{n\to\infty} y_n = \lim_{n\to\infty} \overline{y_n} = \lim_{n\to\infty} (+\infty) = +\infty$.

(c) Note that $\{\underline{z_n}\} = \{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{8}, -\frac{1}{8}, \cdots\}$; thus, $\underline{\lim}_{n\to\infty} z_n = 0$. Similarly, $\{\overline{z_n}\} = \{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{7}, \cdots\}$; thus, $\overline{\lim}_{n\to\infty} z_n = 0$.

(d) $\forall n \in \mathbb{N}$, $\underline{w_n} = -\infty$, so $\varliminf\limits_{n\to\infty} w_n = -\infty$. Also, $\forall n \in \mathbb{N}$, $\overline{w_n} = -n$, so $\varlimsup\limits_{n\to\infty} w_n = -\infty$. □

Remarks 2.9.5 (a) Since $\varlimsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (\sup\{x_k : k \geq n\})$, $\varlimsup\limits_{n\to\infty} x_n$ is often called the **lim sup** (or **limit superior**) of $\{x_n\}$. Similarly, $\varliminf\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (\inf\{x_k : k \geq n\})$, so $\varliminf\limits_{n\to\infty} x_n$ is often called the **lim inf** (or **limit inferior**) of $\{x_n\}$. These quantities are often denoted $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$, respectively.

(b) Since $\{\overline{x_n}\}$ is monotone decreasing, the monotone convergence theorem (2.5.3) tells us that

$$\begin{aligned}\varlimsup_{n\to\infty} x_n &= \inf\{\overline{x_n} : n \in \mathbb{N}\}\\ &= \inf\left\{\sup\{x_k : k \geq n\} : n \in \mathbb{N}\right\}.\end{aligned}$$

Similarly, $\{\underline{x_n}\}$ is monotone increasing, so

$$\begin{aligned}\varliminf_{n\to\infty} x_n &= \sup\{\underline{x_n} : n \in \mathbb{N}\}\\ &= \sup\left\{\inf\{x_k : k \geq n\} : n \in \mathbb{N}\right\}.\end{aligned}$$

Theorem 2.9.6 (*Elementary Properties of Upper and Lower Limits*)

(a) $\varliminf\limits_{n\to\infty} x_n \leq \varlimsup\limits_{n\to\infty} x_n$.

(b) *If $\{x_n\}$ is bounded above by B, then $\varlimsup\limits_{n\to\infty} x_n \leq B$.*

(c) *If $\{x_n\}$ is bounded below by A, then $\varliminf\limits_{n\to\infty} x_n \geq A$.*

(d) *If $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then*

$$\varliminf_{n\to\infty} x_n \leq \varliminf_{k\to\infty} x_{n_k} \leq \varlimsup_{k\to\infty} x_{n_k} \leq \varlimsup_{n\to\infty} x_n.$$

Proof.

(a) $\forall m, n \in \mathbb{N}$, $\underline{x_n} \leq \underline{x_{n+m}} \leq \overline{x_{n+m}} \leq \overline{x_m}$. Thus, $\sup\{\underline{x_n} : n \in \mathbb{N}\} \leq \inf\{\overline{x_n} : n \in \mathbb{N}\}$. (See Exercise 1.6-B.3.) That is, $\lim\limits_{n\to\infty} \underline{x_n} \leq \lim\limits_{m\to\infty} \overline{x_m}$ (by the monotone convergence theorem, 2.5.3). Therefore, $\varliminf\limits_{n\to\infty} x_n \leq \varlimsup\limits_{n\to\infty} x_n$.

(b) Suppose that $\forall n \in \mathbb{N}$, $x_n \leq B$. Then B is an upper bound for every n-tail of $\{x_n\}$, so $\overline{x_n} = \sup\{x_k : k \geq n\} \leq B$. Thus, $\lim\limits_{n\to\infty} \overline{x_n} \leq B$. (Why?) That is, $\varlimsup\limits_{n\to\infty} x_n \leq B$.

(c) Exercise 2.

(d) Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Let $m \in \mathbb{N}$. Then, $\forall k \in \mathbb{N}$, $n_k \geq k$ so $\{x_{n_k} : k \geq m\} \subseteq \{x_n : n \geq m\}$. Thus, $\underline{x_{n_m}} \geq \underline{x_m}$ and $\overline{x_{n_m}} \leq \overline{x_m}$.

Therefore, since limits preserve inequalities, $\underline{\lim}_{m\to\infty} x_{n_m} \geq \underline{\lim}_{m\to\infty} x_m$ and $\overline{\lim}_{m\to\infty} x_{n_m} \leq \overline{\lim}_{m\to\infty} x_m$. Putting these inequalities together with (a), we have the desired inequalities, $\underline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{k\to\infty} x_{m_k} \leq \overline{\lim}_{k\to\infty} x_{n_k} \leq \overline{\lim}_{n\to\infty} x_n$. ■

Theorem 2.9.7 (*ε Criterion for Upper Limit*) *Let $\{x_n\}$ be a bounded sequence. Then $L = \overline{\lim}_{n\to\infty} x_n \Leftrightarrow \forall\varepsilon > 0$,*

(a) *$x_n < L + \varepsilon$, for all but finitely many n, and*

(b) *$x_n > L - \varepsilon$ for infinitely many n.*

Proof. Part 1 ($\Rightarrow$): Suppose $L = \overline{\lim}_{n\to\infty} x_n$ and let $\varepsilon > 0$. By definition, this means $L = \lim_{n\to\infty} \overline{x_n}$. Then

(a) By definition of limit, $\exists\, n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \geq n_0 &\Rightarrow |\overline{x_n} - L| < \varepsilon \\ &\Rightarrow L - \varepsilon < \sup\{x_k : k \geq n\} < L + \varepsilon. \\ &\Rightarrow L + \varepsilon \text{ is an upper bound for } \{x_k : k \geq n\}. \end{aligned}$$

Then, $\forall k \geq n_0$, $x_k < L + \varepsilon$. That is, $x_n < L + \varepsilon$ for all but finitely many n.

(b) By Remark 2.9.5 (b), $L = \inf\{\overline{x_n} : n \in \mathbb{N}\}$. Let $n_0 \in \mathbb{N}$. Then $\overline{x_{n_0}} > L - \varepsilon/2$, and since $\overline{x_{n_0}} = \sup\{x_k : k \geq n_0\}$, $\exists\, k \geq n_0 \ni x_k > \overline{x_{n_0}} - \varepsilon/2$. Thus, $\forall n_0 \in \mathbb{N}$, $\exists\, k \geq n_0 \ni x_k > L - \varepsilon$. That is, $x_n > L - \varepsilon$ for infinitely many n.

Part 2 ($\Leftarrow$): Suppose that $\forall\varepsilon > 0$, the given conditions (a) and (b) hold. Let $\varepsilon > 0$. Then by (a), $\exists\, n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \geq n_0 &\Rightarrow x_n < L + \varepsilon \\ &\Rightarrow \overline{x_n} \leq L + \varepsilon. \end{aligned}$$

Thus, $\forall\varepsilon > 0$, $\lim_{n\to\infty} \overline{x_n} \leq L + \varepsilon$, so by the forcing principle, $\overline{\lim}_{n\to\infty} x_n \leq L$.

On the other hand, by (b), $\forall n \in \mathbb{N}$, $\overline{x_n} = \sup\{x_k : k \geq n\} \geq L - \varepsilon$. Thus, since limits preserve inequalities, $\lim_{n\to\infty} \overline{x_n} \geq L$, so $\overline{\lim}_{n\to\infty} x_n \geq L$. ■

Theorem 2.9.8 (*ε Criterion for Lower Limit*) *Let $\{x_n\}$ be a bounded sequence. Then $L = \underline{\lim}_{n\to\infty} x_n \Leftrightarrow \forall\varepsilon > 0$,*

(a) *$x_n > L - \varepsilon$ for all but finitely many n, and*

(b) *$x_n < L + \varepsilon$ for infinitely many n.*

Proof. Exercise 3. ■

Theorem 2.9.9 *A bounded sequence $\{x_n\}$ converges if and only if $\varliminf_{n\to\infty} x_n$ and $\varlimsup_{n\to\infty} x_n$ are both real numbers and are equal. In fact,*

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \varliminf_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n = L.$$

Proof. Exercise 4. ■

Theorem 2.9.10 *Let $\{x_n\}$ be a bounded sequence. Then $\varliminf_{n\to\infty} x_n$ and $\varlimsup_{n\to\infty} x_n$ are cluster points of $\{x_n\}$; moreover, they are the minimum and maximum cluster points of $\{x_n\}$, respectively.*

Proof. Suppose $\{x_n\}$ is a bounded sequence. Let $L = \varliminf_{n\to\infty} x_n$ and $U = \varlimsup_{n\to\infty} x_n$. Then, $\forall\varepsilon > 0$, Theorems 2.9.7 and 2.9.8 guarantee that the intervals $(L-\varepsilon, L+\varepsilon)$ and $(U-\varepsilon, U+\varepsilon)$ contain x_n for infinitely many n. Thus, L and U are cluster points of $\{x_n\}$ by Definition 2.6.14.

Now, let W be a cluster point of $\{x_n\}$. Then $\exists$ subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to W$. By Theorem 2.9.9,

$$\varliminf_{k\to\infty} x_{n_k} = \varlimsup_{k\to\infty} x_{n_k}.$$

Then, by Theorem 2.9.6 (d), $L \le \lim_{k\to\infty} x_{n_k} \le U$. That is, $L \le W \le U$. ■

EXERCISE SET 2.9

1. Find the upper and lower limits of each of the following sequences:
   (a) $\left\{\frac{1}{2}, -\frac{2}{3}, \frac{1}{3}, -\frac{3}{4}, \frac{1}{4}, -\frac{4}{5}, \frac{1}{5}, -\frac{5}{6}, \frac{1}{6}, \cdots\right\}$
   (b) $\left\{\sin\frac{n\pi}{6}\right\}$ (c) $\left\{n\sin\frac{n\pi}{6}\right\}$
   (d) $\left\{\frac{n\sin\frac{n\pi}{6}}{n+1}\right\}$ (e) $\left\{\frac{n+(-1)^n(2n+1)}{n}\right\}$
   (f) $\left\{\frac{1+\cos\frac{n\pi}{2}}{(-1)^n\, n^2}\right\}$ (g) $\left\{n^{\cos\left(\frac{n\pi}{2}\right)}\right\}$
   (h) $\left\{\left(1+\cos\frac{n\pi}{2}\right)^{\frac{1}{n}}\right\}$
2. Prove Theorem 2.9.6 (c).
3. Prove Theorem 2.9.8.
4. Prove Theorem 2.9.9.

5. For each of the following, prove or find a counterexample for which the given equation is *not* true:

   (a) $\overline{\lim_{n\to\infty}}\,(x_n + y_n) = \overline{\lim_{n\to\infty}}\, x_n + \overline{\lim_{n\to\infty}}\, y_n$

   (b) $\overline{\lim_{n\to\infty}}\,(x_n y_n) = \left(\overline{\lim_{n\to\infty}}\, x_n\right)\left(\overline{\lim_{n\to\infty}}\, y_n\right)$

   (c) $\underline{\lim_{n\to\infty}}\,(x_n + y_n) = \underline{\lim_{n\to\infty}}\, x_n + \underline{\lim_{n\to\infty}}\, y_n$

   (d) $\underline{\lim_{n\to\infty}}\,(x_n y_n) = \left(\underline{\lim_{n\to\infty}}\, x_n\right)\left(\underline{\lim_{n\to\infty}}\, y_n\right)$

6. Prove that

   (a) If $\{x_n\}$ and $\{y_n\}$ are bounded above, then $\overline{\lim_{n\to\infty}}\,(x_n + y_n) \le \overline{\lim_{n\to\infty}}\, x_n + \overline{\lim_{n\to\infty}}\, y_n$.

   (b) If $\{x_n\}$ is bounded above and $r \ge 0$, then $\overline{\lim_{n\to\infty}}\, r x_n = r\, \overline{\lim_{n\to\infty}}\, x_n$.

   (c) If $\{x_n\}$ and $\{y_n\}$ are bounded sequences of nonnegative numbers, then $\overline{\lim_{n\to\infty}}\,(x_n y_n) \le \left(\overline{\lim_{n\to\infty}}\, x_n\right)\left(\overline{\lim_{n\to\infty}}\, y_n\right)$.

   (d) Upper limits preserve inequalities. That is, if $\forall n \in \mathbb{N}$, $x_n \le y_n$, then $\overline{\lim_{n\to\infty}}\, x_n \le \overline{\lim_{n\to\infty}}\, y_n$.

7. State and prove results similar to those of Exercise 2.9.6 for lower limits.

8. Given any sequence $\{x_n\}$, prove that $\overline{\lim_{n\to\infty}}\,(-x_n) = -\underline{\lim_{n\to\infty}}\, x_n$ and $\underline{\lim_{n\to\infty}}\,(-x_n) = -\overline{\lim_{n\to\infty}}\, x_n$.

9. Suppose $\{x_n\}$ and $\{y_n\}$ are sequences of nonnegative numbers such that $x_n \to x \ne 0$ and $\overline{\lim_{n\to\infty}}\, y_n = y$. Prove that $\overline{\lim_{n\to\infty}}\, x_n y_n = xy$. [Hint: Use subsequences and Theorem 2.9.10.]

Chapter 3

Topology of the Real Number System

Sections 3.1 and 3.2 present the concepts of neighborhoods, open and closed sets, interior and boundary points, cluster points, and closures, which are essential tools in modern real analysis. Sections 3.3 and 3.4 can be safely omitted in a one-semester course. For the core of this book, a compact set is one that is closed and bounded, although the open covering approach is given full treatment in Section 3.3.

Mathematical topics often have both algebraic and geometric (or visual) sides. Indeed, students and instructors often find visualization to be an indispensable tool in learning and remembering new mathematical concepts. In this chapter we introduce a powerful geometric tool, called "topology," that has proved invaluable in formulating the ideas of elementary real analysis. It introduces a language that is highly suggestive of visualization. By its very nature, it is qualitative rather than quantitative. Topology is a subject in its own right. Here we barely scratch the surface of this wide and deep subject. It is difficult to say in a few words just what "topology" is and what its achievements have been. To give a complete definition would require us to digress too far from our objective. Briefly, it is a geometric type of mathematics in which neither size nor shape has any significance. In fact, one can say that "distance," so important in the concepts of ordinary geometry, plays no essential role in topology. We shall shortly define what is meant by an "open set" in the real number system. Once we know what an open set is, distance will no longer be necessary in defining limits, continuous functions, and other elementary concepts in analysis. The concept of "open" is as basic to topology as "distance" is to ordinary geometry. Some knowledge of elementary topology is indispensable in learning modern

analysis. Indeed, every aspiring mathematician should eventually take a course in the subject. In this text, we present only enough of the fundamental ideas to get us through elementary analysis. For further knowledge or study, the reader is encouraged to consult any general topology textbook listed in the Bibliography. Especially good introductions can be found in [4], [36], [70], and [94]; for more challenging presentations start with [98] or [137].

3.1 Neighborhoods and Open Sets

We begin with the fundamental concept of *neighborhood*, which leads directly to the concept of *open set*. While our context is the real number system, which is one dimensional, these ideas derive their power from the ease with which they generalize to higher dimensions. Indeed, your instructor may choose to illustrate each of the following ideas with two- or three-dimensional drawings.

Definition 3.1.1 Let $x \in \mathbb{R}$ and $\varepsilon > 0$. The interval $(x-\varepsilon, x+\varepsilon)$ will be called the **ε-neighborhood** of x and denoted $N_\varepsilon(x)$. Geometrically speaking, $N_\varepsilon(x)$ is the set of all points that are within a distance of ε from x.

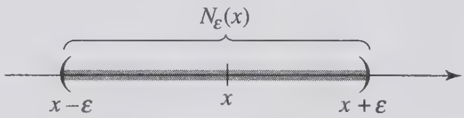


Figure 3.1

We often say simply "neighborhood[1] of x," by which we will always mean "ε-neighborhood of x, for some $\varepsilon > 0$." We shall see that the language of neighborhoods is quite useful in expressing concepts of analysis.

Examples 3.1.2 Uses of the language of neighborhoods:

(a) A sequence $\{x_n\}$ converges to L iff $\forall \varepsilon > 0$, x_n is eventually in $N_\varepsilon(L)$. In words, a sequence converges to L iff it is eventually in every neighborhood of L.

(b) A sequence $\{x_n\}$ has a subsequence converging to L iff $\forall \varepsilon > 0$, x_n is frequently in $N_\varepsilon(L)$. In words, a sequence has a subsequence converging to L iff it is frequently in every neighborhood of L.

1. In topology, the term "neighborhood" has a slightly more general definition.

Definition 3.1.3 A set $U \subseteq \mathbb{R}$ is **open** if $\forall x \in U$, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq U$. In words, a set U is open if and only if each of its points has a neighborhood contained entirely in U.

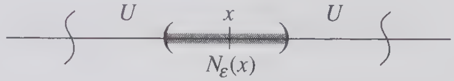


Figure 3.2

Theorem 3.1.4 *Let $a, b \in \mathbb{R}$. The intervals (a, b), $(a, +\infty)$, $(-\infty, a)$, and $(-\infty, +\infty)$ are open sets.*

Proof. (a) Consider the interval (a, b).

Case 1 ($a \geq b$): In this case, $(a, b) = \varnothing$. Since $\nexists x \in \varnothing$, it is true that $\forall x \in \varnothing$, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq \varnothing$. Thus, $\varnothing$ is open, and so (a, b) is open.

Case 2 ($a < b$): Let $x \in (a, b)$. Then $a < x < b$. Let $\varepsilon = \min\{x - a, b - x\}$. Then $N_\varepsilon(x) \subseteq (a, b)$. Thus, (a, b) is open.

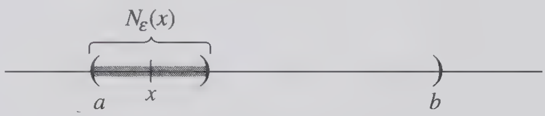


Figure 3.3

(b) Finish the proof by considering each of the other types of intervals given. (Exercise 1.) ■

Corollary 3.1.5 *Every ε-neighborhood $N_\varepsilon(x)$ is open.* ■

The following theorem is the basis for establishing that many other sets are open as well. It is considered fundamental.

Theorem 3.1.6 (*Open Set Theorem*)

Main Theorem

(a) *$\varnothing$ and $\mathbb{R}$ are open.*

(b) *The union of any collection of open sets is open.*

(c) *The intersection of any finite number of open sets is open.*

Proof. (a) In proving Case 1 of Theorem 3.1.4 we proved that $\varnothing$ is open. To see that $\mathbb{R}$ is open, merely observe that $\forall x \in \mathbb{R}$, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq \mathbb{R}$.

(b) Let $\mathcal{C}$ be any collection of open sets. To prove that $\cup\mathcal{C}$ is open, let $x \in \cup\mathcal{C}$. Then $\exists A \in \mathcal{C} \ni x \in A$. Since A is open, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$. But $A \subseteq \cup\mathcal{C}$. Thus, $N_\varepsilon(x) \subseteq \cup\mathcal{C}$. Therefore, $\forall x \in \cup\mathcal{C}$, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq \cup\mathcal{C}$. That is, $\cup\mathcal{C}$ is open.

(c) Let $\{A_1, A_2, \cdots, A_n\}$ be a finite collection of open sets. To prove $\bigcap_{i=1}^{n} A_i$ is open, let $x \in \bigcap_{i=1}^{n} A_i$. Now for each $i = 1, 2, \cdots, n$, A_i is open, so $\exists \varepsilon_i > 0 \ni N_{\varepsilon_i}(x) \subseteq A_i$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$. Then $\varepsilon > 0$ and for each $i = 1, 2, \cdots, n$, $\varepsilon \leq \varepsilon_i$, so

$$N_\varepsilon(x) \subseteq N_{\varepsilon_i}(x) \subseteq A_i, \text{ so}$$

$$N_\varepsilon(x) \subseteq \bigcap_{i=1}^{n} A_i.$$

Therefore, by Definition 3.1.3, $\bigcap_{i=1}^{n} A_i$ is open. ■

Theorem 3.1.7 *A nonempty open set must be an infinite set. That is, a nonempty set with only finitely many elements cannot be open.*

Proof. Let $A = \{a_1, a_2, \cdots, a_n\}$ be a finite set. Let $x \in A$. Now, $\forall \varepsilon > 0$, the set $N_\varepsilon(x)$ is an *infinite* set, hence it cannot be a subset of A, since A is a finite set. Thus, there is no $\varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$. That is, a finite set A cannot be open. Therefore, if A is a nonempty open set, it must be infinite. ■

The open set theorem (Theorem 3.1.6) claims that the union of *any* collection of open sets is open, but only that the intersection of *finitely many* open sets is open. The next example shows that the intersection of infinitely many open sets need not be open.

Example 3.1.8 (A Collection of Open Sets Whose Intersection Is Not Open)

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

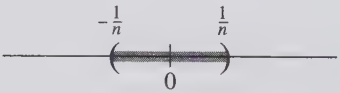


Figure 3.4

Each set $\left(-\frac{1}{n}, \frac{1}{n}\right)$ is open, while the intersection $\{0\}$ is *not* open. □

The open set idea is extremely powerful. We shall see in the remainder of this section that it enables us to define other intrinsically interesting and useful concepts.

INTERIOR, EXTERIOR, AND BOUNDARY

Definition 3.1.9 (Interior of a Set) Let A be a set of real numbers. A real number x is said to be an **interior point** of A if $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$. That is, an interior point of A can be surrounded by a neighborhood contained entirely in A.

The **interior of** A is the set

$$A^\circ = \{x : x \text{ is an interior point of } A\}.$$

Examples 3.1.10 Let $A = [0, 1)$, $B = [0, 1) \cup \{2\}$, and $C = (0, 3) \cup (3, 5)$. Then

(a) $A^\circ = (0, 1)$ (b) $B^\circ = (0, 1)$
(c) $C^\circ = C$ (d) $\mathbb{N}^\circ = \varnothing$
(e) $\mathbb{R}^\circ = \mathbb{R}$ (f) $\mathbb{Q}^\circ = \varnothing$.

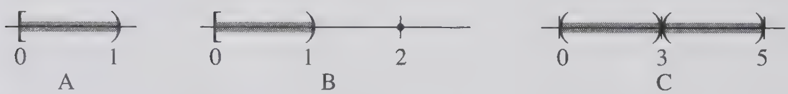


Figure 3.5

Theorem 3.1.11 (***Properties of Interior***) *Let A be a set of real numbers. Then,*

(a) $A^\circ = \cup\{$*all open subsets of* $A\}$;

(b) A° *is the largest open subset of* A, *in the sense that* A° *is open and if* U *is an open subset of* A, *then* $U \subseteq A^\circ$;

(c) A *is open* $\Leftrightarrow A = A^\circ$.

Proof. (a) Part 1: Let $x \in A^\circ$. By definition, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$. Thus x is a member of an open subset of A; namely, $N_\varepsilon(x)$. Hence, $x \in \cup\{$all open subsets of $A\}$. Therefore, $A^\circ \subseteq \cup\{$all open subsets of $A\}$.

Part 2: Let $x \in \cup\{$all open subsets of $A\}$. Then, $\exists$ an open subset U of A such that $x \in U$. Since U is open, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq U \subseteq A$. Thus, $x \in A^\circ$. Therefore, $\cup\{$all open subsets of $A\} \subseteq A^\circ$.

(b) Exercise 3.

(c) Exercise 4. ■

Definition 3.1.12 (Exterior of a Set) Let A be a set of real numbers. The **exterior** of A is the interior of the complement of A. In symbols,

$$A^{ext} = (A^c)^\circ.$$

An element of A^{ext} is called an **exterior point of A**.

It is important to see the difference between the exterior of a set and the complement of a set. The following example may help.

Examples 3.1.13 Let $A = [0,1)$, $B = [0,1) \cup \{2\}$, $C = (0,3) \cup (3,5)$. (See Figure 3.6.)

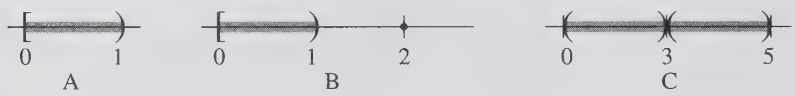


Figure 3.6

Then,

$$\begin{aligned} A^c &= (-\infty,0) \cup [1,\infty), && \text{while } A^{ext} = (-\infty,0) \cup (1,\infty); \\ B^c &= (-\infty,0) \cup [1,2) \cup (2,\infty), && \text{while } B^{ext} = (-\infty,0) \cup (1,2) \cup (2,\infty); \\ C^c &= (-\infty,0] \cup \{3\} \cup [5,\infty), && \text{while } C^{ext} = (-\infty,0) \cup (5,\infty). \end{aligned}$$

Additionally,

$$\mathbb{N}^c = (-\infty,1) \cup \left(\bigcup_{n=1}^{\infty} (n,n+1) \right), \quad \text{and} \quad \mathbb{N}^{ext} = \mathbb{N}^c;$$

$$\mathbb{Q}^{ext} = \varnothing, \quad \text{and} \quad \mathbb{R}^{ext} = \varnothing. \qquad \square$$

Theorem 3.1.14 (*Properties of Exterior*) *Let A be a set of real numbers. Then,*

(a) A^{ext} is an open set.

(b) x is an **exterior point** of A iff $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A^c$; i.e., x has a neighborhood containing no points of A.

Proof. Trivial. ■

Definition 3.1.15 (Boundary of a Set) Let A be a set of real numbers and $x \in \mathbb{R}$. We say that x is a **boundary point** of A if every neighborhood of x contains at least one point of A and at least one point of A^c. The set of all boundary points of A is called the **boundary** of A, and is denoted A^b.

Examples 3.1.16 Some boundary points:

(a) 3 and 6 are boundary points of the intervals $(3,6)$, $(3,6]$, $[3,6)$, and $[3,6]$.

(b) 0 is a boundary point of $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. So is $\frac{1}{n}, \forall n \in \mathbb{N}$.

Examples 3.1.17 Let $A = [0,1)$, $B = [0,1) \cup \{2\}$, and $C = (0,3) \cup (3,5)$. (See Figure 3.6.) Then

$$
\begin{array}{ll}
A^b = \{0,1\}; & B^b = \{0,1,2\}; \\
C^b = \{0,3,5\}; & \mathbb{N}^b = \mathbb{N}; \\
\mathbb{Q}^b = \mathbb{R}; & \mathbb{R}^b = \varnothing.
\end{array}
$$

Theorem 3.1.18 *For any set $A \subseteq \mathbb{R}$,*

(a) *A^b consists of all real numbers that are in neither A° nor A^{ext}.*

(b) *$A^b = (A^c)^b$; that is, a set and its complement have the same boundary.*

(c) *A°, A^b, and A^{ext} are mutually exclusive (i.e., pairwise disjoint) sets whose union is $\mathbb{R}$.*

Proof. Exercise 11. ■

ISOLATED POINTS

Definition 3.1.19 A real number x is an **isolated point** of a set $A \subseteq \mathbb{R}$ if $x \in A$ and $\exists \varepsilon > 0 \ni N_\varepsilon(x)$ contains no point of A other than x; i.e., $N_\varepsilon(x) \cap A = \{x\}$.

In words, an isolated point of A is a member of A that can be surrounded by a neighborhood containing no other members of A.

Examples 3.1.20 Let $A = [0,1)$, $B = [0,1) \cup \{2,3,4\}$, $C = (0,3) \cup (3,5)$, and $D = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. (See Figure 3.6.) The isolated points of these sets and others are as follows:

Set:	A	B	C	D	$\mathbb{N}$	$\mathbb{Q}$	$\mathbb{R}$
Isolated points:	none	$\{2,3,4\}$	none	D	$\mathbb{N}$	none	none

Theorem 3.1.21 *Every isolated point of a set A is a boundary point of A. The converse, however, is false.*

Proof. Exercise 15. ■

Theorem 3.1.22 (***Finite Sets***)

(a) *Finite sets have no interior points.*

(b) *Every point of a finite set A is a boundary point of A.*

(c) *Every point of a finite set A is an isolated point of A.*

Proof. Exercise 16. ■

EXERCISE SET 3.1

1. Finish proving Theorem 3.1.4.

2. Tell whether the following sets are open: (Justify your answer.)

   (a) $(3,5) \cup \{6\}$ (b) $(-\infty,0) \cup (0,1)$
   (c) $\{1,2,3,4,5,6,7,8,9\}$ (d) $(-\infty,0) \cup [0,1)$
   (e) $\mathbb{Z}$ (f) $(-\infty,0) \cup (0,1]$
   (g) $(-\infty,0) \cup [0,1]$ (h) $\mathbb{R} - \{1,2,3\}$
   (i) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ (j) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$
   (k) $\mathbb{Q}$ (l) $\mathbb{Q} \cap (0,1)$

3. Prove Theorem 3.1.11 (b).

4. Prove Theorem 3.1.11 (c).

5. Prove that a set is open iff it is the union of a family of open intervals.

6. Find the interior, exterior, and boundary of each of the sets given in Exercise 3.1.2.

7. In Examples 3.1.10, 3.1.13, and 3.1.17 we asserted that $\mathbb{Q}^\circ = \varnothing$, $\mathbb{Q}^{ext} = \varnothing$, and $\mathbb{Q}^b = \mathbb{R}$. Prove these assertions.

8. Find the interior, exterior, and boundary of the set of irrational numbers.

9. Prove that a set is open iff it contains none of its boundary points; i.e., A is open $\Leftrightarrow A \cap A^b = \varnothing$.

10. Give an example of a collection of bounded open intervals whose intersection is $[0,1]$. (See Example 3.1.8.)

11. Prove Theorem 3.1.18.

12. Suppose A is a bounded, nonempty set of real numbers. Prove that $\sup A$ and $\inf A$ are boundary points of A. Also prove that if A is open then $\sup A \notin A$ and $\inf A \notin A$.

13. Find all the isolated points of the sets given in Exercise 3.1.2.

14. In Example 3.1.20 we asserted that $\mathbb{Q}$ has no isolated points. Prove that assertion. Does the set of all irrational numbers have any isolated points? Justify your answer.

15. Prove Theorem 3.1.21.

16. Prove Theorem 3.1.22.

17. (a) Prove that $(A \cap B)^\circ = A^\circ \cap B^\circ$.

    (b) Prove that $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

    (c) Give an example of sets A and B such that $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

18. Prove that $A^\circ = A - A^b$.

19. Prove that a set $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ (see Definition 1.5.6) iff $\forall x \in \mathbb{R}$, every neighborhood of x contains a point of A.

20. Prove that a set $A \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ iff every nonempty open set of real numbers contains a point of A.

21. Prove that a sequence $\{x_n\}$ converges to a real number L iff every open set containing L contains all but finitely many terms of $\{x_n\}$.

Exercises 22 and 23 are for students who have studied "countable" sets, as in Section 2.8.

22. Prove that every nonempty open set A is the union of countably many open intervals with rational endpoints. [Hint: Consider intervals of the form $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$, where $r \in A \cap \mathbb{Q}$ and $n \in \mathbb{N}$.]

23. Prove that every nonempty open set A is the union of countably many *pairwise disjoint* open intervals. [Suggestion: Define $\mathcal{C} = \{I_a : a \in A\}$, where $I_a = \cup\{\text{all open subintervals of } A \text{ containing } a\}$. Prove that each I_a is an open interval containing a. Then prove that the I_a's are pairwise disjoint and $A = \cup\, \mathcal{C}$. Finally, prove that $\mathcal{C}$ is a countable collection.]

3.2 Closed Sets and Cluster Points

The notion of "closed set" is closely related to the notion of "open set," but the relation is not exactly what one might expect. Be forewarned that "closed" does not mean "not open." Unlike doors and restaurants, sets that are open might also be closed, and sets that are not open are not necessarily closed! Pay careful attention to the following definition.

Definition 3.2.1 A set of real numbers is **closed** if its complement is open. That is, A is closed $\Leftrightarrow$ A^c is open.

Corollary 3.2.2 *$\forall a, b \in \mathbb{R}$, the following sets are closed: $\varnothing$, $\{a\}$, $(-\infty, a]$, $[a, b]$, $[a, +\infty)$, $\mathbb{R}$.*

Proof. Exercise 1. ■

Note 1: if $a < b$, then the sets $(a, b]$ and $[a, b)$ are neither open nor closed.

Proof. Exercise 2. ■

Theorem 3.2.3 *The boundary of any set is closed.*

Proof. Let $A \subseteq \mathbb{R}$. By Theorem 3.1.18, $(A^b)^c = A^\circ \cup A^{ext}$, and by Theorems 3.1.11 and 3.1.14, both A° and A^{ext} are open sets. Thus, $A^\circ \cup A^{ext}$ is open; i.e., $(A^b)^c$ is open. Therefore, by definition, A^b is closed. ■

Theorem 3.2.4 (*Closed Set Theorem*)

(a) *$\varnothing$ and $\mathbb{R}$ are closed.*

(b) *The intersection of any collection of closed sets is closed.*

(c) *The union of any finite number of closed sets is closed.*

Proof. (a) Exercise 4.

(b) Let $\mathcal{C}$ be any collection of closed sets. Recall de Morgan's law for collections of sets (see Appendix B, Theorem B.1.10):

$$(\cap\, \mathcal{C})^c = \left(\bigcap_{A\in\mathcal{C}} A\right)^c = \bigcup_{A\in\mathcal{C}} A^c.$$

Since each set $A \in \mathcal{C}$ is closed, each A^c is open. Thus, $\bigcup_{A\in\mathcal{C}} A^c$ is the union of a collection of open sets, and so by the open set theorem, is open. That is, $(\cap\, \mathcal{C})^c$ is open. By Definition 3.2.1, this means that $\cap\, \mathcal{C}$ is closed.

(c) Exercise 4. ■

The closed set theorem claims that the intersection of *any* collection of closed sets is closed, but only that the union of *finitely many* closed sets is closed. The next example shows that the union of infinitely many closed sets need not be closed. (Compare with Example 3.1.8.)

Example 3.2.5 (A Collection of Closed Sets Whose Union Is Not Closed)

$$\bigcup_{n=2}^{\infty}\left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1).$$

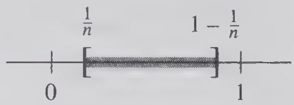


Figure 3.7

Each set $\left[\frac{1}{n}, 1-\frac{1}{n}\right]$ is closed, while the union $(0,1)$ is *not* closed. $\square$

CLUSTER POINTS OF A SET

Definition 3.2.6 (Cluster Points) Suppose $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a **cluster point** of A if every neighborhood of x contains a point of A other than x. [i.e., $\forall \varepsilon > 0,\ N_\varepsilon(x) \cap (A - \{x\}) \neq \varnothing$.]

Examples 3.2.7 (a) $0, 1, \frac{1}{2}$, and 0.789 are cluster points of the interval $(0,1)$.
(b) 0 is a cluster point of $\{\frac{1}{n} : n \in \mathbb{N}\}$ but $\frac{1}{100}$ and $\frac{1}{986}$ are not. $\square$

Theorem 3.2.8 *A set is closed iff it contains all of its cluster points.*

Proof. Part 1 ($\Rightarrow$): Suppose A is closed. Let x be a cluster point of A. We must prove that $x \in A$.

For contradiction, suppose $x \notin A$. Then $x \in A^c$. But A^c is open, so $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A^c$. Then $N_\varepsilon(x)$ is a neighborhood of x containing no point of A. But x is a cluster point of A. Contradiction. Therefore, $x \in A$.

Part 2 ($\Leftarrow$): Suppose A contains all its cluster points. We must prove that A is closed.

For contradiction, suppose A is not closed. Then A^c is not open, so

$$\exists x \in A^c \ni \forall \varepsilon > 0,\ N_\varepsilon(x) \not\subseteq A^c,$$

so, $N_\varepsilon(x)$ contains a point of A
(which can't be x, since $x \notin A$)
so, $N_\varepsilon(x)$ contains a point of A other than x.

Hence, $\exists x \in A^c \ni x$ is a cluster point of A. That is, there is a cluster point of A that does not belong to A. This contradicts our hypothesis that A contains all its cluster points. Therefore, A is closed. ■

Lemma 3.2.9 (*Cluster Points vs. Interior Points and Boundary Points*)

(a) *Every interior point of A is a cluster point of A.*

(b) *If x is a boundary point of A and $x \notin A$, then x is a cluster point of A.*

(c) *If x is a cluster point of A and $x \notin A$, then x is a boundary point of A.*

Proof. Exercise 9. ■

Corollary 3.2.10 *A set is closed iff it contains all of its boundary points.*

Proof. Exercise 10. ■

Theorem 3.2.11 *Suppose $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is a cluster point of A iff every neighborhood of x contains **infinitely many** points of A.*

Proof. Part 1 ($\Rightarrow$): Suppose x is a cluster point of A. For contradiction, suppose $\exists$ neighborhood $N_\varepsilon(x)$ containing only finitely many points of A. Only finitely many of these points will be different from x, say $a_1, a_2, \cdots, a_n$. Let $\delta = \min\{|x-a_1|, |x-a_2|, \cdots, |x-a_n|\}$. Then $\delta > 0$ and $N_\delta(x)$ is a neighborhood of x containing no point of A different from x. This is a contradiction, since x is a cluster point of A. Therefore, every neighborhood of x contains infinitely many points of A.

Part 2 ($\Leftarrow$): Suppose that every neighborhood of x contains infinitely many points of A. Then, every neighborhood of x contains a point of A other than x, so x is a cluster point of A. ■

Corollary 3.2.12 (***Finite Sets***)

(a) *Finite sets have no cluster points.*

(b) *All finite sets are closed.*

Proof. Exercise 11. ■

BOLZANO-WEIERSTRASS THEOREM (AGAIN)

We first encountered the Bolzano-Weierstrass theorem as a result about sequences: every bounded sequence has a convergent subsequence. This theorem has an analogue for bounded sets. Like the sequential version, it is a theorem of considerable power.

Theorem 3.2.13 (***Bolzano-Weierstrass Theorem for Sets***) *Every bounded infinite set of real numbers has a cluster point.*

Proof. Let A be a bounded infinite set of real numbers. Since A is an infinite set, we can find a sequence $\{a_n\}$ of *different* elements of A. Then $\{a_n\}$ is a bounded sequence, so by the Bolzano-Weierstrass Theorem for sequences, $\{a_n\}$ has a convergent subsequence, $\{a_{n_k}\}$. Say $a_{n_k} \to L$. Then, every neighborhood of L contains a_{n_k} for infinitely many k. Since the terms of $\{a_n\}$ are all different, every neighborhood of L contains at least one point of A other than L. Thus, L is a cluster point of A. ■

THE CLOSURE OF A SET

A given set A of real numbers is not necessarily closed; it need not contain all its cluster points. We want to be able to "close" it; that is, adjoin to it just enough points to make the resulting set closed. This set will be called the "closure" of A. Our task will be made easier if we start with a slightly different definition.

Definition 3.2.14 If $A \subseteq \mathbb{R}$, the **closure of** A is the set $\overline{A}$ (or A^{cl}) defined as the intersection of the collection of all closed sets containing A.

Theorem 3.2.15 (***Basic Properties of the Closure of A***) *$\forall A \subseteq \mathbb{R}$,*

(a) *$\overline{A}$ is a closed set;*

(b) $A \subseteq \overline{A}$;

(c) *$\overline{A}$ is the smallest closed set containing A, in the sense that if B is any closed set containing A, then $\overline{A} \subseteq B$;*

(d) *A is closed iff $A = \overline{A}$;*

(e) $\overline{\varnothing} = \varnothing$; $\overline{\mathbb{R}} = \mathbb{R}$.

Proof. (a) By the closed set theorem (3.2.4), the intersection of any collection of closed sets is closed, so $\overline{A}$ is closed.

(b)–(e) Exercise 13. ■

The interior of a set and the closure of a set share a duality relationship. The interior of A is the largest open set contained in A, while the closure of A is the smallest closed set containing A. For other interesting aspects of this duality, compare for yourself the properties of interior and the properties of closure listed in Theorems 3.1.11 and 3.2.15.

Examples 3.2.16 Let $A = [0, 1)$, $B = [0, 1) \cup \{2\}$, and $C = (0, 3) \cup (3, 5)$. (See Examples 3.1.10 and 3.1.13.) Then,

$$\overline{A} = [0, 1];\ \overline{B} = [0, 1] \cup \{2\};\ \overline{C} = [0, 5];\ \overline{\mathbb{N}} = \mathbb{N};\ \overline{\mathbb{Q}} = \mathbb{R};\ \overline{\mathbb{R}} = \mathbb{R}.$$

Figure 3.8

Theorem 3.2.17 *Let* $\mathbf{A}'$ = *the set*[2] *of all cluster points of* A. *Then* $\overline{A} = A \cup A'$. *(Thus, every point of* $\overline{A}$ *is either a point of* A *or a cluster point of* A.)

Proof. (a) First we prove that $\overline{A} \subseteq A \cup A'$. [i.e., $x \in \overline{A} \Rightarrow x \in A \cup A'$.] We prove the contrapositive. Suppose $x \notin A \cup A'$. Then $x \notin A$ and $x \notin A'$, so $\exists \varepsilon > 0 \ni N_\varepsilon(x)$ contains no points of A. Then $N_\varepsilon(x) \subseteq A^c$, so $A \subseteq N_\varepsilon(x)^c$. Now $N_\varepsilon(x)^c$ is a closed set containing A, since $N_\varepsilon(x)$ is open. But $x \notin N_\varepsilon(x)^c$. Thus $x \notin \cap\{$all closed subsets containing $A\}$; i.e., $x \notin \overline{A}$. Therefore, $\overline{A} \subseteq A \cup A'$.

(b) Next we prove that $A \cup A' \subseteq \overline{A}$. Let $x \in A \cup A'$.

Case 1 ($x \in A$): Then $x \in \overline{A}$ since $A \subseteq \overline{A}$.

Case 2 ($x \in A'$): Then x is a cluster point of A, so by definition of cluster point, x is a cluster point of $\overline{A}$. But $\overline{A}$ is closed, so by Theorem 3.2.8, $x \in \overline{A}$.

In either case, $x \in \overline{A}$. Therefore, $A \cup A' \subseteq \overline{A}$.

(c) By (a) and (b) together, $\overline{A} = A \cup A'$. ■

SEQUENTIAL CRITERIA

Theorem 3.2.18 (***Sequential Criterion for Cluster Points***) *x is a cluster point of a set A iff $\exists$ sequence $\{a_n\}$ of points of A other than x, such that $a_n \to x$.*

Proof. (a) ($\Rightarrow$): Suppose x is a cluster point of A. Then, by definition, $\forall n \in \mathbb{N}$, the neighborhood $N_{\frac{1}{n}}(x)$ contains a point $a_n \in A$ other than x. Consider the sequence $\{a_n\}$. Note that $\forall n \in \mathbb{N}$, $a_n \in N_{\frac{1}{n}}(x)$, so $|a_n - x| < \frac{1}{n}$. By the squeeze principle, $a_n \to x$.

2. The set A' is often called the **derived set** of A.

(b) ($\Leftarrow$): Suppose $\exists$ sequence $\{a_n\}$ of points of A other than x, such that $a_n \to x$. Then every neighborhood of x contains infinitely many terms of the sequence, and hence contains a point of A other than x. That is, x is a cluster point of A. ∎

Theorem 3.2.19 (***Sequential Criterion for Closed Sets***) *A set A is closed iff $\forall$ convergent sequences $\{a_n\}$ of points of A, $\lim_{n\to\infty} a_n \in A$.*

Proof. Part 1 ($\Rightarrow$): Suppose A is a closed set. Suppose $\{a_n\}$ is a convergent sequence of points of A. Let $L = \lim_{n\to\infty} a_n$. For contradiction, suppose $L \notin A$. Then, $\forall n$, $a_n \neq L$. By Theorem 3.2.18, L is a cluster point of A, so by Theorem 3.2.8, $L \in A$. Contradiction. Therefore, $L \in A$.

Part 2 ($\Leftarrow$): Suppose that $\forall$ convergent sequence $\{a_n\}$ of points of A, $\lim_{n\to\infty} a_n \in A$. Let x be a cluster point of A. Then, by Theorem 3.2.18, $\exists$ sequence $\{a_n\}$ of points of A other than x, such that $a_n \to x$. Then, by our hypothesis, $x \in A$. That is, A contains all its cluster points. Thus, by Theorem 3.2.8, A is closed. ∎

DENSE SUBSETS OF A SET

Definition 3.2.20 Suppose $A, B \subseteq \mathbb{R}$. We say that A **is dense in** B if $B \subseteq \overline{A}$. Equivalently, A is dense in $B \Leftrightarrow$ every member of B is either a member of A or a cluster point of A. (Cf. Definition 1.5.6, Exercises 3.1.19–20, and Exercises 3.2.27–29.)

Theorem 3.2.21 (***Sequential Criterion for Denseness***) *A set A is dense in a set B iff $\forall b \in B$, $\exists$ sequence $\{a_n\}$ of points of A such that $a_n \to b$.*

Proof. Exercise 30. ∎

Compare this result with Theorem 2.3.6.

EXERCISE SET 3.2

1. Prove Corollary 3.2.2.

2. Prove Note 1, following Corollary 3.2.2.

3. Tell whether the following sets are open, closed, both, or neither:

   (a) $(3,5) \cup \{6\}$
   (b) $(-\infty,0) \cup (0,1)$
   (c) $\{1,2,3,4,5,6,7,8,9\}$
   (d) $(-\infty,0) \cup [0,1)$
   (e) $\mathbb{Z}$
   (f) $(-\infty,0) \cup (0,1]$
   (g) $(-\infty,0) \cup [0,1]$
   (h) $\mathbb{R} - \{1,2,3\}$
   (i) $\{\frac{1}{n} : n \in \mathbb{N}\}$
   (j) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$
   (k) $\mathbb{Q}$
   (l) $\mathbb{Q} \cap (0,1)$

4. Prove Theorem 3.2.4 (a) and (c). [Hint: Use the open set theorem and, for (c), use de Morgan's law.]

5. Find all the cluster points of each set given in Exercise 3.2.3.

6. Give an example of a collection of bounded closed intervals whose union is unbounded and not closed.

7. Suppose $A \neq \varnothing$ and A is bounded above. Is $\sup A$ necessarily a cluster point of A? Prove that if $\sup A \notin A$, then it is a cluster point of A. State and prove analogous results for $\inf A$.

8. Prove that every nonempty closed set that is bounded above contains a maximum element, and every nonempty closed set that is bounded below contains a minimum element.

9. Prove Lemma 3.2.9.

10. Prove Corollary 3.2.10.

11. Prove Corollary 3.2.12.

12. Prove that

    (a) If A is open and B is closed, then $A - B$ is open;

    (b) If A is closed and B is open, then $A - B$ is closed.

13. Finish proving Theorem 3.2.15.

14. Prove that $\overline{A} = A \cup A^b$. [Show how this follows from Theorem 3.2.17.]

15. Suppose A is a nonempty set of real numbers. Prove that

    (a) If A is bounded above, then $\sup A \in \overline{A}$.

    (b) If A is bounded below, then $\inf A \in \overline{A}$.

16. Find the closure of each of the following sets:

(a) $(3,5) \cup \{6\}$	(b) $(-\infty,0) \cup (0,1)$
(c) $\{1,2,3,4,5,6,7,8,9\}$	(d) $(-\infty,0) \cup [0,1)$
(e) $\mathbb{Z}$	(f) $(-\infty,0) \cup (0,1]$
(g) $(-\infty,0) \cup [0,1]$	(h) $\mathbb{R} - \{1,2,3\}$
(i) $\{\frac{1}{n} : n \in \mathbb{N}\}$	(j) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$
(k) $\mathbb{Q}$	(l) $\mathbb{Q} \cap (0,1)$

17. Prove that $x \in \overline{A}$ iff every neighborhood of x contains a point of A.

18. Prove that $A^b = \overline{A} \cap \overline{A^c}$, which yields an alternate proof that A^b is closed.

19. Prove that if $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

20. Suppose $A, B \subseteq \mathbb{R}$. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Show by example that $\overline{A \cap B}$ and $\overline{A} \cap \overline{B}$ are not necessarily equal.

21. Prove that $\forall A \subseteq \mathbb{R}$, the set A' of all its cluster points is closed.

22. Prove that the set of cluster points of a bounded sequence (see Definition 2.6.14) is a closed set.

23. Prove the following identities:

(a) $A^{\circ\circ} = A^\circ$	(b) $A^{cl\ cl} = A^{cl}$	(c) $A^{cl} = \mathbb{R} - A^{ext}$
(d) $A^{c\ cl\ c} = A^\circ$	(e) $A^b = A^{cl} - A^\circ$	(f) $A^{cl\ \circ\ cl\ \circ} = A^{cl\ \circ}$

24. Prove that $x \in A$ is an isolated point of A iff it is not a cluster point of A.

25. Prove that $x \in \overline{A}$ iff $\exists$ sequence of points of A converging to x.

26. Prove the **sequential criterion for open sets**: A set A is open iff $\nexists$ sequence in A^c converging to a point in A. Use this criterion to prove that the interval $(0,1)$ is open, but the interval $[0,1)$ is not open.

27. Prove that A is dense in $\mathbb{R}$ (in the sense of Definition 1.5.6) iff every real number is a cluster point of A.

28. Prove that A is dense in $\mathbb{R}$ iff $\overline{A} = \mathbb{R}$. [See Exercise 3.1.20.]

29. Given $A, B \subseteq \mathbb{R}$, prove that A is dense in B if and only if every neighborhood of every point of B contains a point of A. (Equivalently, every open set containing a point of B contains a point of A.)

30. Prove Theorem 3.2.21.

31. **(Project)** In this exercise we shall denote the closure of a set A by A^{cl} rather than by $\overline{A}$. Start with a set A. Then perform the operations of complement and closure, alternating in succession, forming the sequences A, A^c, $A^{c\,cl}$, $A^{c\,cl\,c}$, $A^{c\,cl\,c\,cl}$, $\cdots$, and A, A^{cl}, $A^{cl\,c}$, $A^{cl\,c\,cl}$, $A^{cl\,c\,cl\,c}$, $\cdots$.
   (a) Prove that this process can never yield more than 14 different sets.
   (b) Find a set A for which this process yields exactly 14 different sets.

3.3 *Compact Sets

> Instructors may omit this entire section and replace it by the following definition:

Definition: A set of real numbers is **compact** if it is closed and bounded.

We shall not need to use the concept of compactness until Section 5.3. When we do use the concept, the above definition will suffice. Instructors who choose to cover the remainder of this section will do so out of a desire to place the concept of compactness upon a solid topological foundation. **Please disregard the above definition if you are studying this section.**

TOPOLOGICAL TERMS

Definition 3.3.1 A **topological term or concept**[3] is a term or concept definable using only the terminology of sets and open sets. Some topological terms are:

Term:	Attribute of:
open	a set
closed	a set
interior	a point, relative to a set
exterior	a point, relative to a set
boundary	a point, relative to a set
isolated	a point, relative to a set
cluster	a point, relative to a set
dense in	a set, relative to $\mathbb{R}$ or another set (see Exercise 3.2.28)
closure of	a set
$\lim\limits_{x \longrightarrow \infty} x_n = L$	a sequence, and a real number (see Exercise 3.1.21)

3. This definition would not satisfy a topologist, but it suffices for our purposes.

*An asterisk before a theorem, proof, or other item in this chapter indicates that the item is challenging or can be omitted, especially in a one-semester course.

Notice that the following are *not* topological terms: "interval," "bounded," "sup A," "inf A," "Archimedean," and "complete."

"Compact" is another topological term, although you cannot tell that from the definition we gave above. It turns out that the notion of a "compact set" is a very powerful tool in analysis, but its topological definition is rather complicated. After developing the abstract machinery necessary to give a topological definition, we shall give that definition and develop some consequences of it.

Compact sets share some essential features with finite sets, which make them especially well-suited for expressing some of the results of real analysis. In fact, we shall see that all finite sets are compact.

Definition 3.3.2 Suppose A is a set of real numbers.

(a) A family $\mathbf{U}$ of open sets of real numbers is said to be an **open cover** of A if every element of A belongs to at least one set in $\mathbf{U}$; that is,

$$\forall a \in A, \exists\, U \in \mathbf{U} \ni a \in U;$$
$$\text{i.e., } A \subseteq \cup\, \mathbf{U}.$$

(b) If $\mathbf{U}$ is an open cover of A and there is a finite subcollection of $\mathbf{U}$ that covers A $\left[\text{i.e., } \exists U_1, U_2, \cdots, U_n \in \mathbf{U} \ni A \subseteq \bigcup_{k=1}^{n} U_k\right]$, then we say that $\mathbf{U}$ has a **finite subcover** of A.

Example 3.3.3 Let $A = [0, 1]$ and $\mathbf{U} = \left\{N_{\frac{1}{n}}(r) : r \in \mathbb{Q},\, n \in \mathbb{N}\right\}$.

(a) Then $\mathbf{U}$ consists of open intervals $\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ that collectively "cover" $[0, 1]$.

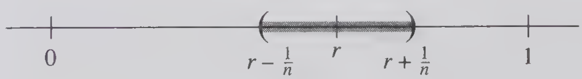


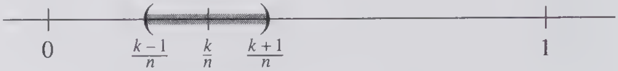
Figure 3.9

Thus, $\mathbf{U}$ is an open cover of A. Notice that $\mathbf{U}$ contains infinitely many sets, so $\mathbf{U}$ is not a finite open cover of A.

(b) We shall now produce a finite subcollection of $\mathbf{U}$ that covers A. (Hence, $\mathbf{U}$ has a finite subcover of A.)

Let n be a fixed natural number. Then the sets $N_{\frac{1}{n}}\left(\frac{k}{n}\right) = \left(\frac{k-1}{n}, \frac{k+1}{n}\right)$ for $k = 0, 1, 2, \cdots, n$ form a collection of $n+1$ sets in $\mathbf{U}$. These sets collectively cover A, since they overlap as follows:

$N_{\frac{1}{n}}\left(\frac{k}{n}\right)$



$N_{\frac{1}{n}}\left(\frac{k+1}{n}\right)$

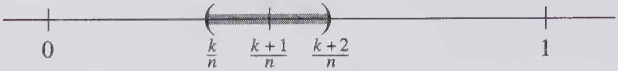


Figure 3.10

Thus, $\mathbf{U}$ has a finite subcover of A. □

Definition 3.3.4 A set A of real numbers is **compact** if every open cover of A has a finite subcover of A.

The notion of compactness is not as complicated in the setting of the real number system as this definition seems to imply. Examples, and the Heine-Borel Theorem below, will make the concept more concrete.

Theorem 3.3.5 *Every finite set is compact.*

Proof. Exercise 1. ■

Theorem 3.3.6 *Every compact set is bounded.*

Proof. Exercise 2. ■

Example 3.3.7 Is the open interval $A = (0, 1)$ a compact set?

Discussion: The collection $\mathbf{U} = \left\{N_{\frac{1}{n}}(r) : r \in \mathbb{Q},\ n \in \mathbb{N}\right\}$ is an open cover of $(0, 1)$. We have seen in Example 3.3.3 that $\mathbf{U}$ has a finite subcover. Does this mean that $(0, 1)$ is compact? No!

To show that $(0, 1)$ is compact we would have to show that *every* open cover of $(0, 1)$ has a finite subcover of $(0, 1)$. In fact, we shall now show that $(0, 1)$ is *not* compact.

Solution: Consider the collection $\mathbf{U} = \left\{\left(\frac{1}{n}, 1 - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$. This is a collection of open sets, and $\cup\, \mathbf{U} = (0, 1)$.

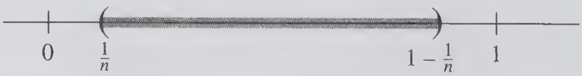


Figure 3.11

Thus, $\mathbf{U}$ is an open cover of $(0,1)$. Clearly, no finite subcollection of $\mathbf{U}$ will cover $(0,1)$. Thus, $\mathbf{U}$ has no finite subcover of $(0,1)$. Therefore, $(0,1)$ is not compact. □

In fact, it would have been obvious that an open interval is not compact if we had known the following theorem.

Theorem 3.3.8 *Every compact set is closed.*

Proof. (We shall prove the contrapositive.) Suppose A is a set of real numbers that is *not* closed. Then, by Theorem 3.2.8, there is at least one cluster point x of A that does not belong to A. Consider the collection $\mathbf{U} = \{J_n : n \in \mathbb{N}\}$, where $\forall n \in \mathbb{N}$,

$$J_n = \left(-\infty, x - \tfrac{1}{n}\right) \cup \left(x + \tfrac{1}{n}, +\infty\right).$$

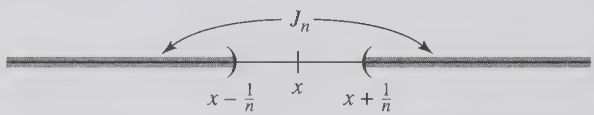


Figure 3.12

Then $\cup\, \mathbf{U} = \mathbb{R} - \{x\}$, so $\mathbf{U}$ covers A since $x \notin A$. Thus, $\mathbf{U}$ is an open cover of A.

To see that no finite subcollection of $\mathbf{U}$ covers A, consider any finite subcollection of $\mathbf{U}$, say $\mathbf{V} = \{J_{n_1}, J_{n_2}, \cdots, J_{n_k}\}$. Then

$$\cup\, \mathbf{V} = \bigcup_{i=1}^{k} J_{n_i} = \left(-\infty, x - \tfrac{1}{M}\right) \cup \left(x + \tfrac{1}{M}, +\infty\right) = J_M,$$

where $M = \max\{n_1, n_2, \cdots, n_k\}$. Thus the subcollection $\mathbf{V}$ does not contain any point of the interval

$$\left[x - \tfrac{1}{M}, x + \tfrac{1}{M}\right].$$

However, since x is a cluster point of A, this interval must contain a point of A, call it a. Thus, $a \notin \cup\, \mathbf{V}$. That is, no finite subcollection $\mathbf{V}$ of $\mathbf{U}$ can cover A.

Therefore, A is not compact. ■

Corollary 3.3.9 *None of the following sets is compact (assuming $a < b$):*

(a) $\mathbb{R}$
(b) (a, b)
(c) $(a, b]$
(d) $[a, b)$
(e) $(-\infty, a)$
(f) $(-\infty, a]$
(g) $(a, +\infty)$
(h) $[a, +\infty)$
(i) $\{\frac{1}{n} : n \in \mathbb{N}\}$
(j) $\mathbb{N}$
(k) $\mathbb{Z}$
(l) $\mathbb{Q}$

Proof. Exercise 3. ■

Theorem 3.3.10 (***Heine-Borel***) *Every closed, bounded interval of real numbers is compact.*

Proof. Let $I = [a, b]$, where $a < b$. Let $\mathbf{U}$ denote any open cover of I. We want to show that $\mathbf{U}$ has a finite subcover of I. We define the set

$$S = \{x \in I : \text{some finite subcollection of } \mathbf{U} \text{ covers } [a, x]\}.$$

Then S is nonempty since $a \in S$. Moreover, S is bounded since $S \subseteq [a, b]$. By completeness of $\mathbb{R}$, $\exists\ u = \sup S$.

Claim #1: $u \in S$.

Proof: Since $a \in S$, $a \leq u$; and since b is an upper bound for S, $u \leq b$. Thus, $a \leq u \leq b$. Thus, u is a member of some set V in $\mathbf{U}$; i.e., $\exists\ V \in \mathbf{U} \ni u \in V$. But V is open, so $\exists \varepsilon > 0 \ni$

$$(u - \varepsilon, u + \varepsilon) \subseteq V.$$

By the ε-criterion for supremum (Theorem 1.6.6) $\exists x \in S \ni u - \varepsilon < x \leq u$.

Case 1 ($x = u$): Then $u \in S$ as desired.

Case 2 ($x < u$): Then $(x, u] \subseteq (u - \varepsilon, u + \varepsilon) \subseteq V$. Since $x \in S$, the definition of S says that some finite subcollection of $\mathbf{U}$ covers $[a, x]$. Adding V to this subcollection gives us a finite subcollection of $\mathbf{U}$ that covers $[a, u]$. Thus, $u \in S$. (See Figure 3.13.)

In either case, $u \in S$.

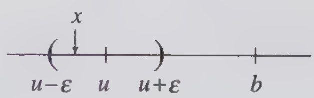


Figure 3.13

Claim #2: $u = b$.

Proof: For contradiction, suppose $u \neq b$. Then $u < b$, since $a \leq u \leq b$. Since $u \in S$, some finite subcollection $\mathbf{V}$ of $\mathbf{U}$ covers $[a, u]$. Thus, some open set $V \in \mathbf{V}$ contains u. Since V is open, $\exists\, \varepsilon > 0 \ni$

$$(u - \varepsilon, u + \varepsilon) \subseteq V.$$

Thus, the finite subcollection $\mathbf{V}$ of $\mathbf{U}$ covers $[a, u+\varepsilon)$. Let c be any real number satisfying $u < c < \min\{u + \varepsilon, b\}$. Then

$$[u, c] \subseteq (u - \varepsilon, u + \varepsilon) \subseteq V,$$

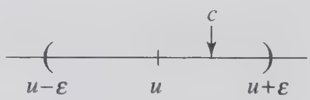


Figure 3.14

so the finite subcollection $\mathbf{V}$ of $\mathbf{U}$ covers $[a, c]$. But then

$$c \in S, \text{ and } c > u = \sup S.$$

This is a contradiction. Therefore, $u = b$.

By Claim #1 and Claim #2, $b \in S$. That is, $\mathbf{U}$ has a finite subcover of $[a, b]$. Therefore, $[a, b]$ is compact. ■

Theorem 3.3.11 *A closed subset of a compact set is compact.*

Proof. Suppose A is a closed subset of a compact set C. Let $\mathbf{U}$ be any open cover of A. Since A is closed, A^c is open. Then $\mathbf{U} \cup \{A^c\}$ is an open cover of $\mathbb{R}$, hence of C. But C is compact. Hence $\mathbf{U} \cup \{A^c\}$ has a finite subcover of C. But $A \subseteq C$. Hence $\mathbf{U}$ has a finite subcover of A. That is, A is compact. ■

Corollary 3.3.12 ***A set of real numbers is compact if and only if it is closed and bounded.*** Main Result

Proof. A compact set is closed and bounded, by Theorems 3.3.6 and 3.3.8. To prove the converse, suppose A is closed and bounded. Then A is a closed subset of a closed interval. By the Heine-Borel Theorem (3.3.10) and Theorem 3.3.11, it follows that A is compact. ■

In the remainder of this section, we shall show that compactness is closely related to several important concepts previously studied. We begin by showing a connection with the Bolzano-Weierstrass Theorems.

Theorem 3.3.13 (***Sequential Criterion for Compactness***) *A set A of real numbers is compact if and only if every sequence of points of A has a subsequence that converges to a point of A.*

Proof. Let A be a set of real numbers.

Part 1 ($\Rightarrow$): Suppose A is compact. Let $\{a_n\}$ be a sequence of points of A. Now A is a bounded set, so $\{a_n\}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem for sequences, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Let $L = \lim\limits_{k\to\infty} a_{n_k}$. Now A is closed, since it is compact. So, by the sequential criterion for closed sets (3.2.19), $L \in A$. Thus, $\{a_n\}$ has a subsequence converging to a point of A.

Part 2 ($\Leftarrow$): Suppose every sequence of points of A has a subsequence converging to a point of A. We want to prove that A is compact; i.e., closed and bounded.

Suppose A is not bounded. Then $\forall n \in \mathbb{N}$, $\exists\, a_n \in A \ni |a_n| > n$. By our hypothesis, the sequence $\{a_n\}$ has a convergent subsequence, $\{a_{n_k}\}$. Now, $\forall k$, $|a_{n_k}| > n_k \geq k$. This means $\{a_{n_k}\}$ is unbounded. But every convergent sequence is bounded. Contradiction. Therefore, A is bounded.

We shall prove that A is closed using the sequential criterion for closed sets. Suppose $\{b_n\}$ is a sequence of points of A that converges; say, $b_n \to M$. Then $\{b_n\}$ is bounded. So, by our hypothesis, $\{b_n\}$ must have a convergent subsequence $\{b_{n_k}\}$ whose limit is in A. By Theorem 2.6.8, this limit must be M. Therefore, $M \in A$. So, by the sequential criterion for closed sets, A is closed.

Therefore, A is compact. ■

Theorem 3.3.14 *A set A of real numbers is compact if and only if every infinite subset of A has a cluster point in A.*

Proof. Part 1 ($\Rightarrow$): Suppose A is compact. Let S be an infinite subset of A. Then S is a bounded, infinite set. By the Bolzano-Weierstrass Theorem, S has a cluster point, say x. From the definition of cluster point, x is also a cluster point of A. But A is closed, so $x \in A$ (Theorem 3.2.8). Thus, every infinite subset of A has a cluster point in A.

Part 2 ($\Leftarrow$): Suppose every infinite subset of A has a cluster point in A. We are going to apply Theorem 3.3.13 to show that A is compact. Suppose $\{x_n\}$ is a sequence of points of A.

Case 1: The set $\{x_n : n \in \mathbb{N}\}$ is finite. Then, $\exists\, c \in A \ni x_n = c$ for infinitely many n. So, $\{x_n\}$ has a constant subsequence. This subsequence converges to c, a point of A.

Case 2: The set $\{x_n : n \in \mathbb{N}\}$ is infinite. By our hypothesis, this set must have a cluster point, $a \in A$. Then, by Theorem 3.2.11, every neighborhood of a contains x_n for infinitely many n. By Theorem 2.6.7, this means that $\{x_n\}$ has a subsequence converging to a, and we already know that $a \in A$.

In either case, $\{x_n\}$ has a subsequence converging to a point of A. Therefore, by Theorem 3.3.13, A is compact. ■

Compactness is also closely related to Cantor's Nested Intervals Theorem, as we shall see from the following results.

Definition 3.3.15 A collection $\mathcal{C}$ of sets is said to have the **finite intersection property** if every finite subcollection of sets of $\mathcal{C}$ has nonempty intersection.

Example 3.3.16 A nested sequence of nonempty intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$, such as occurs in Cantor's Nested Intervals Theorem, has the finite intersection property. In fact, $\bigcap_{i=1}^{n} I_{k_i} = I_m$, where $m = \max\{k_1, k_2, \cdots, k_n\}$.

Theorem 3.3.17 *If $\mathcal{C}$ is a collection of compact sets with the finite intersection property, then $\cap\, \mathcal{C}$ is nonempty.*

Proof. Suppose $\mathcal{C}$ is a collection of compact sets with the finite intersection property. For contradiction, suppose $\cap\, \mathcal{C} = \varnothing$. Consider the family

$$\mathbf{U} = \{C^c : C \in \mathcal{C}\}.$$

Since each set C in $\mathcal{C}$ is compact, it is closed. Hence, $\mathbf{U}$ is a family of open sets.

Consider a fixed set $K \in \mathcal{C}$. Since $\cap\, \mathcal{C} = \varnothing$, no point of K belongs to every C in $\mathcal{C}$. Thus, every point of K belongs to one of the sets C^c in $\mathbf{U}$. Thus, $\mathbf{U}$ is an open cover of K. But K is compact. Hence, $\mathbf{U}$ has a finite subcover C_1^c, C_2^c, $\cdots$, C_n^c of K; i.e.,

$$K \subseteq C_1^c \cup C_2^c \cup \cdots \cup C_n^c.$$

By de Morgan's law, this says

$$K \subseteq (C_1 \cap C_2 \cap \cdots \cap C_n)^c.$$

That is, $K \cap (C_1 \cap C_2 \cap \cdots \cap C_n) = \varnothing$. But this intersection must be nonempty, since $\mathcal{C}$ has the finite intersection property. Contradiction!

Therefore, $\cap\, \mathcal{C} \neq \varnothing$. ■

Definition 3.3.18 A collection $\mathcal{C}$ of sets is said to have the **finite intersection property relative to a set** A if every finite subcollection of sets of $\mathcal{C}$ has nonempty intersection with A.

Theorem 3.3.19 *A set A of real numbers is compact if and only if, for every collection $\mathcal{C}$ of closed sets with the finite intersection property relative to A, $\cap\, \mathcal{C}$ contains at least one point of A.*

Proof. Part 1 ($\Rightarrow$): Suppose A is a compact set of real numbers, and let $\mathcal{C}$ be a collection of closed sets with the finite intersection property relative to A. For contradiction, suppose $\cap\, \mathcal{C}$ contains no points of A. Then

$$\cap\, \mathcal{C} \subseteq A^c.$$

Therefore, $A \subseteq (\cap\, \mathcal{C})^c$, and by de Morgan's law,

$$A \subseteq \cup \{C^c : C \in \mathcal{C}\}.$$

But then $\{C^c : C \in \mathcal{C}\}$ is an open cover of A. Since A is compact, A can be covered by finitely many of these sets:

$$A \subseteq C_1^c \cup C_2^c \cup \cdots \cup C_n^c.$$

Then, $(C_1^c \cup C_2^c \cup \cdots \cup C_n^c)^c \subseteq A^c$. By de Morgan's law, this says

$$C_1 \cap C_2 \cap \cdots \cap C_n \subseteq A^c,$$

which implies that $A \cap (C_1 \cap C_2 \cap \cdots \cap C_n) = \varnothing$. But this intersection must be nonempty, since $\mathcal{C}$ has the finite intersection property relative to A. Contradiction!

Therefore, $\cap\, \mathcal{C}$ contains at least one point of A.

Part 2 ($\Leftarrow$): Suppose that for every collection $\mathcal{C}$ of closed sets with the finite intersection property relative to A, $\cap\, \mathcal{C}$ contains a point of A.

For contradiction, suppose that A is not compact. Then there is an open cover $\mathbf{U}$ of A that has no finite subcover of A. Define

$$\mathcal{C} = \{U^c : U \in \mathbf{U}\}.$$

Then $\mathcal{C}$ is a collection of closed sets. Suppose $\{U_1^c, U_2^c, \cdots, U_n^c\}$ is a finite subcollection of $\mathcal{C}$. By de Morgan's law,

$$\bigcap_{i=1}^{n} U_i^c = \left(\bigcup_{i=1}^{n} U_i\right)^c \tag{1}$$

Since $\mathbf{U}$ has no finite subcover of A, $\bigcup\limits_{i=1}^{n} U_i$ cannot contain all of A. That is, some point of A is in $\left(\bigcup\limits_{i=1}^{n} U_i\right)^c$. In view of (1), this says that $\bigcap\limits_{i=1}^{n} U_i^c$ has a nonempty intersection with A. Thus, $\mathcal{C}$ is a collection of closed sets with the finite intersection property relative to A.

By our hypothesis, $\cap\, \mathcal{C}$ contains a point of A. But,

$$\cap\, \mathcal{C} = \cap\, \{U^c : U \in \mathbf{U}\} = (\cup\, \mathbf{U})^c.$$

Thus, $(\cup\, \mathbf{U})^c$ contains a point of A. But then $\cup\, \mathbf{U}$ does not contain all of A; i.e., $\mathbf{U}$ is not a cover of A. Contradiction!

Therefore, A is compact. ■

Applications of compactness will be found at various places later in the course. See especially Sections 5.3 and 5.4.

EXERCISE SET 3.3

1. Prove Theorem 3.3.5.

2. Prove Theorem 3.3.6. [Hint: for an unbounded set, try the cover $\mathbf{U} = \{(n-1, n+1) : n \in \mathbb{N}\}$.]

3. For each of the sets given in Corollary 3.3.9,

   (a) give an open cover that has no finite subcover;

   (b) without using open covers, give a simple reason and a theorem showing that the set is not compact.

4. Prove that a set of real numbers is bounded iff its closure is compact.

5. Prove that the union of finitely many compact sets is compact.

6. Prove that the boundary of a bounded set is compact.

7. Suppose that $\{x_n\}$ is a convergent sequence; say $\lim\limits_{n \to \infty} x_n = L$. Prove that $\{x_n : n \in \mathbb{N}\} \cup \{L\}$ is compact.

8. Prove that the intersection of any collection of compact sets is compact. Is the union of any collection of compact sets necessarily compact? Justify your answer.

9. Prove that every nonempty compact set contains both its supremum and its infimum. [Thus, it contains both a maximum element and a minimum element.]

10. Show that the first part of Cantor's Nested Intervals Theorem (See Alternate 2.5.17) is an easy corollary of Theorem 3.3.17.

11. If A is a bounded set, we define its **diameter** to be the real number $d(A) = \sup\{|x - y| : x, y \in A\}$. Prove that if A is compact, then $\exists\, x_0, y_0 \in A \ni d(A) = |x_0 - y_0|$.

12. Let $f : \mathcal{D}(f) \to \mathbb{R}$ be a function with domain $\mathcal{D}(f)$, and let $A \subseteq \mathcal{D}(f)$. We say that f is **bounded** on A if $\exists B > 0 \ni \forall x \in A$, $|f(x)| \le B$. We say that f is **locally bounded** at a point x if $\exists \delta > 0 \ni f$ is bounded on $N_\delta(x)$.

    (a) Prove that if f is locally bounded at every point of a *compact* set A, then f is bounded on A.

    (b) Find a function f that is locally bounded at every point of $(0, 1)$ but not bounded on $(0, 1)$.

3.4 *The Cantor Set

The Cantor set is a most remarkable set of real numbers. It is a subset of $[0, 1]$ obtained by removing, in successive steps, a sequence of subsets of $[0, 1]$. In some sense, it is easier to visualize the complement of the Cantor set than it is to visualize the set itself; the Cantor set is what is "left over" after the removal process.

Definition 3.4.1 The Cantor Set:

Let $C_0 = [0, 1]$ and $C_1 =$ the set remaining after removing $\left(\frac{1}{3}, \frac{2}{3}\right)$, the "open middle third" of C_0. Thus,

$$C_1 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right].$$

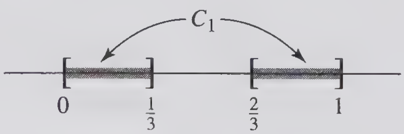


Figure 3.15

Similarly, let $C_2 =$ the set remaining after removing the "open middle thirds" $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, each of length $\frac{1}{9}$, from the intervals comprising C_1.

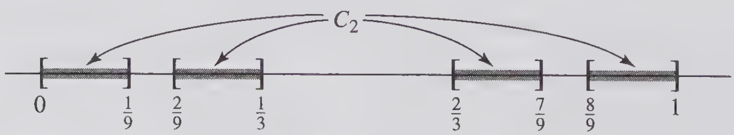


Figure 3.16

Thus, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

Continuing inductively, if C_n is the union of 2^n disjoint closed intervals of length $\frac{1}{3^n}$, we define C_{n+1} to be the result of removing from C_n the open middle thirds of these intervals, each of length $\frac{1}{3^{n+1}}$. For example,

$C_3 = \left[0, \frac{1}{27}\right] \cup \left[\frac{2}{27}, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{7}{27}\right] \cup \left[\frac{8}{27}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{19}{27}\right] \cup \left[\frac{20}{27}, \frac{7}{9}\right] \cup \left[\frac{24}{27}, \frac{25}{27}\right] \cup \left[\frac{26}{27}, 1\right]$.

At each stage, to get C_{n+1} we remove the open middle thirds of the 2^n disjoint closed intervals comprising C_n, and C_{n+1} is the union of the resulting 2^{n+1} disjoint closed intervals, each of length $\frac{1}{3^{n+1}}$. Notice that

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq C_{n+1} \supseteq \cdots.$$

We define the Cantor set to be

$$\mathbf{C} = \bigcap_{n=1}^{\infty} C_n.$$

That is, $\mathbf{C}$ is what is left over in $[0,1]$ after removing successively all the "open middle third" sets, as described above. □

Theorem 3.4.2 *The Cantor set is compact.*

Proof. Exercise 1. ■

Theorem 3.4.3 *The Cantor set contains no nonempty open interval.*

Proof. Exercise 2. ■

THE CANTOR SET AND TERNARY DECIMALS

Let us consider what numbers belong to the Cantor set. The Cantor set is clearly nonempty; for example, $\frac{1}{3} \in \mathbf{C}$. In fact,

Lemma 3.4.4 *If a is an endpoint of one of the disjoint closed intervals comprising some C_n, then $a \in \mathbf{C}$.*

Proof. Exercise 4. ■

To characterize the numbers that belong to the Cantor set $\mathbf{C}$, we resort to **ternary** (base-three) decimal-like representation of real numbers. While the word "decimal" signals base-ten, any natural number $b > 1$ can be used to represent real numbers in decimal-like form. For lack of a better term, we shall refer to these expressions as **base-*b* decimals**.

Definition 3.4.5 (Base-*b* "Decimals") Let b be any natural number greater than 1. Then a "base-b decimal" is any expression of the form

$$K.d_1 d_2 \cdots d_n d_{n+1} \cdots \quad (\text{base } b) \quad \text{or}$$
$$-K.d_1 d_2 \cdots d_n d_{n+1} \cdots \quad (\text{base } b)$$

where K is a natural number and $\forall i \in \mathbb{N}$, $d_i \in \{0, 1, 2, \cdots, b-1\}$.

Each base-b decimal represents a unique real number. As in Section 2.5, $\forall n \in \mathbb{N}$, we define

$$D_n = K.d_1d_2\cdots d_n \text{ (base } b) = K + \sum_{i=1}^{n} \frac{d_i}{b^i}$$

and observe that $\{D_n\}$ is a bounded monotone sequence. By the monotone convergence theorem, this sequence has a limit, which we shall call D. It is in this sense that we say

$$K.d_1d_2\cdots d_nd_{n+1}\cdots \text{ (base } b) = D. \qquad \square$$

We can modify the proofs of Theorems 2.5.5 and 2.5.7, and Example 2.5.6 to yield:

Theorem 3.4.6 *Given any natural number $b > 1$,*

(a) *Every base-b decimal represents a unique real number.*

(b) *Every real number can be represented by a base-b decimal.*

(c) *Some real numbers can be represented by two base-b decimals, one ending in all 0's, and another ending in all $(b-1)$'s.*

Proof. (Omitted) ■

For example, in base-3,

$2.10022222\cdots$ (base 3) $= 2.1010000\cdots$ (base 3) $= 2 + \frac{1}{3} + \frac{1}{27} = 2\frac{10}{27}$.

Definition 3.4.7 A **terminating decimal** in base-b is one ending in all 0's. (All others are called **nonterminating**.)

Theorem 3.4.8 *Let b be a natural number greater than* 1. *There is a* 1-1 *correspondence between $\mathbb{R}$ and all nonterminating base-b decimals. (That is, every real number has a unique nonterminating decimal in base-b.)*

Proof. (Omitted) ■

Examples 3.4.9 In base-3, only $0, 1,$ and 2 are used as "digits." We have

(a) 0.1 (base 3) $= \frac{1}{3}$;

(b) 0.12 (base 3) $= \frac{1}{3} + \frac{2}{9} = \frac{5}{9}$;

(c) 0.021 (base 3) $= \frac{0}{3} + \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$;

(d) $0.1111111\cdots$ (base 3) $= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{3-1} = \frac{1}{2};$$

$$\left(\begin{array}{c}\text{Recall the geometric series from calculus:} \\ \displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if } |r| < 1\end{array}\right)$$

(e) $0.020202020\cdots$ (base 3) $= \frac{2}{9} + \frac{2}{9^2} + \frac{2}{9^3} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{2}{9}\left(\frac{1}{9}\right)^n = \frac{\frac{2}{9}}{1-\frac{1}{9}} = \frac{2}{9-1} = \frac{1}{4}. \qquad \square$$

Theorem 3.4.10 *The Cantor set consists of all those real numbers in* $[0,1]$ *that can be represented by a base-3 decimal consisting of only* 0*'s and* 2*'s.*

Proof. Exercise 5. ■

Theorem 3.4.10 allows us to show that the Cantor set contains many real numbers besides endpoints of intervals removed in the construction.

Example 3.4.11 $\frac{1}{4}$ is in the Cantor set, but is not an endpoint of a "removed interval."

Proof. From Example 3.4.9, $\frac{1}{4} = 0.202020\cdots$ (base-3), so by Theorem 3.4.10, $\frac{1}{4} \in \mathbf{C}$. However, $\frac{1}{4}$ is not an endpoint of a "removed interval," since any such endpoint is a fraction whose denominator is a power of 3. $\square$

Theorem 3.4.10 allows us to prove other interesting facts about the Cantor set.

PROPERTIES OF THE CANTOR SET

We have already seen that the Cantor set is compact.

Theorem 3.4.12 *The Cantor set has the same number of elements as* $[0,1]$*; hence, it is uncountable.*

Proof. Define the function $f : \mathbf{C} \to [0,1]$ as follows. Let $x \in \mathbf{C}$. By Theorem 3.4.10, x has a ternary decimal representation consisting of all 0's and 2's. Replace all 2's by 1's. The result can be regarded as a base-2 decimal

of some number in $[0, 1]$ since it consists of all 0's and 1's. Moreover, there is only one such number. Let $f(x) =$ this number. Thus, $\forall x \in \mathbf{C}$,

$$f(x) = \left\{ \begin{array}{l} \text{the real number in } [0,1] \text{ whose base-2} \\ \text{decimal representation results when all} \\ \text{2's in the nonterminating ternary expan-} \\ \text{sion of } x \text{ are replaced by 1's.} \end{array} \right\}$$

It is easy to see that $f : \mathbf{C} \to [0, 1]$ is a 1-1 correspondence. ■

Remark: While Example 3.4.11 shows that the Cantor set contains points other than endpoints of "removed intervals," Theorem 3.4.12 allows us to show the far more remarkable fact that the Cantor set contains *uncountably many* points that are not "endpoints," whereas there are only countably many such endpoints. (See Exercise 6.)

Definition 3.4.13 A set A of real numbers is **perfect** if $A' = A$; that is, A consists of all its cluster points.

Examples 3.4.14 The following sets are perfect:

(a) $\varnothing$ and $\mathbb{R}$;
(b) $(-\infty, a]$ and $[a, +\infty)$, for any real number a;
(c) $[a, b]$, for any real numbers a, b with $a < b$.

Proof. Exercise 8. □

Theorem 3.4.15 *The Cantor set is a perfect set.*

Proof. Let $\mathbf{C}$ denote the Cantor set.

(a) $\mathbf{C}' \subseteq \mathbf{C}$, since $\mathbf{C}$ is closed.

(b) To prove that $\mathbf{C} \subseteq \mathbf{C}'$, choose any $x \in \mathbf{C}$. Let $\varepsilon >$. Then $\exists n \in \mathbb{N} \ni \dfrac{1}{3^n} < \varepsilon$, and by definition of $\mathbf{C}$, $x \in C_n$. Then x belongs to exactly one of the 2^{n+1} disjoint closed intervals of length $\dfrac{1}{3^n}$ comprising C_n; call it $I_n = [a_n, b_n]$.

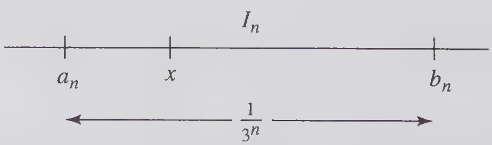


Figure 3.17

Since $\frac{1}{3^n} < \varepsilon$, either $a_n \in N_\varepsilon(x)$ or $b_n \in N_\varepsilon(x)$. Recall that both a_n and b_n are in the Cantor set. Hence, $\forall \varepsilon > 0$, $N_\varepsilon(x)$ contains a point of the Cantor set other than x. Therefore, x is a cluster point of $\mathbf{C}$; i.e., $x \in \mathbf{C}'$.

Therefore, $\mathbf{C} \subseteq \mathbf{C}'$. ■

Definition 3.4.16 A set A of real numbers is **nowhere dense** if $\overline{A}$ contains no nonempty open intervals.

Theorem 3.4.17 *The Cantor set is nowhere dense.*

Proof. Exercise 10. ■

The following theorem is an interesting application of compactness as characterized by Theorem 3.3.17.

Theorem 3.4.18 *Every nonempty perfect set is uncountable.*

Proof. Let A be a perfect set. Since A has cluster points it must be infinite, by Corollary 3.2.12. For contradiction, suppose A is countable, say $A = \{a_1, a_2, \cdots, a_n, \cdots\}$. We construct the following sequence $\{N_k\}$ of neighborhoods in A:

Let $N_1 = N_\varepsilon(a_1)$, for some $\varepsilon > 0$.

Since a_1 is a cluster point of A, N_1 contains a point a_1' of A other than a_1. Since a_1' must be a cluster point of A, it has a neighborhood N_2 such that $N_2 \subseteq N_1$, $a_1 \notin N_2$, and $N_2 \cap A \neq \varnothing$.

Since a_1' is a cluster point of A, N_2 contains a point a_2' of A other than a_2. Since a_2' must be a cluster point of A, it has a neighborhood N_3 such that $N_3 \subseteq N_2$, $a_2 \notin N_3$, and $N_3 \cap A \neq \varnothing$.

Continuing inductively, we obtain a sequence $\{N_k\}$ of neighborhoods (open intervals) such that

(1) $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_k \supseteq \cdots$;
(2) $\forall k \in \mathbb{N}$, $a_k \notin N_{k+1}$;
(3) Each N_k contains a point of A; i.e., $N_k \cap A \neq \varnothing$.

Now, $\forall k \in \mathbb{N}$, define $C_k = \overline{N_k} \cap A$. Then each C_k is compact. The collection $\{C_k : k \in \mathbb{N}\}$ is a collection of compact sets with the "finite intersection property"[4] since $\bigcap_{i=1}^{n} C_{k_i} = C_m$, where $m = \max\{k_1, k_2, \cdots, k_n\}$. So, by Theorem 3.3.17, $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$. But $\forall k \in \mathbb{N}$, $a_k \notin C_{k+1}$, so $a_k \notin \bigcap_{k=1}^{\infty} C_k$. Therefore, $\bigcap_{k=1}^{\infty} C_k = \varnothing$, since $\bigcap_{k=1}^{\infty} C_k \subseteq A$. Contradiction.

Therefore, A cannot be countable. ■

4. See Definition 3.3.15.

SETS OF MEASURE ZERO

Is the Cantor set a large or small subset of $[0,1]$? In one sense it is a large subset. It is uncountable; in Theorem 3.4.12 we showed that it has the same number of elements as $[0,1]$. In another sense, it is small in comparison with $[0,1]$. As we shall see below, it has total length (measure) zero, whereas $[0,1]$ has total length 1. These two views of the Cantor set, as simultaneously large in one sense and small in another, suggest that it is a peculiar set. Indeed, it is one of the most peculiar sets in all of mathematics. It is a source of many intriguing, non-intuitive examples and counterexamples in analysis.

Definition 3.4.19 A set A of real numbers has **measure zero** if $\forall \varepsilon > 0$, A can be covered by a countable collection of open intervals of total length less than ε. That is, A has measure zero iff $\forall \varepsilon > 0\ \exists$ collection $\{I_n : n \in \mathbb{N}\}$ of open intervals $I_n = (a_n, b_n)$ such that $\sum_{n=1}^{\infty} length(I_n) < \varepsilon$, where $length(I_n) = (b_n - a_n)$.

Theorem 3.4.20 *A countable set must have measure zero.*

Proof. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable set. Let $\varepsilon > 0$. Then $\forall n \in \mathbb{N}$, let

$$I_n = \left(x_n - \frac{\varepsilon}{2^{n+2}}, \quad x_n + \frac{\varepsilon}{2^{n+2}}\right).$$

Then $\{I_n : n \in \mathbb{N}\}$ is a countable collection of open intervals that covers A. Moreover, $length(I_n) = \left(x_n + \frac{\varepsilon}{2^{n+2}}\right) - \left(x_n - \frac{\varepsilon}{2^{n+2}}\right) = \frac{2\varepsilon}{2^{n+2}} = \frac{\varepsilon}{2^{n+1}}$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} length(I_n) &= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \frac{\varepsilon}{16} + \cdots \\
&= \frac{\varepsilon}{4}\left[1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots\right] = \frac{\varepsilon}{4} \cdot 2 \\
&\qquad \text{(sum of geometric series)} \\
&= \frac{\varepsilon}{2} < \varepsilon.
\end{aligned}$$

Thus, A has measure zero. ■

Theorem 3.4.21 *The Cantor set has measure zero.*

Proof. Recall the construction of the Cantor set:

$$\mathbf{C} = \bigcap_{n=1}^{\infty} C_n,$$

where each C_n is the union of 2^n disjoint closed intervals, all of length $\frac{1}{3^n}$.

Let $\varepsilon > 0$. Each of the 2^n closed intervals comprising C_n can be covered by an open interval of somewhat longer length, say $\frac{1}{3^n} + \frac{\varepsilon}{2^{n+1}}$.

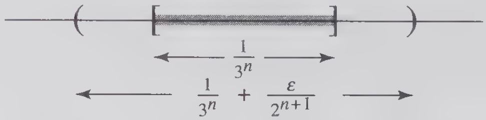


Figure 3.18

Thus, C_n can be covered by 2^n open intervals whose combined length is

$$2^n\left[\frac{1}{3^n} + \frac{\varepsilon}{2^{n+1}}\right] = \left(\frac{2}{3}\right)^n + \frac{\varepsilon}{2}.$$

Since $\left(\frac{2}{3}\right)^n \to 0$, we can choose $n \in \mathbb{N} \ni \left(\frac{2}{3}\right)^n < \frac{\varepsilon}{2}$. For *this* n, C_n can be covered by 2^n open intervals whose combined length is $\left(\frac{2}{3}\right)^n + \frac{\varepsilon}{2} < \varepsilon$. But $\mathbf{C} \subseteq C_n$. Therefore, $\mathbf{C}$ can be covered by a *finite* collection of open intervals of total length less than ε. Thus, $\mathbf{C}$ has measure zero. ∎

"FAT" CANTOR-LIKE SETS OF POSITIVE MEASURE

The concept of measure zero is a special case of the more general concept of "measurability" of sets, developed in the early twentieth century by Henri Lebesgue and others. To present this concept here would involve complications beyond the scope of this course. For a full treatment, the reader may consult any real analysis book[5] containing "measure theory." It suffices to say that it is possible to define a class $\mathcal{M}$ of "measurable" subsets of $\mathbb{R}$ and a "measure function" $\mu : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ such that

(μ1) $\forall A \in \mathcal{M}$, $\mu(A) \geq 0$.

(μ2) $\varnothing$ and $\mathbb{R}$ are measurable; $\mu(\varnothing) = 0$ and $\mu(\mathbb{R}) = +\infty$.

(μ3) If A and B are measurable, then $A \cup B$, $A \cap B$, and $A - B$ are measurable.

(μ4) The union of a countable collection $\{A_n : n \in \mathbb{N}\}$ of measurable sets is measurable. If the sets are pairwise disjoint then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$. (If the collection is finite, then "∞" in this formula is replaced by some "n".)

5. An excellent source is Royden [116] listed in the Bibliography.

(μ5) If A and B are measurable, then

$$\mu(A - B) = \mu(A) - \mu(A \cap B), \text{ and}$$
$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(μ6) If A, B are measurable and $B \subseteq A$, then $\mu(B) \leq \mu(A)$.

(μ7) If $\{A_n : n \in \mathbb{N}\}$ is a (countable) collection of measurable sets such that $\mu(A_1) < \infty$ and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, then

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(μ8) Every interval I is measurable, and $\mu(I) =$ the length of I.

(μ9) A set A has "measure zero" by Definition 3.4.19 iff $\mu(A) = 0$.

These properties are somewhat redundant; see Exercises 15 and 16.

Assuming the existence of such a "measure function," we are able to construct intriguing Cantor-like sets that do not have measure zero. As a warm-up, we use these ideas to give an alternative proof of Theorem 3.4.21.

Theorem 3.4.21 (Again) *The Cantor set has measure zero.*

Alternate Proof. From the definition of the Cantor set, $\mathbf{C} = \bigcap_{n=1}^{\infty} C_n$, with $\mu(C_{n+1}) = \frac{2}{3}\mu(C_n)$, and $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$. Thus, by ($\mu$7),

$$\mu(\mathbf{C}) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \left(\tfrac{2}{3}\right)^n = 0. \quad \blacksquare$$

Corollary 3.4.22 *The complement of the Cantor set in* $[0, 1]$ *has measure 1.*

Proof. Exercise 17. ■

Definition 3.4.23 ("Fat" Cantor-like Sets) Pick any number $0 < a < 1$. We construct a Cantor-like set of measure a.

We shall do so by removing intervals from $[0, 1]$, analogous to the "middle thirds" intervals removed in Definition 3.4.1, leaving a set whose complement in $[0, 1]$ has measure $1 - a$.

We begin with a geometric series $\sum_{n=1}^{\infty} r^n$ with $r < 1$ whose sum is $1 - a$. $\left(\text{We want } \sum_{n=1}^{\infty} r^n = 1 - a; \text{ i.e., } \dfrac{r}{1-r} = 1 - a, \text{ so take } r = \dfrac{1-a}{2-a}.\right)$ We shall describe the sequence $\{R_n\}$ of sets of intervals removed at each step, as we did in Definition 3.4.1. First, note that $0 < r < \frac{1}{2}$.

Let $R_1 =$ the open interval of length r centered in $[0, 1]$. Then the set $C_1 = [0, 1] - R_1$ consists of 2 disjoint closed intervals, each of length $\frac{1-r}{2}$.

Then, let $R_2 =$ the union of the 2 open intervals of length $\frac{r^2}{2}$, one centered in each of the 2 closed intervals comprising C_1. [Note that $0 < r < \frac{1}{2} \Rightarrow 0 < r < 1 - r$ and $0 < \frac{r^2}{2} < \frac{1-r}{2}$.] Then the set $C_2 = [0,1] - (R_1 \cup R_2)$ consists of 4 ($= 2^2$) disjoint closed intervals. Moreover, $\mu(R_2) = 2 \cdot \frac{r^2}{2} = r^2$ and $\mu(C_2) = 1 - (r + r^2)$.

Continuing inductively, we let R_{n+1} consist of the 2^n open intervals of length $\frac{r^{n+1}}{2^n}$, one centered in each of the 2^n disjoint closed intervals comprising C_n. Then

$$\mu(R_{n+1}) = 2^n \cdot \frac{r^{n+1}}{2^n} = r^{n+1}.$$

We define $C_{n+1} = [0,1] - \bigcup_{i=1}^{n} R_i$, and calculate its measure:

$$\mu(C_{n+1}) = 1 - \mu\left(\bigcup_{i=1}^{n} R_i\right) = 1 - \sum_{i=1}^{n} r^i.$$

Finally, we define

$$\mathbf{C}(\mathbf{a}) = \bigcap_{n=1}^{\infty} C_n.$$

Then by (μ7), since $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \cdots$,

$$\begin{aligned}\mu(\mathbf{C}(\mathbf{a})) &= \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty} \mu(C_{n+1}) \\ &= \lim_{n\to\infty} \left[1 - \sum_{i=1}^{n} r^i\right] \\ &= 1 - \sum_{i=1}^{\infty} r^i = 1 - (1 - a) \\ &= a.\end{aligned}$$

The set $\mathbf{C}(\mathbf{a})$ is a "fat" Cantor-like set of measure a. ■

In Exercise 3.4.19 you will explore some of the properties of such "fat" sets.

EXERCISE SET 3.4

1. Prove Theorem 3.4.2.
2. Prove Theorem 3.4.3.
3. Prove that every point of $\mathbf{C}$ is a boundary point of $\mathbf{C}$.
4. Prove Lemma 3.4.4.
5. Prove Theorem 3.4.10.
6. (a) Prove that the set of endpoints of all the open intervals removed from $[0,1]$ in forming $\mathbf{C}$ is a countable set.

   (b) Prove that there are uncountably many members of $\mathbf{C}$ that are *not* endpoints of removed intervals.

7. $\forall n \in \mathbb{N}$, let L_n denote the set of all left endpoints of the disjoint closed intervals comprising C_n. For example, $L_1 = \{0, \frac{2}{3}\}$ and $L_2 = \{0, \frac{2}{9}, \frac{2}{3}, \frac{8}{9}\}$.

   (a) Show that $\forall n \in \mathbb{N}$, L_n consists of those numbers in $[0,1]$ having a terminating base-3 decimal of no more than n digits using only 0's and 2's. [Use mathematical induction.]

   (b) Let $L = \bigcup_{n=1}^{\infty} L_n$. Prove that L is dense in $\mathbf{C}$.

8. Prove that $\varnothing$, $\mathbb{R}$, $(-\infty, a]$, $[a, +\infty)$, and $[a, b]$, are perfect sets, for any real numbers a, b with $a < b$.

9. Prove that a set of real numbers is perfect iff it is closed and has no isolated points.

10. Prove Theorem 3.4.17.

11. Prove that if A is nowhere dense, then A^c is dense in $\mathbb{R}$. Give a counterexample to show that the converse is not true.

12. Prove that A is closed, then A is nowhere dense iff A^c is dense in $\mathbb{R}$.

13. That $\mathbf{C}$ has measure zero may seem intuitively consistent with the fact that $\mathbf{C}$ is nowhere dense. Show by an example that a set can be dense in $\mathbb{R}$ and still have measure zero.

14. Use mathematical induction to prove that if $A_1, A_2, \cdots, A_n$ are measurable, then $A_1 \cup A_2 \cup \cdots \cup A_n$ is measurable, and $\mu\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{n=1}^{n} \mu(A_i)$.

15. Show that (μ5) follows from (μ3) and (μ4), and that (μ6) follows from (μ1) and (μ5).

16. Show that the ($\Rightarrow$) direction of (μ9) follows from (μ1), (μ6), and (μ7).

17. Prove Corollary 3.4.22.

18. **(Project) "Open Middle n^{th}" Cantor-like Sets:** Repeat the construction of the Cantor set given in Definition 3.4.1, but at each stage remove the "open middle fourth" (or fifth, or ...). Prove that the resulting Cantor-like set has measure zero. [Hint: Don't bother with the actual intervals comprising each C_n; only $\mu(C_n)$ is significant.] Which of Theorems 3.4.2 and 3.4.3, Lemma 3.4.4, and Theorems 3.4.12, 3.4.15, and 3.4.17 remain true for the resulting set?

19. **(Project) Fat Cantor-like Sets:** For "fat" Cantor-like sets $\mathbf{C}(\mathbf{a})$ defined in Definition 3.4.23, which of Theorems 3.4.2 and 3.4.3, Lemma 3.4.4, and Theorems 3.4.12, 3.4.15, and 3.4.17 remain true?

20. **Generalized Cantor Sets:** Notice that the Cantor set and "Cantor-like" sets are nonempty, bounded, perfect, and nowhere dense. A set with these properties is called a **generalized Cantor set**. Prove that the union of a finite collection of generalized Cantor sets is a generalized Cantor set.

Chapter 4

Limits of Functions

Limits are basic to analysis. But we have done much of the hard work in Chapter 2. In this chapter we carry over the tools developed there to the limits of functions. We shall encounter very little difficulty and only a few new ideas. The basic ε-δ techniques, discussed in Section 4.1, are extremely important. The sequential criterion is shown to be very useful. Sections 4.2 and 4.4, on the algebra of limits, parallel Sections 2.2–2.3 closely. One-sided limits are covered in Section 4.3.

4.1 Definition of Limit for Functions

As you will recall, the idea of limits of functions underlies the entire subject of calculus. Without an understanding of limits, the concepts of derivative and integral cannot be made rigorous. Thus, the subject of this chapter is of fundamental significance. What may be a new insight for you is that the theory of *sequences* plays an important role in the theory of limits of functions.

Our first task is to define the statement $\lim_{x \to x_0} f(x) = L$. To help our definition make sense, we reflect a little on the intuitive notion of limit. First, a word about notation. To indicate that f is a real-valued function[1] with domain $\mathcal{D}(f)$, we write $f : \mathcal{D}(f) \to \mathbb{R}$. Remember that in taking the limit "as x approaches x_0," we do not really care about the value of $f(x_0)$, nor even whether $f(x_0)$ exists. We only care about $f(x)$ for values of x "close to" (but different from) x_0. Thus, we do not require that x_0 be in the domain of f. But we do require that x_0 be a *cluster point* of the domain—otherwise values of x in the domain

1. For a review of the fundamental ideas and notation of functions, see Appendix B.

of f could not get "close to" x_0. Now, saying that $f(x)$ gets close to L is saying that $|f(x) - L|$ gets small. Similarly, saying that x gets close to (but not equal to) x_0 is saying that $|x - x_0|$ gets small without equaling 0. Now, we are ready for the definition.

Definition 4.1.1 If $f : \mathcal{D}(f) \to \mathbb{R}$, and x_0 is a cluster point of $\mathcal{D}(f)$, then

$$\boxed{\begin{array}{l}\lim\limits_{x\to x_0} f(x) = L \text{ if} \\ \forall \varepsilon > 0, \exists\, \delta > 0 \ni \forall x \in \mathcal{D}(f),\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.\end{array}}$$

That is, $\lim\limits_{n\to\infty} x_n$ equal to L means:

$|f(x) - L|$ is arbitrarily small whenever $x \in \mathcal{D}(f)$ is sufficiently close to x_0 (but not equal to x_0),

or

$\forall \varepsilon > 0$, there is some $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in \mathcal{D}(f)$ and $0 < |x - x_0| < \delta$. ☐

Verbal Paraphrase of Definition 4.1.1.[2]

$$\boxed{\begin{array}{r}\lim\limits_{x\to x_0} f(x) = L \Leftrightarrow f(x) \text{ can be made arbitrarily close to } L \text{ by making} \\ x \in \mathcal{D}(f) \text{ sufficiently close to } x_0 \text{ (but not equal to } x_0).\end{array}}$$

Notes on Definition 4.1.1:

(1) If $\exists L \in \mathbb{R} \ni \lim\limits_{x\to x_0} f(x) = L$, we say that $\lim\limits_{x\to x_0} f(x)$ **exists**; otherwise, we say that $\lim\limits_{x\to x_0} f(x)$ **does not exist**.

(2) We shall never say that $\lim\limits_{x\to x_0} f(x)$ exists unless x_0 is a cluster point of $\mathcal{D}(f)$. For example, $\lim\limits_{x\to 0} \sqrt{x^3 - x^2}$ does not exist. [The domain of the function $f(x) = \sqrt{x^3 - x^2}$ is $\{0\} \cup [1, +\infty)$, and 0 is not a cluster point of this set.]

(3) Even if $x_0 \in \mathcal{D}(f)$, the value of $f(x_0)$ is irrelevant to the consideration of whether $\lim\limits_{x\to x_0} f(x) = L$. The condition "$0 < |x - x_0|$" in Definition 4.1.1 guarantees that we are not letting $x = x_0$.

2. Although Definition 4.1.1 is officially correct, and should be memorized, it is equally important (for the sake of understanding) to be able to paraphrase it in words.

(4) If x_0 is an interior point of $\mathcal{D}(f)$, then Definition 4.1.1 simplifies to:

$$\boxed{\lim_{x \to x_0} f(x) = L \text{ if } \forall \varepsilon > 0,\ \exists\, \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.}$$

(5) There is actually a third quantifier here. The universal quantifier on x is understood to be present, even when left out in the interest of simplicity.

(6) The following statements are interchangeable, and each one will find use at one time or another:

(a) $\lim_{x \to x_0} f(x) = L$. (b) f has limit L as $x \to x_0$.

(c) f has limit L at x_0. (d) $f(x) \to L$ as $x \to x_0$.

(7) The definitions of $\lim_{n \to \infty} x_n = L$ and $\lim_{x \to x_0} f(x) = L$ are really quite similar. Much of the development of the theory of limits in this chapter will closely parallel the theory of limits of sequences developed in Chapter 2. This chapter is deliberately laid out in a way that helps you see that relationship.

THE ε-δ GAME

The process of using Definition 4.1.1 to prove that $\lim_{x \to x_0} f(x) = L$ may be viewed as an "ε-δ game." Player #1 chooses an arbitrary $\varepsilon > 0$ and challenges Player #2 (that's you) to find a $\delta > 0$ that guarantees that $f(x)$ will be within a distance ε from L whenever x is within a distance δ from x_0 (without equalling x_0). See Figure 4.1.

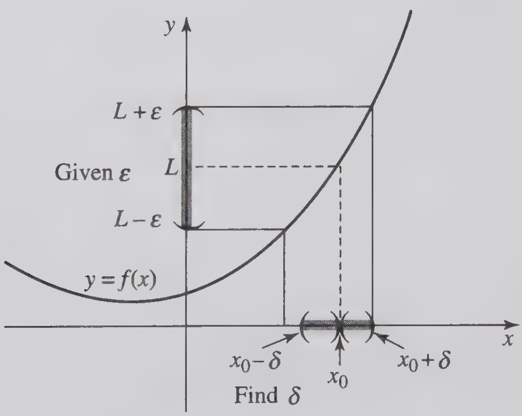


Figure 4.1

Example 4.1.2 Consider the limit statement $\lim_{x\to 2}(4x-5)=3$.

(a) Find a value of $\delta > 0$ that will guarantee that whenever x is within a distance δ from 2 (but not equal to 2) $4x-5$ is within a distance .01 from 3.

(b) Find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ from 2 (but not equal to 2) $4x-5$ will approximate the limit accurately to 3 decimal places.

(c) For arbitrary $\varepsilon > 0$, find a value of $\delta > 0$ that will guarantee that whenever x is within a distance δ from 2 (but not equal to 2) $4x-5$ is within a distance ε from 3.

Solution: This example should remind you of Example 2.1.5. You are advised to take a look at that example. We begin by letting $f(x) = 4x-5$.

(a) We want a real number $\delta > 0$ such that $0 < |x-2| < \delta \Rightarrow |f(x)-3| < .01$. Now,

$$\begin{aligned} |f(x)-3| &= |(4x-5)-3| = |4x-8| \\ &= 4|x-2|. \end{aligned}$$

Thus, our objective is to find a $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow 4|x-2| < .01.$$

The latter inequality will be true if $|x-2| < \dfrac{.01}{4} = .0025$. Thus, we take $\delta = .0025$.

(b) This time, we want a real number $\delta > 0$ such that $0 < |x-2| < \delta \Rightarrow |f(x)-3| < .0005$. Thus, our objective is to find a $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow 4|x-2| < .0005.$$

The latter inequality will be true if $|x-2| < \dfrac{.0005}{4} = .000125$. Thus, we take $\delta = .000125$.

(c) Let $\varepsilon > 0$. This time, we want a real number $\delta > 0$ such that $0 < |x-2| < \delta \Rightarrow |f(x)-3| < \varepsilon$. Thus, our objective is to find a $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow 4|x-2| < \varepsilon.$$

The latter inequality will be true if $|x-2| < \dfrac{\varepsilon}{4}$. Thus, we take $\delta = \dfrac{\varepsilon}{4}$. □

Example 4.1.3 Prove that $\lim_{x \to 2}(4x - 5) = 3$.

(This example should remind you of Example 2.1.6.)

Proof. First, let $f(x) = 4x - 5$. Note that 2 is an interior point of $\mathcal{D}(f)$. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{4}$ (as suggested by Part (c) of Example 4.1.2). Then

$$\begin{aligned} 0 < |x - 2| < \delta &\Rightarrow |x - 2| < \delta \\ &\Rightarrow |x - 2| < \frac{\varepsilon}{4} \\ &\Rightarrow 4|x - 2| < \varepsilon \\ &\Rightarrow |4x - 8| < \varepsilon \\ &\Rightarrow |(4x - 5) - 3| < \varepsilon. \end{aligned}$$

Therefore, by Definition 4.1.1, $\lim_{x \to 2}(4x - 5) = 3$. □

The next example demonstrates the role of "$0 < |x - x_0|$" in the definition of $\lim_{x \to x_0} f(x) = L$.

Example 4.1.4 Use Definition 4.1.1 to prove that $\lim_{x \to 3} \frac{2x^2 - 18}{x - 3} = 12$.

Proof. Consider $f(x) = \frac{2x^2 - 18}{x - 3}$. Let $\varepsilon > 0$. This time $3 \notin \mathcal{D}(f)$. Note that when $x \neq 3$, $f(x) = \frac{2(x - 3)(x + 3)}{x - 3} = 2x + 6$. Choose $\delta = \frac{\varepsilon}{2}$. Then

$$\begin{aligned} 0 < |x - 3| < \delta &\Rightarrow x \neq 3 \text{ and } |x - 3| < \delta \\ &\Rightarrow f(x) = 2x + 6 \text{ and } |x - 3| < \frac{\varepsilon}{2} \\ &\Rightarrow |f(x) - 12| = |(2x + 6) - 12| = |2x - 6| \\ &\Rightarrow |f(x) - 12| = 2|x - 3| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon \\ &\Rightarrow |f(x) - 12| < \varepsilon. \end{aligned}$$

$-1 < x + 3 < 1$
$-4 < x < -2$
$-2 < 2x + 6 < 2$

Therefore, $\lim_{x \to 3} \frac{2x^2 - 18}{x - 3} = 12$. □

Example 4.1.5 Consider the limit statement: $\lim_{x \to 2} x^2 = 4$.

(a) Find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ from 2 (but not equal to 2) x^2 is within distance .01 from 4.

(b) For arbitrary $\varepsilon > 0$, find a value of $\delta > 0$ that will guarantee that whenever x is within a distance δ from 2 (but not equal to 2) x^2 is within a distance ε from 4.

Solution: (a) We want a real number $\delta > 0$ such that $0 < |x - 2| < \delta \Rightarrow |x^2 - 4| < .01$. That is, we want to have

$$\begin{gathered}
-.01 < x^2 - 4 < .01 \\
3.99 < x^2 < 4.01 \\
\sqrt{3.99} < x < \sqrt{4.01} \\
1.99749\cdots < x < 2.0024\cdots ,
\end{gathered}$$

which will be true if $1.998 < x < 2.002$; that is,

$$\begin{gathered}
2 - .002 < x < 2 + .002 \\
|x - 2| < .002.
\end{gathered}$$

Thus, we take $\delta = .002$.

(b) We want a real number $\delta > 0$ such that $0 < |x - 2| < \delta \Rightarrow |x^2 - 4| = |x + 2||x - 2| < \varepsilon$. The reasoning used here will probably be new to you; follow it closely. First, suppose we have $\delta \leq 1$. Then

$$\begin{aligned}
|x - 2| < \delta &\Rightarrow |x - 2| < 1 \\
&\Rightarrow -1 < x - 2 < 1 \\
&\Rightarrow 1 < x < 3 \\
&\Rightarrow 3 < x + 2 < 5 \\
&\Rightarrow |x + 2| < 5.
\end{aligned}$$

Thus, when $|x - 2| < 1$, we will have $|x + 2| < 5$. Remember that we want

$$|x + 2||x - 2| < \varepsilon.$$

Thus, in addition to $|x - 2| < 1$, we want $|x - 2| < \dfrac{\varepsilon}{5}$ in order to guarantee that $|x + 2||x - 2| < 5 \cdot \dfrac{\varepsilon}{5} = \varepsilon$. Therefore, we take $\delta = \min\{1, \frac{\varepsilon}{5}\}$. □

In the next example we show how to use the result of Example 4.1.5 (b) to *prove* that $\lim_{x \to 2} x^2 = 4$. As we did in Chapter 2, we regard the work done in Example 4.1.5 (b) as "scratchwork," and work backwards to produce our proof.

Example 4.1.6 Prove that $\lim_{x \to 2} x^2 = 4$.

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \frac{\varepsilon}{5}\}$. Then

$$\begin{aligned}
|x-2| < \delta &\Rightarrow |x-2| < 1 \text{ and } |x-2| < \frac{\varepsilon}{5} \\
&\Rightarrow -1 < x-2 < 1 \text{ and } |x-2| < \frac{\varepsilon}{5} \\
&\Rightarrow 3 < x+2 < 5 \text{ and } |x-2| < \frac{\varepsilon}{5} \\
&\Rightarrow |x+2| < 5 \text{ and } |x-2| < \frac{\varepsilon}{5} \\
&\Rightarrow |x+2||x-2| < 5 \cdot \frac{\varepsilon}{5} \\
&\Rightarrow |x^2-4| < \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to 2} x^2 = 4$. $\square$

SUMMARY: HOW TO PROVE $\lim_{x \to x_0} f(x) = L$

1. Let $\varepsilon > 0$.
2. Find a value of $\delta > 0$ that will guarantee that whenever x is within a distance δ from x_0 (but not equal to x_0), $f(x)$ is within a distance ε from L. [This is what we did in Part (c) of Example 4.1.2 and Part (b) of Example 4.1.5 above.]
3. Let $\delta =$ the value found in Step 2.
4. Prove that for this value of δ,
   $\forall x \in \mathcal{D}(f),\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
   [This is what we did in Examples 4.1.3, 4.1.4, and 4.1.6 above.]

Note: Step 2, although of critical importance in finding δ, is not considered part of the *proof* of $\lim_{x \to x_0} f(x) = L$. It is never included when the proof is written up. It may be discarded once Step 4 is completed. In fact, step 4 is usually done by working Step 2 backwards, as demonstrated in Examples 4.1.3, 4.1.4, and 4.1.6.

Lemma 4.1.7 (*Negation of* $\lim_{x \to x_0} f(x) = L$): *$f$ does not have limit L at x_0 if and only if*

$$\boxed{\exists \varepsilon > 0 \ni \forall \delta > 0\ ,\ \exists x \in \mathcal{D}(f) \ni 0 < |x - x_0| < \delta \text{ and } |f(x) - L| \geq \varepsilon.}$$

Proof. Apply the rules of quantifier negation[3] to Definition 4.1.1. ■

Theorem 4.1.8 (*Uniqueness of Limits*) *A function cannot have more than one limit as* $x \to x_0$.

Proof. Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} f(x) = M$. We want to prove that $L = M$. We shall use the "forcing principle" [Theorem 1.5.9 (d)]. Let $\varepsilon > 0$.
Since $\lim_{x\to x_0} f(x) = L$, $\exists\, \delta_1 > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \dfrac{\varepsilon}{2}$.
Since $\lim_{x\to x_0} f(x) = M$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - M| < \dfrac{\varepsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $\forall x \in \mathcal{D}(f)$,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow 0 < |x - x_0| < \delta_1 \text{ and } 0 < |x - x_0| < \delta_2 \\ &\Rightarrow |f(x) - L| < \frac{\varepsilon}{2} \text{ and } |M - f(x)| < \frac{\varepsilon}{2} \\ &\Rightarrow |f(x) - L| + |M - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\Rightarrow |f(x) - L + M - f(x)| \le |f(x) - L| + |M - f(x)| < \varepsilon \\ &\qquad\qquad \text{(by } \triangle\text{-inequality)} \\ &\Rightarrow |M - L| < \varepsilon. \end{aligned}$$

Thus, $\forall \varepsilon > 0$, $|M - L| < \varepsilon$. By the forcing principle, $L = M$. ■

The next theorem suggests that the extensive work with limits of sequences in Chapter 2 will yield dividends in our study of limits of functions.

Theorem 4.1.9 (*Sequential Criterion for Limits of Functions*)
$\lim_{x\to x_0} f(x) = L \Leftrightarrow \forall$ *sequences* $\{x_n\}$ *in* $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0$, $f(x_n) \to L$.

Proof. Part 1 ($\Rightarrow$): Suppose that $\lim_{x\to x_0} f(x) = L$, and suppose that $\{x_n\}$ is a sequence in $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0$. To prove: $f(x_n) \to L$. Let $\varepsilon > 0$.
Since $\lim_{x\to x_0} f(x) = L$, $\exists\, \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
Since $x_n \to x_0$ and $x_n \neq x_0$, $\exists\, n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow 0 < |x_n - x_0| < \delta$.
Thus, $n \ge n_0 \Rightarrow |f(x_n) - L| < \varepsilon$. That is, $f(x_n) \to L$.
Therefore, $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0$, $f(x_n) \to L$.

Part 2 ($\Leftarrow$): Suppose that $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0$, $f(x_n) \to L$. We want to prove that $\lim_{x\to x_0} f(x) = L$. For contradiction, suppose it is not true that $\lim_{x\to x_0} f(x) = L$. By Lemma 4.1.7, this means that $\exists\, \varepsilon > 0 \ni$

3. See Appendix A.2 for rules governing quantifiers and their negations.

$\forall \delta > 0, \exists x \in \mathcal{D}(f) \ni 0 < |x - x_0| < \delta$ but $|f(x) - L| \geq \varepsilon$. Keep this ε fixed throughout the remainder of the proof. Taking $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots$ we have

$$\begin{array}{l}
\exists x_1 \in \mathcal{D}(f) \ni 0 < |x_1 - x_0| < 1 \text{ but } |f(x_1) - L| \geq \varepsilon; \\
\exists x_2 \in \mathcal{D}(f) \ni 0 < |x_2 - x_0| < \frac{1}{2} \text{ but } |f(x_2) - L| \geq \varepsilon; \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\exists x_n \in \mathcal{D}(f) \ni 0 < |x_n - x_0| < \frac{1}{n} \text{ but } |f(x_n) - L| \geq \varepsilon; \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{array}$$

In this way we generate a sequence $\{x_n\}$ in $\mathcal{D}(f)$ such that $\forall n \in \mathbb{N}$, $0 < |x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - L| \geq \varepsilon$. Thus, $\{x_n\}$ is a sequence in $\mathcal{D}(f) - \{x_0\}$ and, by the squeeze principle, $x_n \to x_0$. On the other hand, the sequence $\{f(x_n)\}$ cannot converge to L since $\forall n \in \mathbb{N}$, $|f(x_n) - L| \geq \varepsilon$. This contradicts our hypothesis. Therefore, $\lim_{x \to x_0} f(x) = L$. ■

USING THE SEQUENTIAL CRITERION TO DISPROVE $\lim_{x \to x_0} f(x) = L$

Corollary 4.1.10 *If $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\}$ such that $x_n \to x_0$, but the sequence $\{f(x_n)\}$ does not converge to L, then $f(x)$ does not have limit L at x_0.*

Proof. This is the contrapositive of Theorem 4.1.9. ■

Corollary 4.1.11 *If $\exists$ sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{D}(f) - \{x_0\}$, which both converge to x_0, but the sequences $\{f(x_n)\}$ and $\{f(y_n)\}$ do not both converge to the same number, then $\lim_{x \to x_0} f(x)$ does not exist.*

Proof. Immediate consequence of Theorem 4.1.9 and the uniqueness of limits of functions. ■

Example 4.1.12 Prove[4] that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.[5] [See Figure 4.2.]

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4. For an alternate proof, see Exercise 11.

5. We shall define $\sin x$ formally in Chapters 7 and 9. Until then, we assume the usual algebraic properties of $\sin x$.

Proof. Let $f(x) = \sin\left(\frac{1}{x}\right)$. Consider the sequences $\{x_n\} = \left\{\frac{1}{n\pi}\right\}$ and $\{y_n\} = \left\{\frac{1}{\frac{\pi}{2} + 2n\pi}\right\}$. Then both $x_n \to 0$ and $y_n \to 0$, but

$$f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin(n\pi) = 0 \to 0, \text{ while}$$

$$f(y_n) = \sin\left(\frac{1}{y_n}\right) = \sin(\tfrac{\pi}{2} + 2n\pi) = 1 \to 1.$$

Thus, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. Therefore, by Corollary 4.1.11, $\lim_{x\to 0} f(x)$ does not exist. $\square$

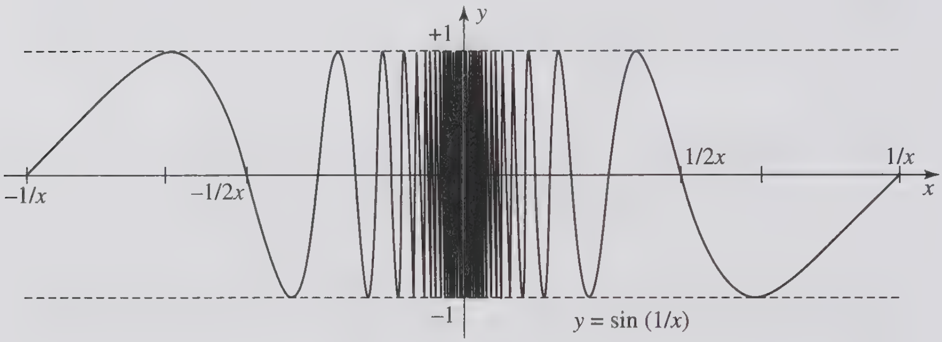

Figure 4.2

EXERCISE SET 4.1

1. For each of the following limit statements $\lim_{x\to x_0} f(x) = L$, do the following:

   (i) Find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ from x_0 (but $\neq x_0$) $f(x)$ is within distance .01 from L.

   (ii) Find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ from x_0 (but not $\neq x_0$) $f(x)$ will approximate the limit accurately to 3 decimal places.

   (iii) For arbitrary but unknown $\varepsilon > 0$, find a value of $\delta > 0$ that will guarantee that whenever x is within distance δ of x_0 (but $\neq x_0$) $f(x)$ is within distance of ε of L.

   (iv) Prove the given limit statement using Definition 4.1.1

   (a) $\lim_{x\to 3}(5x - 11) = 4$ (b) $\lim_{x\to 1}(3x - 8) = -5$

   (c) $\lim_{x\to 3} x^2 = 9$ (d) $\lim_{x\to 2} x^3 = 8$

2. Prove each of the following limit statements using Definition 4.1.1:

(a) $\lim_{x\to -4}(2x+13) = 5$ (b) $\lim_{x\to -2} x^2 = 4$

(c) $\lim_{x\to 1}(3x^2+2x-3) = 2$ (d) $\lim_{x\to -2}(3x^2-2x) = 16$

(e) $\lim_{x\to -3}(2x^2+5x+1) = 4$ (f) $\lim_{x\to -1} x^3 = -1$

(g) $\lim_{x\to 1}(x^3+5x) = 6$ (h) $\lim_{x\to -1} \dfrac{x^2-1}{x+1} = -2$

(i) $\lim_{x\to -2} \dfrac{3x^2-12}{x+2} = -12$ (j) $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3} = 4$

3. Use the sequential criterion for limits of functions to prove each of the limit statements given in Exercise 2.

4. In each of the following, a function $f(x)$ and a number x_0 are given. Use the sequential criterion to prove that $\lim_{x\to x_0} f(x)$ does not exist.

(a) $f(x) = \dfrac{|x|}{x}$; $x_0 = 0$. (b) $f(x) = \left\{\begin{matrix} 5, \text{ if } x < 3 \\ 6, \text{ if } x \geq 3 \end{matrix}\right\}$; $x_0 = 3$.

(c) **Dirichlet's Function:** $f(x) = \left\{\begin{matrix} 1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{matrix}\right\}$; $x_0 \in \mathbb{R}$.

5. Prove that for constants a and $b \in \mathbb{R}$, and $\forall x_0 \in \mathbb{R}$, $\lim_{x\to x_0}(ax+b) = ax_0+b$.

6. Prove that $\lim_{x\to x_0} f(x) = L \Leftrightarrow \lim_{h\to 0} f(x_0+h) = L$.

7. Prove that the function $f(x) = \left\{\begin{matrix} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{matrix}\right\}$ has a limit at x_0 if and only if $x_0 = 0$.

8. Prove that $\lim_{x\to 2}\sqrt{4x^2-4x-x^3}$ does not exist. [See 4.1.1, Note (2).]

9. Prove that $\lim_{x\to 1}\sqrt{x^4-4x^3+5x^2-2x}$ does not exist. [See 4.1.1, Note (2).]

10. Use Exercise 2.6.20 and the sequential criterion for limits of functions to prove that $\lim_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist.

4.2 Algebra of Limits of Functions

The material of this section runs parallel with the algebra of limits of sequences, as presented in Section 2.2. There are some differences, primarily due to the fact that when given an $\varepsilon > 0$ we must find a $\delta > 0$ instead of an $n_0 \in \mathbb{N}$, and

we look for the implications of $0 < |x - x_0| < \delta$ instead of $n \geq n_0$. As you go through this section, you are encouraged to observe how this section parallels Section 2.2. That will help you learn and remember these ideas.

Theorem 4.2.1 (***Absolute Value and Limits***)

(a) $\lim_{x \to x_0} f(x) = 0 \Leftrightarrow \lim_{x \to x_0} |f(x)| = 0;$

(b) $\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0} |f(x) - L| = 0;$

(c) $\lim_{x \to x_0} f(x) = L \Rightarrow \lim_{x \to x_0} |f(x)| = |L|$, *but the converse is not true.*

Proof. Exercise 1. ■

SOME TERMINOLOGY USEFUL FOR LIMITS

Some additional language and terminology of sets and functions will be useful. We review the basic terminology[6] here, and relate it to the theory of limits.

Definition 4.2.2 Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ is a function, $A \subseteq \mathcal{D}(f)$ and $B \subseteq \mathbb{R}$. Then

$$f(A) = \{f(x) : x \in A\};$$
$$f^{-1}(B) = \{x : f(x) \in B\}.$$

The set $f(A)$ is called the **image of** A under f and the set $f^{-1}(B)$ is called the **inverse image of** B under f.

Definition 4.2.3 (**Deleted Neighborhoods**) Let $x_0 \in \mathbb{R}$. $\forall \varepsilon > 0$, we define the **deleted ε-neighborhood** of x_0 to be the set (see Figure 4.3)

$$\begin{aligned} N'_\varepsilon(x_0) &= N_\varepsilon(x_0) - \{x_0\} \\ &= \{x : 0 < |x - x_0| < \varepsilon\} \\ &= (x_0 - \varepsilon,\ x_0) \cup (x_0,\ x_0 + \varepsilon). \end{aligned}$$

6. For a review of the language and terminology of sets and functions, see Appendices B.2 and B.3.

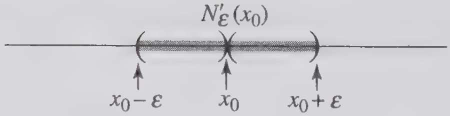

Figure 4.3

The language of neighborhoods and deleted neighborhoods provides a useful way of verbalizing the limit statement. If x_0 is a cluster point of $\mathcal{D}(f)$,

$$\begin{aligned}\lim_{x\to x_0} f(x) = L &\Leftrightarrow \forall \varepsilon > 0, \exists\, \delta > 0 \ni x \in N'_\delta(x_0) \cap \mathcal{D}(f) \Rightarrow f(x) \in N_\varepsilon(L) \\ &\Leftrightarrow \forall \text{ nbd. } N \text{ of } L, \exists \text{ deleted nbd. } M \text{ of } x_0 \ni f(M) \subseteq N \\ &\Leftrightarrow \forall \text{ nbd. } N \text{ of } L, \exists \text{ deleted nbd. } M \text{ of } x_0 \ni M \subseteq f^{-1}(N) \\ &\Leftrightarrow \text{the inverse image of every neighborhood of } L \text{ contains a} \\ &\qquad \text{deleted neighborhood of } x_0.\end{aligned}$$

Definition 4.2.4 A function f is said to be **constant on a set** A if $\exists\, c \in \mathbb{R} \ni \forall x \in A, f(x) = c$.

Theorem 4.2.5 *If f is constant, say $f(x) = c$, on some deleted neighborhood of x_0, then $\lim_{x\to x_0} f(x) = c$.*

Proof. Exercise 2. ■

Definition 4.2.6 A function f is said to be **bounded on a set** $A \subseteq \mathcal{D}(f)$ if $\exists\, B > 0 \ni \forall x \in A$, $|f(x)| \le B$. [Equivalently, $\exists\, a, b \in \mathbb{R} \ni \forall x \in A$, $a \le f(x) \le b$.]

Theorem 4.2.7 *If $\lim_{x\to x_0} f(x) = L \in \mathbb{R}$, then there is some neighborhood N of x_0 such that f is bounded on $N \cap \mathcal{D}(f)$.*

Proof. Suppose $\lim_{x\to x_0} f(x) = L \in \mathbb{R}$. Letting $\varepsilon = 1$ in Definition 4.1.1, $\exists\, \delta > 0 \ni \forall x \in \mathcal{D}(f)$,

$$\begin{aligned}0 < |x - x_0| < \delta &\Rightarrow |f(x) - L| < 1 \\ &\Rightarrow -1 < f(x) - L < 1 \\ &\Rightarrow L - 1 < f(x) < L + 1.\end{aligned}$$

Therefore, f is bounded on $N_\delta(x_0) \cap \mathcal{D}(f)$. ■

Definition 4.2.8 A function f is said to be **bounded away from**[7] **0 on a set** $A \subseteq \mathcal{D}(f)$ if $\exists C > 0 \ni \forall x \in A$, $|f(x)| \geq C$.

Theorem 4.2.9 *Suppose* $\lim_{x \to x_0} f(x) = L \neq 0$. *Then there is a deleted neighborhood* $N'_\delta(x_0)$ *such that* f *is bounded away from* 0 *on* $N'_\delta(x_0) \cap \mathcal{D}(f)$. *In fact,*[8]

(a) *If* $0 < C < L$, *then* $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow f(x) > C$.

(b) *If* $L < C < 0$, *then* $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow f(x) < C$.

(c) *If* $0 < C < |L|$, *then* $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow |f(x)| > C$.

Proof. (a) Suppose $\lim_{x \to x_0} f(x) = L \neq 0$, and $0 < C < L$ (see Figure 4.4). Then $L - C > 0$. By Definition 4.1.1 with $\varepsilon = L - C$, $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow |f(x) - L| < L - C \\ &\Rightarrow C - L < f(x) - L < L - C \\ &\Rightarrow C < f(x) < 2L - C \\ &\Rightarrow f(x) > C. \end{aligned}$$

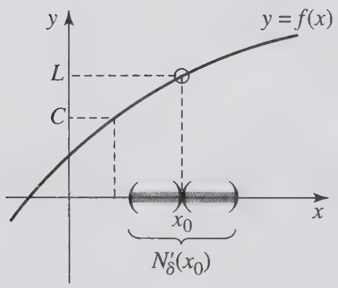

Figure 4.4

Thus, $x \in N'_\delta(x_0) \cap \mathcal{D}(f) \Rightarrow f(x) > C$.

The proofs of (b) and (c) are similar. ■

7. For an extension of this idea, see Exercise 20.

8. You are encouraged to express these statements less formally, in words, to yourself. For example, read (a) as, "If C is less than L, then $f(x)$ is greater than C on some deleted neighborhood of x_0."

Lemma 4.2.10 (***Fundamental Limit***) *For every $x_0 \in \mathbb{R}$, $\lim\limits_{x \to x_0} x = x_0$.*

Proof. Exercise 3. ■

Theorem 4.2.11 (***Algebra of Limits of Functions***)
Suppose $\lim\limits_{x \to x_0} f(x) = L$, $\lim\limits_{x \to x_0} g(x) = M$, and $c \in \mathbb{R}$. Then

(a) $\lim\limits_{x \to x_0} cf(x) = cL$;

(b) $\lim\limits_{x \to x_0} (f(x) + g(x)) = L + M$;

(c) $\lim\limits_{x \to x_0} (f(x) - g(x)) = L - M$;

(d) $\lim\limits_{x \to x_0} (f(x) \cdot g(x)) = LM$;

(e) $\lim\limits_{x \to x_0} \left(\dfrac{1}{g(x)}\right) = \dfrac{1}{M}$ (*if* $M \neq 0$);

(f) $\lim\limits_{x \to x_0} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$ (*if* $M \neq 0$);

(g) $\lim\limits_{x \to x_0} \sqrt{f(x)} = \sqrt{L}$ (*if $f(x) \geq 0$ for all x in some $N'_\delta(x_0)$*).

[*In (b), (c), (d), and (f), we assume that x_0 is a cluster point of $\mathcal{D}(f) \cap \mathcal{D}(g)$.*]

Proof. Suppose $\lim\limits_{x \to x_0} f(x) = L$, $\lim\limits_{x \to x_0} g(x) = M$, and $c \in \mathbb{R}$. Then

(a) Case 1 ($c \neq 0$): Let $\varepsilon > 0$. Since $\lim\limits_{x \to x_0} f(x) = L$, $\exists\, \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \dfrac{\varepsilon}{|c|}$. Then, $\forall x \in \mathcal{D}(f)$,

$$\begin{aligned} 0 < |x - x_0| < \delta \Rightarrow |cf(x) - cL| &= |c||f(x) - L| \\ &< |c| \cdot \frac{\varepsilon}{|c|} \\ &< \varepsilon. \end{aligned}$$

Case 2 ($c = 0$): Exercise 4.

(b) Let $\varepsilon > 0$.

Since $\lim\limits_{x \to x_0} f(x) = L$, $\exists\, \delta_1 > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L| < \dfrac{\varepsilon}{2}$.

Since $\lim\limits_{x \to x_0} g(x) = M$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(g)$, $0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \dfrac{\varepsilon}{2}$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $\forall x \in \mathcal{D}(f+g)$,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow 0 < |x - x_0| < \delta_1 \text{ and } 0 < |x - x_0| < \delta_2 \\ &\Rightarrow |f(x) - L| < \frac{\varepsilon}{2} \text{ and } |g(x) - M| < \frac{\varepsilon}{2} \\ &\Rightarrow |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\Rightarrow |f(x) - L + g(x) - M| < \varepsilon, \text{ by } \triangle\text{-inequality} \\ &\Rightarrow |(f(x) + g(x)) - (L + M)| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to x_0} (f(x) + g(x)) = L + M$.

(c) Exercise 5.

(d) Let $\varepsilon > 0$.

Since $\lim_{x \to x_0} f(x) = L$, f is bounded on some deleted nbd. of x_0; i.e., $\exists\, B > 0$ and $\exists\, \delta_1 > 0 \ni 0 < |x - x_0| < \delta_1 \Rightarrow |f(x)| \le B$. Since $\lim_{x \to x_0} f(x) = L$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - L| < \dfrac{\varepsilon}{2(|M|+1)}$. Finally, since $\lim_{x \to x_0} g(x) = M$, $\exists\, \delta_3 > 0 \ni \forall x \in \mathcal{D}(g)$, $0 < |x - x_0| < \delta_3 \Rightarrow |g(x) - M| < \dfrac{\varepsilon}{2B}$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then, $\forall x \in \mathcal{D}(f \cdot g)$,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow 0 < |x - x_0| < \delta_1 \text{ and } \delta_2 \text{ and } \delta_3 \\ &\Rightarrow |f(x)| \le B,\ |f(x) - L| < \frac{\varepsilon}{2(|M|+1)}, \text{ and } |g(x) - M| < \frac{\varepsilon}{2B} \\ &\Rightarrow (|M|+1)|f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad B|g(x) - M| < \frac{\varepsilon}{2} \\ &\Rightarrow |M||f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x)||g(x) - M| < \frac{\varepsilon}{2} \\ &\Rightarrow |Mf(x) - ML| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x)g(x) - f(x)M| < \frac{\varepsilon}{2} \\ &\Rightarrow |f(x)g(x) - f(x)M| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x)M - LM| < \frac{\varepsilon}{2} \\ &\Rightarrow |f(x)g(x) - f(x)M + f(x)M - LM| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by } \triangle\text{-inequality} \\ &\Rightarrow |f(x)g(x) - LM| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to x_0} (f(x)g(x)) = LM$.

(e) Let $\varepsilon > 0$. Since $\lim_{x \to x_0} g(x) = M \neq 0$, g is bounded away from 0 on some deleted nbd. of x_0. In fact, by Theorem 4.2.9, $\exists\, \delta_1 > 0 \ni x \in N'_{\delta_1}(x_0) \cap \mathcal{D}(g) \Rightarrow |g(x)| > \frac{|M|}{2}$. Also, since $\lim_{x \to x_0} g(x) = M$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(g)$, $0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - M| < \dfrac{\varepsilon |M|^2}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $\forall x \in \mathcal{D}\left(\dfrac{1}{g}\right)$,

$$0 < |x - x_0| < \delta \Rightarrow 0 < |x - x_0| < \delta_1 \text{ and } 0 < |x - x_0| < \delta_2$$

$$\Rightarrow |g(x)| > \frac{|M|}{2} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon |M|^2}{2}$$

$$\Rightarrow \frac{1}{|g(x)|} < \frac{2}{|M|} \quad \text{and} \quad |g(x) - M| < \frac{\varepsilon |M|^2}{2}$$

$$\Rightarrow \frac{|g(x) - M|}{|g(x)||M|} = \frac{1}{|g(x)|} \cdot \frac{|g(x) - M|}{|M|} < \frac{2}{|M|} \cdot \frac{\varepsilon |M|^2}{2|M|} = \varepsilon$$

$$\Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x)M} \right| = \frac{|g(x) - M|}{|g(x)||M|} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon.$$

Therefore, $\lim_{x \to x_0} \left(\dfrac{1}{g(x)} \right) = \dfrac{1}{M}$.

(f) Exercise 8.

(g) We postpone the proof of (g) until we have discussed an alternate method of proof, which follows. ■

Alternate proof of Theorem 4.2.11 using the "sequential criterion." The sequential criterion for limits of functions (Theorem 4.1.9) provides a very powerful technique for proving theorems about limits of functions. It enables us to use the power of the theory of sequences developed in Chapter 2. As examples, we use it to give alternate (much easier) proofs of Theorem 4.2.11 Parts (d) and (g).

Theorem 4.2.11 (d): *If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then $\lim_{x \to x_0} (f(x)g(x)) = LM$. (Assuming x_0 is a cluster point of $\mathcal{D}(f) \cap \mathcal{D}(g)$.)*

Proof. (Alternate) Suppose $\{x_n\}$ is a sequence in $[\mathcal{D}(f)\cap\mathcal{D}(g)]-\{x_0\} \ni x_n \to x_0$. Since $\lim\limits_{x\to x_0} f(x) = L$, the sequential criterion for limits of functions (Theorem 4.1.9) guarantees that $f(x_n) \to L$. Since $\lim\limits_{x\to x_0} g(x) = M$, the sequential criterion guarantees that $g(x_n) \to M$. By the algebra of limits for sequences,

$$f(x_n)\cdot g(x_n) \to LM.$$

Thus, $\forall$ sequences in $[\mathcal{D}(f)\cap\mathcal{D}(g)]-\{x_0\} \ni x_n \to x_0$, $f(x_n)\cdot g(x_n) \to LM$. By the sequential criterion, we conclude that $\lim\limits_{x\to x_0} (f(x)g(x)) = LM$. ■

Theorem 4.2.11 (g): *If* $\lim\limits_{x\to x_0} f(x) = L$ *and* $f(x) \geq 0$ *for all* x *in some* $N'_\delta(x_0)$, *then* $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{L}$.

Proof. Suppose $\{x_n\}$ is a sequence in $\mathcal{D}(f)-\{x_0\} \ni x_n \to x_0$. Since $f(x) \geq 0$ for all x in some $N'_\delta(x_0)$, we know that $\sqrt{x}$ exists as $x \to x_0$. Since $\lim\limits_{x\to x_0} f(x) = L$, the sequential criterion (Theorem 4.1.9) guarantees that $f(x_n) \to L$. By the algebra of limits for sequences, $\sqrt{f(x_n)} \to \sqrt{L}$.

Thus, $\forall$ sequences in $\mathcal{D}(f) - \{x_0\} \ni x_n \to x_0$, $\sqrt{f(x_n)} \to \sqrt{L}$. By the sequential criterion, this tells us that $\lim\limits_{x\to x_0} \sqrt{f(x)} = \sqrt{L}$. ■

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS

Definition 4.2.12 A **polynomial** (in one variable) is a function of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \cdots, a_n$ are (constant) real numbers.

Theorem 4.2.13 (***Limits of Polynomials***) *For any polynomial* $p(x)$ *and any* $x_0 \in \mathbb{R}$, $\lim\limits_{x\to x_0} p(x) = p(x_0)$.

Proof. By the "algebra of limits of functions," $\forall k = 0, 1, 2, \cdots, n$, $\lim\limits_{x\to x_0} x^k = \left(\lim\limits_{x\to x_0} x\right)^k$ since x^k is a product. Thus, by Lemma 4.2.10, $\lim\limits_{x\to x_0} x^k = x_0^k$. Then, by the algebra of limits, $\forall k = 0, 1, 2, \cdots, n$,

$$\begin{aligned}\lim_{x\to x_0} a_k x^k &= a_k \lim_{x\to x_0} x^k \\ &= a_k x_0^k.\end{aligned}$$

Finally, we apply the algebra of limits again to conclude that

$$\begin{aligned}
\lim_{x \to x_0} p(x) &= \lim_{x \to x_0} \sum_{k=0}^{n} a_k x^k \\
&= \sum_{k=0}^{n} \left(\lim_{x \to x_0} a_k x^k \right) \\
&= \sum_{k=0}^{n} a_k x_0^k \\
&= p(x_0). \quad \blacksquare
\end{aligned}$$

Example 4.2.14 $\lim_{x \to 2} (3x^3 - 7x^2 + x + 11) = 3(8) - 7(4) + 2 + 11 = 9.$

Note: Theorem 4.2.13 tells us that limits are of no essential significance in the study of polynomials. The limit of a polynomial at x_0 is found by merely "plugging in" x_0. Obviously, limits were introduced to study functions more complicated than polynomials.

Definition 4.2.15 A **rational function** (of one variable) is any function of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

Theorem 4.2.16 (***Limits of Rational Functions***) *For any rational function $r(x) = \dfrac{p(x)}{q(x)}$, and any $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} r(x) = r(x_0)$ provided that $q(x_0) \neq 0$.*

Proof. Apply Theorem 4.2.11(f) and Theorem 4.2.13. ■

Example 4.2.17 $\lim_{x \to 3} \dfrac{5x - 3}{2x^2 + 1} = \dfrac{12}{19}.$

In Chapter 2 we saw that in discussing convergence of a sequence, "only the tail matters." (See Theorem 2.2.16.) Similarly, in discussing the limit of a function as $x \to x_0$, only what happens in a deleted neighborhood of x_0 matters. That principle is formalized in the next theorem.

Theorem 4.2.18 (***Only What Happens in a Deleted Neighborhood of x_0 Matters***) *Suppose $\lim_{x \to x_0} f(x) = L$, and $f(x) = g(x)$ for all x in some deleted neighborhood of x_0. Then $\lim_{x \to x_0} g(x) = L$.*

Proof. Exercise 14. ■

The following example shows how Theorem 4.2.18 is used in practice.

Example 4.2.19 Find $\lim\limits_{x\to 3} \dfrac{2x^2-18}{x-3}$.

Solution: For $x \neq 3$, $\dfrac{2x^2-18}{x-3} = \dfrac{2(x-3)(x+3)}{x-3} = 2x+6$. Thus, by Theorem 4.2.18, $\lim\limits_{x\to 3} \dfrac{2x^2-18}{x-3} = \lim\limits_{x\to 3}(2x+6) = 12$. □

INEQUALITIES AND LIMITS

Theorem 4.2.20 (***The "Squeeze" Principle for Functions***)

(a) ***The First Squeeze Principle***: *Suppose $f(x) \le g(x) \le h(x)$ for all x in some deleted nbd. of x_0, and $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = L$. Then $\lim\limits_{x\to x_0} g(x) = L$.*

(b) ***The Second Squeeze Principle***: *Suppose $\lim\limits_{x\to x_0} g(x) = 0$. If $|f(x)-L| \le |g(x)|$, for all x in some deleted nbd. of x_0, then $\lim\limits_{x\to x_0} f(x) = L$.*

Proof. (a) Suppose $f(x) \le g(x) \le h(x)$ for all x in some deleted nbd. of x_0, and $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = L$. Then, $\exists\, \delta_1 > 0 \ni \forall x \in N'_{\delta_1}(x_0)$, $f(x) \le g(x) \le h(x)$. Let $\varepsilon > 0$.

Since $\lim\limits_{x\to x_0} f(x) = L$, $\exists\, \delta_2 > 0 \ni \forall x \in N'_{\delta_2}(x_0)$, $|f(x)-L| < \varepsilon$.

Since $\lim\limits_{x\to x_0} h(x) = L$, $\exists\, \delta_3 > 0 \ni \forall x \in N'_{\delta_3}(x_0)$, $|h(x)-L| < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$
\begin{aligned}
0 < |x-x_0| < \delta &\Rightarrow 0 < |x-x_0| < \delta_1, \delta_2 \text{ and } \delta_3 \\
&\Rightarrow f(x) \le g(x) \le h(x),\ |f(x)-L| < \varepsilon \text{ and } |h(x)-L| < \varepsilon \\
&\Rightarrow f(x) \le g(x) \le h(x),\ L-\varepsilon < f(x) \text{ and } h(x) < L+\varepsilon \\
&\Rightarrow L-\varepsilon < f(x) \le g(x) \le h(x) < L+\varepsilon \\
&\Rightarrow L-\varepsilon < g(x) < L+\varepsilon \Rightarrow |g(x)-L| < \varepsilon.
\end{aligned}
$$

Therefore, $\lim\limits_{x\to x_0} g(x) = L$.

(b) Exercise 16. ■

In Example 4.1.12 we proved that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist. In the next example, we consider a closely related example.

Example 4.2.21 Use the squeeze principle to prove that $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right) = 0$.

Solution: We start with the inequality, $|\sin t| \leq 1$, $\forall t \in \mathbb{R}$. Then $\forall x \neq 0$, $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$. Multiplying both sides by $|x|$, we have

$$\begin{aligned} |x| \left|\sin\left(\tfrac{1}{x}\right)\right| &\leq |x|; \text{ i.e.,} \\ \left|x \sin\left(\tfrac{1}{x}\right)\right| &\leq |x|. \end{aligned}$$

But $\lim_{x\to 0} x = 0$, so by the second squeeze principle, $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. □

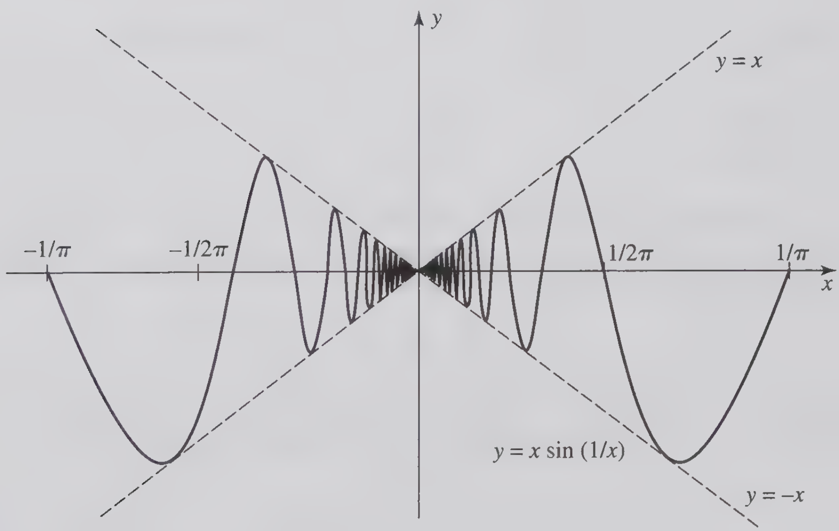

Figure 4.5

Theorem 4.2.22 (***Limits Preserve Inequalities***)

(a) *If* $\lim_{x\to x_0} f(x) = L$ *and* $f(x) \le K$ *for all* x *in some deleted nbd. of* x_0*, then* $L \le K$*.*

(b) *If* $\lim_{x\to x_0} f(x) = L$ *and* $f(x) \ge K$ *for all* x *in some deleted nbd. of* x_0*, then* $L \ge K$*.*

(c) *If* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist, and* $f(x) \le g(x)$ *for all* x *in some deleted nbd. of* x_0*, then* $\lim_{x\to x_0} f(x) \le \lim_{x\to x_0} g(x)$*.*

Proof. (a) Suppose $\lim_{x\to x_0} f(x) = L$ and $\exists\, \delta_1 > 0 \ni \forall x \in N'_{\delta_1}(x_0)$, $f(x) \le K$. We want to prove that $L \le K$. For contradiction, suppose $L > K$. Let $\varepsilon = L - K$. Then $\varepsilon > 0$. Since $\lim_{x\to x_0} f(x) = L$, $\exists\, \delta_2 > 0 \ni \forall x \in N'_{\delta_2}(x_0)$, $|f(x) - L| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$
\begin{aligned}
x \in N'_\delta(x_0) &\Rightarrow x \in N'_{\delta_2}(x_0) \text{ and } x \in N'_{\delta_1}(x_0) \\
&\Rightarrow |f(x) - L| < \varepsilon \text{ and } f(x) \le K \\
&\Rightarrow -\varepsilon < f(x) - L < \varepsilon \text{ and } f(x) \le K \\
&\Rightarrow L - \varepsilon < f(x) < L + \varepsilon \text{ and } f(x) \le K \\
&\Rightarrow L - (L - K) < f(x) \text{ and } f(x) \le K \\
&\Rightarrow K < f(x) \text{ and } f(x) \le K. \quad \text{Contradiction!}
\end{aligned}
$$

Therefore, $L \le K$.

(b) Exercise 17.

(c) Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, and $f(x) \le g(x)$ for all x in some deleted nbd. of x_0. Define the function $h(x) = g(x) - f(x)$. Then, by the algebra of limits theorem,

$$\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x) - \lim_{x\to x_0} f(x).$$

Now, $\forall x \ne x_0$, $h(x) \ge 0$, so by Part (b) above, $\lim_{x\to x_0} h(x) \ge 0$. That is,

$$
\begin{gathered}
\lim_{x\to x_0} g(x) - \lim_{x\to x_0} f(x) \ge 0; \text{ i.e.,} \\
\lim_{x\to x_0} g(x) \ge \lim_{x\to x_0} f(x). \quad \blacksquare
\end{gathered}
$$

*CHANGE OF VARIABLES IN LIMITS

"Change of variables" is a technique you have used frequently to evaluate limits in a completely natural way. To formulate this technique formally and justify it rigorously, however, is a task requiring some subtlety. We start with an example. Suppose we want to obtain

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x}.$$

We notice that if we substitute $u = \frac{\pi}{2} - x$, our desired limit has the form

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin u}{u}.$$

We know that $\lim\limits_{x \to \frac{\pi}{2}} u = \lim\limits_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) = 0$, and we recall from calculus that $\lim\limits_{u \to 0} \dfrac{\sin u}{u} = 1$. We thus ask whether we are justified in asserting that

$$\lim_{x \to \frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x} = \lim_{u \to 0} \frac{\sin u}{u} = 1.$$

Now, let us pose the problem in a more general framework. Suppose we want to obtain

$$\lim_{x \to x_0} f\left(g(x)\right)$$

at some cluster point x_0 of $\mathcal{D}(g)$. Suppose we know that $\lim\limits_{x \to x_0} g(x) = u_0$ and $\lim\limits_{u \to u_0} f(u) = L$, and we want to know whether we are justified in asserting that

$$\lim_{x \to x_0} f\left(g(x)\right) = \lim_{u \to u_0} f(u).$$

The answer is "Yes," subject to certain technical conditions. The following theorem provides these conditions and the justification.

*An asterisk with a theorem, proof, or other item in this chapter indicates that the item is optional and can be omitted, especially in a one-semester course.

Theorem 4.2.23 (***Change of Variables in Limits***) *Suppose* $\lim_{x \to x_0} g(x) = u_0$ *and* $\lim_{u \to u_0} f(u) = L$*, where* x_0 *and* u_0 *are cluster points of* $\mathcal{D}(g)$ *and* $\mathcal{D}(f)$*, respectively, and* $g(x) \in \mathcal{D}(f) - \{u_0\}$ *for all* $x \in \mathcal{D}(g)$ *in some deleted neighborhood of* x_0*. Then*

$$\lim_{x \to x_0} f(g(x)) = \lim_{u \to u_0} f(u) = L.$$

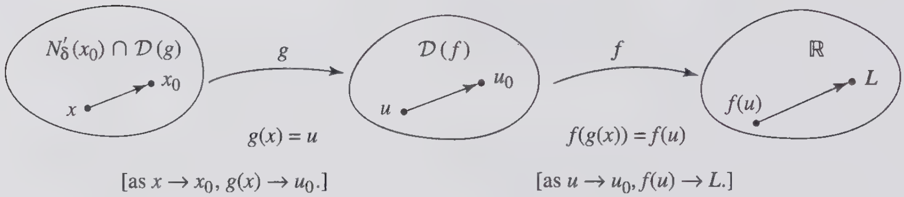

Figure 4.6

Proof. Suppose $\lim_{x \to x_0} g(x) = u_0$ and $\lim_{u \to u_0} f(u) = L$, where x_0 and u_0 are cluster points of $\mathcal{D}(g)$ and $\mathcal{D}(f)$, respectively, and $\exists\, \delta > 0 \ni \forall x \in N'_\delta(x_0) \cap \mathcal{D}(g)$, $g(x) \in \mathcal{D}(f) - \{u_0\}$.

Let $\varepsilon > 0$. Since $\lim_{u \to u_0} f(u) = L$, $\exists\, \delta_1 > 0 \ni \forall u \in \mathcal{D}(f)$,

$$0 < |u - u_0| < \delta_1 \Rightarrow |f(u) - L| < \varepsilon.$$

Since $\lim_{x \to x_0} g(x) = u_0$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(g)$,

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - u_0| < \delta_1.$$

Choose $\delta_3 = \min\{\delta, \delta_2\}$. Then $\forall x \in \mathcal{D}(f(g))$, $x \in \mathcal{D}(g)$ and

$$\begin{aligned} 0 < |x - x_0| < \delta_3 &\Rightarrow 0 < |x - x_0| < \delta \text{ and } 0 < |x - x_0| < \delta_2 \\ &\Rightarrow g(x) \in \mathcal{D}(f) - u_0, \text{ and } |g(x) - u_0| < \delta_1 \\ &\Rightarrow g(x) \in \mathcal{D}(f), \text{ and } 0 < |g(x) - u_0| < \delta_1 \\ &\Rightarrow |f(g(x)) - L| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to x_0} f(g(x)) = L = \lim_{u \to u_0} f(u)$. ■

EXERCISE SET 4.2

1. Prove Theorem 4.2.1.

2. Prove Theorem 4.2.5.

3. Prove Lemma 4.2.10.

4. Prove Case 2 of Theorem 4.2.11 (a).

5. Prove Theorem 4.2.11 (c).

6. Prove Theorem 4.2.11 (f).

7. Suppose $\lim_{x \to x_0} f(x) = 2$ and $\lim_{x \to x_0} g(x) = 3$. Find each of the following, and justify your answers:

   (a) $\lim_{x \to x_0} [4f(x) - g(x)]$ (b) $\lim_{x \to x_0} \sqrt{f(x)g(x)}$

   (c) $\lim_{x \to x_0} |f(x) - 3g(x)|$ (d) $\lim_{x \to x_0} \dfrac{f(x)}{f(x) + 2g(x)}$

8. Use the "algebra of limits of functions" (Theorem 4.2.11 and 4.2.18 where necessary) to find each of the following limits. Justify each step.

   (a) $\lim_{x \to -1} (5x^2 + 8x + 7)$ (b) $\lim_{x \to 0} (4x^6 + 3x^4 - 8)$

   (c) $\lim_{x \to 3} \dfrac{3x^2 - 1}{x^2 + 9}$ (d) $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 + x - 2}$

   (e) $\lim_{x \to 3} \dfrac{3x - 9}{x^2 - 9}$ (f) $\lim_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 - x - 2}$

   (g) $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, where $f(x) = 3x + 10$

   (h) $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, where $f(x) = x^2 - 5x$

   (i) $\lim_{x \to 1} \sqrt{\dfrac{5x + 4}{x + 3}}$ (j) $\lim_{x \to 0} \dfrac{(x+1)^2 - 1}{x}$

   (k) $\lim_{x \to a} \dfrac{\sqrt{x} - \sqrt{a}}{x - a}$ $(a > 0)$ (l) $\lim_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h}$

   (m) $\lim_{x \to a} \dfrac{x^3 - a^3}{x - a}$ (n) $\lim_{x \to a} \dfrac{x^4 - a^4}{x - a}$

   (o) $\lim_{x \to 1} \dfrac{x^n - 1}{x - 1}$ $(n \in \mathbb{N})$ (p) $\lim_{x \to 1} \dfrac{x^m - 1}{x^n - 1}$ $(m, n \in \mathbb{N})$

9. In each of the following, give an example of functions f and g and a cluster point x_0 of $\mathcal{D}(f) \cap \mathcal{D}(g)$ satisfying the stated property:

   (a) $\lim_{x \to x_0} [f(x) + g(x)]$ exists, but $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ do not.

(b) $\lim\limits_{x \to x_0} [f(x)g(x)]$ exists, but $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ do not.

(c) $\lim\limits_{x \to x_0} \left[\dfrac{f(x)}{g(x)}\right]$ exists, but $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ do not.

In each case, explain your answers.

10. Use the sequential criterion to prove Theorem 4.2.11 (a).
11. Use the sequential criterion to prove Theorem 4.2.11 (b).
12. Use the sequential criterion to prove Theorem 4.2.11 (e).
13. Use the sequential criterion to prove Theorem 4.2.11 (f).
14. Prove Theorem 4.2.18.
15. Suppose that f and g are defined in some $N'_\delta(x_0)$. Use Theorem 4.2.11 to prove that if $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ exists and $\lim\limits_{x \to x_0} g(x) = 0$, then $\lim\limits_{x \to x_0} f(x) = 0$.
16. Prove Theorem 4.2.20 (b).
17. Prove Theorem 4.2.22 (b).
18. Consider the function $f(x) = \left\{ \begin{array}{rl} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{array} \right\}$. Prove that $\lim\limits_{x \to 0} f(x) = 0$.
19. Prove that if $\lim\limits_{x \to x_0} f(x) = 0$, and $g(x)$ is defined and bounded on a deleted nbd. of x_0, then $\lim\limits_{x \to x_0} [f(x)g(x)] = 0$. (Compare with Exercise 2.2.6.)
20. **Neighborhood Inequality Property of Limits, I:** Prove the following extension of Theorem 4.2.9. If $\lim\limits_{x \to x_0} f(x) > M$ then $f(x) > M$ for all x in some deleted neighborhood of x_0. That is, if $\lim\limits_{x \to x_0} f(x) > M$ then $\exists$ deleted neighborhood $N'_\varepsilon(x_0) \ni \forall x \in \mathcal{D}(f) \cap N'_\varepsilon(x_0)$, $f(x) > M$. State and prove a similar result that holds if $\lim\limits_{x \to x_0} f(x) = L < M$.
21. **Neighborhood Inequality Property of Limits, II:** Prove that if $\lim\limits_{x \to x_0} f(x) < \lim\limits_{x \to x_0} g(x)$, then $f(x) < g(x)$ for all x in some deleted neighborhood of x_0.
22. **Cauchy Criterion for Limits of Functions:** Suppose x_0 is a cluster point of $\mathcal{D}(f)$. Prove that $\lim\limits_{x \to x_0} f(x)$ exists $\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni \forall x, y \in \mathcal{D}(f)$, $x, y \in N'_\delta(x_0) \Rightarrow |f(x) - f(y)| < \varepsilon$.
23. Suppose $f : \mathbb{R} \to \mathbb{R}$ is **periodic with period** $p > 0$. That is, $\forall x \in \mathbb{R}$, $f(x+p) = f(x)$. Prove that $\forall x_0 \in \mathbb{R}$, $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow \lim\limits_{x \to x_0+p} f(x) = L$.

4.3 One-Sided Limits

As you recall from calculus, "one-sided" limits frequently make sense in situations in which the ordinary limit does not exist.

Definition 4.3.1 (Limit from the Left) If x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$, then the limit of f from the left is L, written $f(x_0^-) = \lim\limits_{x \to x_0^-} f(x) = L$, if

$$\boxed{\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon.}$$

Notes on Definition 4.3.1:

(1) We shall never say that $\lim\limits_{x \to x_0^-} f(x)$ exists unless x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$.

(2) Even if $x_0 \in \mathcal{D}(f)$, the value of $f(x_0)$ is irrelevant to the consideration of whether $\lim\limits_{x \to x_0^-} f(x) = L$. The condition "$x_0 - \delta < x < x_0$" in Definition 4.3.1 guarantees that when we consider whether $\lim\limits_{x \to x_0^-} f(x) = L$, we are never letting $x = x_0$.

(3) If $\mathcal{D}(f)$ contains some interval of the form $(x_0 - \gamma, x_0)$, for some $\gamma > 0$, then Definition 4.3.1 simplifies to:

$$\boxed{\lim_{x \to x_0^-} f(x) = L \text{ if } \forall \varepsilon > 0, \exists \delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon.}$$

There is actually a third quantifier here. The universal quantifier on x is understood to be present, even when left out in the interest of simplicity.

(4) The following statements are interchangeable, and each one will find use at one time or another:

(i) $\lim\limits_{x \to x_0^-} f(x) = L$.

(ii) $f(x_0^-) = L$.

(iii) f has limit L as x approaches x_0 from the left.

(iv) f has left-hand limit L at x_0.

(v) f has limit L from the left at x_0.

(vi) $f(x) \to L$ as $x \to x_0^-$.

Example 4.3.2 Prove that $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$.

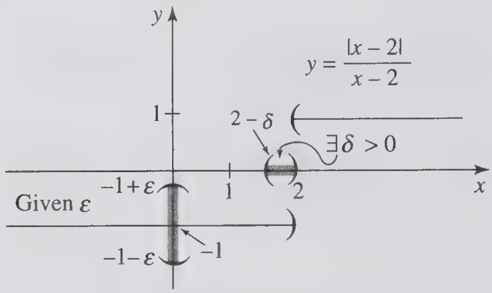

Figure 4.7

Proof. Let $\varepsilon > 0$. Let δ = any positive number. Then, $2 - \delta < x < 2 \Rightarrow \left|\frac{|x-2|}{x-2} - (-1)\right| = \left|\frac{-(x-2)}{x-2} - (-1)\right| = |(-1) - (-1)| = 0 < \varepsilon$. Therefore, $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = -1$. □

Definition 4.3.3 (Limit from the Right) If x_0 is a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$, then the limit of f from the right is L, written $f(x_0^+) = \lim_{x \to x_0^+} f(x) = L$, if

$$\boxed{\forall \varepsilon > 0,\ \exists \delta > 0 \ni \forall x \in \mathcal{D}(f),\ x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon.}$$

Notes on Definition 4.3.3:

(1) We shall never say that $\lim_{x \to x_0+} f(x)$ exists unless x_0 is a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$.

(2) Even if $x_0 \in \mathcal{D}(f)$, the value of $f(x_0)$ is irrelevant to the consideration of whether $\lim_{x \to x_0^+} f(x) = L$. The condition "$x_0 < x < x_0 + \delta$" in Definition 4.3.3 guarantees that, when we consider whether $\lim_{x \to x_0^+} f(x) = L$, we are never letting $x = x_0$.

(3) If $\mathcal{D}(f)$ contains some interval of the form $(x_0,\ x_0 + \gamma)$, for some $\gamma > 0$, then Definition 4.3.3 simplifies to:

$$\boxed{\lim_{x \to x_0^+} f(x) = L \text{ if } \forall \varepsilon > 0,\ \exists \delta > 0 \ni x_0 < x < x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon.}$$

As with Definition 4.3.1, the universal quantifier on x is understood to be present, even when left out in the interest of simplicity.

(4) The following statements are interchangeable, and each one will find use at one time or another:

(i) $\lim_{x \to x_0^+} f(x) = L$.

(ii) $f(x_0^+) = L$.

(iii) f has limit L as x approaches x_0 from the right.

(iv) f has right-hand limit L at x_0.

(v) f has limit L from the right at x_0.

(vi) $f(x) \to L$ as $x \to x_0^+$.

Example 4.3.4 Prove that $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$.

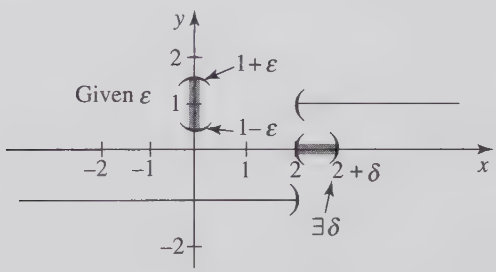

Figure 4.8

Proof. Let $\varepsilon > 0$. Let $\delta =$ any positive number. Then, $2 < x < 2 + \delta \Rightarrow \left| \frac{|x-2|}{x-2} - 1 \right| = \left| \frac{x-2}{x-2} - 1 \right| = |1-1| = 0 < \varepsilon$. Thus, $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = 1$. $\square$

LIMIT THEOREMS FOR ONE-SIDED LIMITS

Theorems 4.1.7 through 4.2.23 express the basic facts about the algebra of limits of functions. Each of these theorems can be revised to express an analogous fact about one-sided limits of functions. For example,

Theorem 4.3.5 (***Sequential Criterion for One-Sided Limits of Functions***)

(a) $\lim_{x \to x_0^-} f(x) = L \Leftrightarrow \forall$ *sequences* $\{x_n\}$ *in* $\mathcal{D}(f) \cap (-\infty, x_0) \ni x_n \to x_0$, $f(x_n) \to L$.

(b) $\lim_{x \to x_0^+} f(x) = L \Leftrightarrow \forall$ *sequences* $\{x_n\}$ *in* $\mathcal{D}(f) \cap (x_0, \infty) \ni x_n \to x_0$, $f(x_n) \to L$.

(Compare with Theorem 4.1.9.)

Theorem 4.3.6 (a) *If* $\lim_{x \to x_0^-} f(x) = L \in \mathbb{R}$, *then* f *is bounded on some interval of the form* $(x_0 - \delta, x_0)$ *where* $\delta > 0$.

(b) *If* $\lim_{x \to x_0^+} f(x) = L \in \mathbb{R}$, *then* f *is bounded on some interval of the form* $(x_0, x_0 + \delta)$ *where* $\delta > 0$.

(Compare with Theorem 4.2.7.)

Theorem 4.3.7 (***Limits from the Left Preserve Inequalities***)

(a) *If* $\lim_{x \to x_0^-} f(x) = L$ *and* $\exists\, \delta > 0 \ni f(x) \le K$ *for all* $x \in (x_0 - \delta, x_0) \cap \mathcal{D}(f)$, *then* $L \le K$.

(b) *If* $\lim_{x \to x_0^-} f(x) = L$ *and* $\exists\, \delta > 0 \ni f(x) \ge K$ *for all* $x \in (x_0 - \delta, x_0) \cap \mathcal{D}(f)$, *then* $L \ge K$.

(c) *If* $\lim_{x \to x_0^-} f(x)$ *and* $\lim_{x \to x_0^-} g(x)$ *exist, and* $\exists\, \delta > 0 \ni f(x) \le g(x)$ *for all* $x \in (x_0 - \delta, x_0)$ *in* $\mathcal{D}(f) \cap \mathcal{D}(g)$, *then* $\lim_{x \to x_0^-} f(x) \le \lim_{x \to x_0^-} g(x)$.

(Compare with Theorem 4.2.22.)

LIMITS VS. ONE-SIDED LIMITS

The following theorem expresses an important relationship between limits and one-sided limits.

Theorem 4.3.8 *If* x_0 *is a cluster point of* $\mathcal{D}(f) \cap (-\infty, x_0)$, *and a cluster point of* $\mathcal{D}(f) \cap (x_0, \infty)$, *then* $\lim_{x \to x_0} f(x) = L \Leftrightarrow$ *both* $\lim_{x \to x_0^-} f(x) = L$ *and* $\lim_{x \to x_0^+} f(x) = L$.

Proof. Suppose x_0 is a cluster point of $\mathcal{D}(f) \cap(-\infty, x_0)$, and a cluster point of $\mathcal{D}(f) \cap(x_0, \infty)$.

Part 1 ($\Rightarrow$): Suppose $\lim\limits_{x \to x_0} f(x) = L$. Let $\varepsilon > 0$. Since $\lim\limits_{x \to x_0} f(x) = L$, $\exists\, \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Then $\forall x \in \mathcal{D}(f)$,

$$x_0 - \delta < x < x_0 \Rightarrow 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon; \text{ and}$$
$$x_0 < x < x_0 + \delta \Rightarrow 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Therefore, $\lim\limits_{x \to x_0^-} f(x) = L$ and $\lim\limits_{x \to x_0^+} f(x) = L$.

Part 2 ($\Leftarrow$): Suppose $\lim\limits_{x \to x_0^-} f(x) = L$ and $\lim\limits_{x \to x_0^+} f(x) = L$. Let $\varepsilon > 0$. Since $\lim\limits_{x \to x_0^-} f(x) = L$, $\exists\, \delta_1 > 0 \ni \forall x \in \mathcal{D}(f)$, $x_0 - \delta_1 < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$. Since $\lim\limits_{x \to x_0^+} f(x) = L$, $\exists\, \delta_2 > 0 \ni \forall x \in \mathcal{D}(f)$, $x_0 < x < x_0 + \delta_2 \Rightarrow |f(x) - L| < \varepsilon$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, $\forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow$ either $x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$. In either of these cases, $|f(x) - L| < \varepsilon$. Thus, $\forall x \in \mathcal{D}(f)$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Therefore, $\lim\limits_{x \to x_0} f(x) = L$. ■

Since the hypothesis of the previous theorem is rather complicated, we restate the theorem with a slightly simpler hypothesis. The theorem is often applicable in this form.

Corollary 4.3.9 *If $\mathcal{D}(f)$ contains a deleted nbd. of x_0, then $\lim\limits_{x \to x_0} f(x) = L \Leftrightarrow$ both $\lim\limits_{x \to x_0^-} f(x) = L$ and $\lim\limits_{x \to x_0^+} f(x) = L$.*

Theorem 4.3.8 and its corollary are often useful in proving that $\lim\limits_{x \to x_0} f(x)$ does not exist, as shown in the following example.

Example 4.3.10 Prove that $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ does not exist.

Solution: In Example 4.3.2, we saw that $\lim\limits_{x \to 2^-} \dfrac{|x-2|}{x-2} = -1$ and in Example 4.3.4 we saw that $\lim\limits_{x \to 2^+} \dfrac{|x-2|}{x-2} = 1$. Since these two one-sided limits at 2 are not equal, Corollary 4.3.9 above tells us that $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ does not exist. □

EXERCISE SET 4.3

1. In each of the following, a function f and a number x_0 are given. Investigate $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$, and where possible, use the results to determine whether $\lim_{x\to x_0} f(x)$ exists.

   (a) $f(x) = \dfrac{|x|}{x}$; $x_0 = 0$ (b) $f(x) = \dfrac{|x+1|}{x+1}$; $x_0 = -1$

   (c) $f(x) = \dfrac{x^2}{|x|}$; $x_0 = 0$ (d) $f(x) = \sqrt{x}$; $x_0 = 0$

   (e) $f(x) = \lfloor x \rfloor =$ the greatest integer[9] $\leq x$; $x_0 = 3$

   (f) $f(x) = x\lfloor x \rfloor$; $x_0 = 0$ (g) $f(x) = x\lfloor x \rfloor$; $x_0 = 1$

2. Revise Theorem 4.1.8 to a correct theorem about limits from the left; limits from the right.

3. Revise Corollary 4.1.10 to a correct statement about limits from the left; limits from the right.

4. Revise Corollary 4.1.11 to a correct statement about limits from the left; limits from the right.

5. Revise Theorem 4.2.1 to a correct theorem about limits from the left; limits from the right.

6. Revise Theorem 4.2.5 to a correct theorem about limits from the left; limits from the right.

7. Revise Theorem 4.2.9 to a correct theorem about limits from the left; limits from the right.

8. Revise Lemma 4.2.10 to a correct statement about limits from the left; limits from the right.

9. Revise Theorem 4.2.11 to a correct theorem about limits from the left; limits from the right.

10. Revise Theorem 4.2.13 to a correct theorem about limits from the left; limits from the right.

11. Revise Theorem 4.2.16 to a correct theorem about limits from the left; limits from the right.

12. Revise Theorem 4.2.18 to a correct theorem about limits from the left; limits from the right.

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9. f is called the "greatest integer function," "bracket function," or "integer floor function."

13. Revise Theorem 4.2.20 to a correct theorem about limits from the left; limits from the right.

14. Revise Theorem 4.2.22 to a correct theorem about limits from the right.

15. **Generalization of One-Sided Limits:** Suppose $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, where A and B are disjoint and x_0 is a cluster point of both A and B. Define $h : A \cup B \to \mathbb{R}$ by $h(x) = \left\{ \begin{matrix} f(x) \text{ if } x \in A \\ g(x) \text{ if } x \in B \end{matrix} \right\}$. Prove that $\lim_{x\to x_0} h(x)$ exists and equals L iff $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = L$.

4.4 *Infinity in Limits

This section is included here in the interest of completeness, to present a rigorous justification of certain limit techniques involving "infinity." Most students in this course have already gained a working knowledge of these concepts in their elementary calculus courses. Thus, students may be encouraged to read this section on their own if coverage is desired. The most significant items may be the definitions and Theorems 4.4.3, 4.4.19, and 4.4.21.

INFINITY AS A LIMIT

Definition 4.4.1 Suppose $f : \mathcal{D}(f) \to \mathbb{R}$, and x_0 is a cluster point of $\mathcal{D}(f)$. Then

(a) $\lim_{x\to x_0} f(x) = +\infty$ if

$$\forall M > 0,\ \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) > M.$$

(b) $\lim_{x\to x_0} f(x) = -\infty$ if

$$\forall M > 0,\ \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), 0 < |x - x_0| < \delta \Rightarrow f(x) < -M.$$

Note: If $\mathcal{D}(f)$ contains a deleted neighborhood of x_0, then Definition 4.4.1 simplifies to:

$$\lim_{x\to x_0} f(x) = +\infty \text{ if } \forall M > 0, \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow f(x) > M;$$

$$\lim_{x\to x_0} f(x) = -\infty \text{ if } \forall M > 0, \exists \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow f(x) < -M.$$

In words, $\lim_{x\to x_0} f(x) = +\infty$ if for every M, $f(x) > M$ whenever x is sufficiently close to, but not equal to, x_0. Similarly for $\lim_{x\to x_0} f(x) = -\infty$.

Example 4.4.2 Consider the limit statement $\lim_{x\to 2} \dfrac{1}{(x-2)^2} = +\infty$.

(a) Find $\delta > 0 \ni 0 < |x - 2| < \delta \Rightarrow \dfrac{1}{(x-2)^2} > 1,000$.

(b) For arbitrary $M > 0$, find $\delta > 0 \ni 0 < |x-2| < \delta \Rightarrow \dfrac{1}{(x-2)^2} > M$.

Solution. (a) We want to find a $\delta > 0$ such that $0 < |x-2| < \delta \Rightarrow \dfrac{1}{(x-2)^2} > 1,000$. Equivalently, $(x-2)^2 < \frac{1}{1000} = .001$. Thus, we want $|x-2| < \sqrt{.001} = 0.031632\cdots$. So, we take $\delta = 0.03163$, or even $\delta = 0.03$.

(b) We want to find a $\delta > 0$ such that $0 < |x-2| < \delta \Rightarrow \dfrac{1}{(x-2)^2} > M$. We want $(x-2)^2 < \dfrac{1}{M}$. Equivalently, $|x-2| < \sqrt{\frac{1}{M}}$. We take $\delta = \sqrt{\frac{1}{M}}$. □

In general, using Definition 4.4.1 to prove that $\lim_{x\to x_0} f(x) = +\infty$ is more difficult than Example 4.4.2 would lead you to believe. For an example of the difficulty, try to prove $\lim_{x\to 2} \dfrac{3x-5}{(x-2)^2} = +\infty$ using only Definition 4.4.1. The following theorem provides a useful technique that will often be helpful.

Theorem 4.4.3 *Suppose $f(x) > 0$ for all x in some deleted neighborhood of x_0. Then $\lim_{x\to x_0} f(x) = +\infty \Leftrightarrow \lim_{x\to x_0} \dfrac{1}{f(x)} = 0$.*

Proof. Suppose $f(x) > 0$ for all x in some deleted neighborhood of x_0.

(a) Part 1 ($\Rightarrow$): Suppose $\lim_{x\to x_0} f(x) = +\infty$. Let $\varepsilon > 0$. Then $\dfrac{1}{\varepsilon} > 0$. Since $\lim_{x\to x_0} f(x) = +\infty$, $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$,

$$\begin{aligned} 0 < |x-x_0| < \delta &\Rightarrow f(x) > \frac{1}{\varepsilon} \\ &\Rightarrow \frac{1}{f(x)} < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x\to x_0} \dfrac{1}{f(x)} = 0$.

(b) Part 2 ($\Leftarrow$): Suppose $\lim_{x\to x_0} \dfrac{1}{f(x)} = 0$. Let $M > 0$. Then $\dfrac{1}{M} > 0$. Since $\lim_{x\to x_0} \dfrac{1}{f(x)} = 0$, $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$,

$$\begin{aligned} 0 < |x-x_0| < \delta &\Rightarrow 0 < \frac{1}{f(x)} < \frac{1}{M} \\ &\Rightarrow f(x) > M. \end{aligned}$$

Therefore, $\lim_{x\to x_0} f(x) = +\infty$. ∎

Example 4.4.4 below shows the power of using Theorem 4.4.3 in proving that $\lim\limits_{x \to x_0} f(x) = +\infty$.

Example 4.4.4 Prove that $\lim\limits_{x \to 2} \dfrac{3x-5}{(x-2)^2} = +\infty$.

Proof. By the algebra of limits, established in Section 4.2, we have $\lim\limits_{x \to 2} \dfrac{(x-2)^2}{3x-5} = \dfrac{0}{1} = 0$. Moreover, as $x \to 2$, $3x - 5 \to 1$, so for x sufficiently close to 2, $\dfrac{3x-5}{(x-2)^2} > 0$. Thus, by Theorem 4.4.3, $\lim\limits_{x \to 2} \dfrac{3x-5}{(x-2)^2} = +\infty$. □

INFINITY AS A ONE-SIDED LIMIT

Definition 4.4.1 can be altered to define $\lim\limits_{x \to x_0^-} f(x) = +\infty$, $\lim\limits_{x \to x_0^-} f(x) = -\infty$, $\lim\limits_{x \to x_0^+} f(x) = +\infty$, and $\lim\limits_{x \to x_0^+} f(x) = -\infty$. (Exercise 3)

Example 4.4.5 Investigate $\lim\limits_{x \to 1^-} \dfrac{x-2}{x-1}$ and $\lim\limits_{x \to 1^+} \dfrac{x-2}{x-1}$.

Solution: In Exercise 4, we modify Theorem 4.4.3 to cover one-sided limits. We show here how to apply these simple modifications.

(a) First, observe that $\lim\limits_{x \to 1^-} \dfrac{x-1}{x-2} = \dfrac{0}{-1} = 0$. Moreover, as $x \to 1^-$, $x < 1$ and $x < 2$, so $x - 1 < 0$ and $x - 2 < 0$; thus as $x \to 1^-$, $\dfrac{x-1}{x-2} > 0$. Thus, using Exercise 4, $\lim\limits_{x \to 1^-} \dfrac{x-2}{x-1} = +\infty$.

(b) Next, observe that $\lim\limits_{x \to 1^+} \dfrac{x-1}{x-2} = \dfrac{0}{-1} = 0$. Moreover, as $x \to 1^+$, $x > 1$ and $x < 2$, so $x - 1 > 0$ and $x - 2 < 0$; thus as $x \to 1^+$, $\dfrac{x-1}{x-2} < 0$. Thus, using Exercise 4, $\lim\limits_{x \to 1^+} \dfrac{x-2}{x-1} = -\infty$. □

Theorem 4.4.6 *If x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$, and a cluster point of $\mathcal{D}(f) \cap (x_0, \infty)$, then*

(a) $\lim\limits_{x \to x_0} f(x) = +\infty \Leftrightarrow$ *both* $\lim\limits_{x \to x_0^-} f(x) = +\infty$ *and* $\lim\limits_{x \to x_0^+} f(x) = +\infty$;

(b) $\lim\limits_{x \to x_0} f(x) = -\infty \Leftrightarrow$ *both* $\lim\limits_{x \to x_0^-} f(x) = -\infty$ *and* $\lim\limits_{x \to x_0^+} f(x) = -\infty$.

Proof. Exercise 7. ■

Example 4.4.7 Investigate $\lim\limits_{x\to 1} \dfrac{x-2}{x-1}$.

Solution: In Example 4.4.5, we showed that $\lim\limits_{x\to 1^-} \dfrac{x-2}{x-1} = +\infty$ and $\lim\limits_{x\to 1^+} \dfrac{x-2}{x-1} = -\infty$. Theorem 4.4.6 then tells us that $\lim\limits_{x\to 1} \dfrac{x-2}{x-1}$ is neither $+\infty$ nor $-\infty$. The most we can say about $\lim\limits_{x\to 1} \dfrac{x-2}{x-1}$ is that it does not exist. □

ALGEBRA OF INFINITE LIMITS

Theorem 4.4.8 *Suppose* $\lim\limits_{x\to x_0} f(x) = +\infty$, $\lim\limits_{x\to x_0} g(x) = +\infty$, $\lim\limits_{x\to x_0} h(x) = -\infty$, *and* $\lim\limits_{x\to x_0} k(x) = -\infty$. *Then*

(a) $\lim\limits_{x\to x_0} (f(x) + g(x)) = +\infty$;

(b) $\lim\limits_{x\to x_0} (h(x) + k(x)) = -\infty$;

(c) $\lim\limits_{x\to x_0} (f(x)g(x)) = +\infty$;

(d) $\lim\limits_{x\to x_0} (h(x)k(x)) = +\infty$;

(e) $\lim\limits_{x\to x_0} (f(x)h(x)) = -\infty$.

Proof. (a) Suppose $\lim\limits_{x\to x_0} f(x) = +\infty$ and $\lim\limits_{x\to x_0} g(x) = +\infty$. Let $M > 0$. Since $\lim\limits_{x\to x_0} f(x) = +\infty$, $\exists\, \delta_1 > 0 \ni 0 < |x - x_0| < \delta_1 \Rightarrow f(x) > M$. Since $\lim\limits_{x\to x_0} g(x) = +\infty$, $\exists\, \delta_2 > 0 \ni 0 < |x - x_0| < \delta_2 \Rightarrow g(x) > M$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow f(x) > M \text{ and } g(x) > M \\ &\Rightarrow f(x) + g(x) > 2M > M. \end{aligned}$$

Therefore, by Definition 4.4.1, $\lim\limits_{x\to x_0} (f(x) + g(x)) = +\infty$.

(b) Exercise 8.

(c) Exercise 9.

(d) Exercise 10.

(e) Suppose $\lim_{x \to x_0} f(x) = +\infty$ and $\lim_{x \to x_0} h(x) = -\infty$. Let $M > 0$. Since $\lim_{x \to x_0} f(x) = +\infty$, $\exists\, \delta_1 > 0 \ni 0 < |x - x_0| < \delta_1 \Rightarrow f(x) > M$. Since $\lim_{x \to x_0} h(x) = -\infty$, $\exists\, \delta_2 > 0 \ni 0 < |x - x_0| < \delta_2 \Rightarrow h(x) < -1$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then,

$$\begin{aligned} 0 < |x - x_0| < \delta &\Rightarrow f(x) > M \text{ and } h(x) < -1 \\ &\Rightarrow f(x) > M \text{ and } -h(x) > 1 \\ &\Rightarrow f(x)\,(-h(x)) > M \cdot 1 \\ &\Rightarrow -\left(f(x)h(x)\right) > M \\ &\Rightarrow f(x)h(x) < -M. \end{aligned}$$

Therefore, by Definition 4.4.1, $\lim_{x \to x_0} (f(x)h(x)) = -\infty$. ■

Corollary 4.4.9 *Theorem 4.4.8 remains true when $x \to x_0$ is replaced by $x \to x_0^-$ or $x \to x_0^+$.*

Symbolic Shorthand: The results of Theorem 4.4.8 and its corollary are often expressed as a kind of "algebra" of $+\infty$ and $-\infty$, summarized in Table 4.1 as follows:

Table 4.1

Algebra of Infinite Limits
$(+\infty) + (+\infty) = +\infty$
$(-\infty) + (-\infty) = -\infty$
$(+\infty) \cdot (+\infty) = +\infty$
$(-\infty) \cdot (-\infty) = +\infty$
$(+\infty) \cdot (-\infty) = -\infty$

Caution: In Table 4.1, the symbols $+\infty$ and $-\infty$ are not to be regarded as numbers. They cannot be manipulated as numbers, nor can they be expected to obey the usual rules of algebra. They represent limits only.

Indeterminate Forms: The forms $(+\infty) + (-\infty)$ and $(+\infty) - (+\infty)$ are "indeterminate" in the sense that no answer can be given that is always true. That is, there are pairs of functions, $f(x)$ and $g(x)$, such that $\lim f(x) = +\infty$ and $\lim g(x) = -\infty$ for which $\lim[f(x) + g(x)] = +\infty$, others for which $\lim[f(x) + g(x)] = -\infty$, others for which $\lim[f(x) + g(x)]$ is a finite number, and still others for which $\lim[f(x) + g(x)]$ does not exist.

Similarly, we can combine finite and infinite limits algebraically. Suppose $P > 0$ and $N < 0$ represent positive and negative real numbers, respectively, which are limits of functions. Table 4.2 summarizes the results:

Table 4.2

Algebra of Infinite Limits
$(+\infty) + P(\text{or } N) = +\infty$
$(-\infty) + P(\text{or } N) = -\infty$
$(\pm\infty) \cdot P = \pm\infty$
$(\pm\infty) \cdot N = \mp\infty$
$(\pm\infty) \cdot 0$ is indeterminate
$\dfrac{1}{\pm\infty} = 0$

The indeterminate $\frac{1}{0}$, covered by Theorem 4.4.3 along with Exercises 2 and 4, can be refined a bit further. Suppose $\lim f(x) = 0$. We shall write $\lim f(x) = 0^+$ if $f(x) > 0$ throughout appropriate interval(s), and $\lim f(x) = 0^-$ if $f(x) < 0$ throughout appropriate interval(s). With this understanding, we have

Table 4.3

Algebra of Infinite Limits
$\dfrac{1}{0^+} = +\infty$
$\dfrac{1}{0^-} = -\infty$

In Section 4.2 we proved the "Squeeze Principle" (Theorem 4.2.20) for [finite] limits of functions. The analogous result for *infinite* limits is called the "Comparison Principle," and is stated in the following theorem.

Theorem 4.4.10 (***Comparison Test***) *Suppose that $f(x) \le g(x)$ for all x in some deleted neighborhood of x_0.*

(a) *If* $\lim\limits_{x \to x_0} f(x) = +\infty$, *then* $\lim\limits_{x \to x_0} g(x) = +\infty$;

(b) $\lim\limits_{x \to x_0} g(x) = -\infty$, *then* $\lim\limits_{x \to x_0} f(x) = -\infty$.

Proof. (a) Exercise 11.
(b) Exercise 12. ∎

ALWAYS REMEMBER that $+\infty$ and $-\infty$ are *not* real numbers. We should not expect them to obey all the rules of the algebra of real numbers. They are merely convenient symbols, which seem to obey *some* common algebraic rules. They are intended for use only in connection with limits.

EXERCISE SET 4.4-A

1. Use Definition 4.4.1 to prove the following limit statements:

   (a) $\lim_{x\to 0} \frac{1}{x^2} = +\infty$ (b) $\lim_{x\to 1} \frac{-1}{(x-1)^4} = -\infty$

   (c) $\lim_{x\to -1} \frac{1}{(x+1)^2} = +\infty$ (d) $\lim_{x\to 1} \frac{x}{(x-1)^2} = +\infty$

   (e) $\lim_{x\to 2} \frac{1-x}{(x-2)^2} = -\infty$ (f) $\lim_{x\to -3} \frac{x+1}{(x+3)^2} = -\infty$

2. Modify Theorem 4.4.3 to yield a correct theorem about $\lim_{x\to x_0} f(x) = -\infty$.

3. Define each of the following:

   (a) $\lim_{x\to x_0^-} f(x) = +\infty$ (b) $\lim_{x\to x_0^-} f(x) = -\infty$

   (c) $\lim_{x\to x_0^+} f(x) = +\infty$ (d) $\lim_{x\to x_0^+} f(x) = -\infty$

4. Modify Theorem 4.4.3 to yield correct theorems about $\lim_{x\to x_0^-} f(x) = +\infty$, $\lim_{x\to x_0^-} f(x) = -\infty$, $\lim_{x\to x_0^+} f(x) = +\infty$, and $\lim_{x\to x_0^+} f(x) = -\infty$.

   In Exercises 8–20, the generic symbolic statements $\lim f(x) = +\infty$ and $\lim f(x) = -\infty$ will be understood to cover all three possibilities: $\lim_{x\to x_0^-}$, $\lim_{x\to x_0^+}$, or $\lim_{x\to x_0}$.

5. Revise Theorem 4.1.9 to a correct theorem about $\lim f(x) = +\infty$ and a correct theorem about $\lim f(x) = -\infty$.

6. Revise Corollary 4.1.10 to a correct theorem about $\lim f(x) = +\infty$ and a correct theorem about $\lim f(x) = -\infty$.

7. Prove Theorem 4.4.6.

8. Prove Theorem 4.4.8 (b).

9. Prove Theorem 4.4.8 (c).

10. Prove Theorem 4.4.8 (d).

11. Prove Theorem 4.4.10 (a).

12. Prove Theorem 4.4.10 (b).

13. Show by examples that the form $(+\infty) + (-\infty)$ is indeterminate. That is, in each of the following, find functions f and g satisfying the given condition, and such that $\lim_{x \to x_0} f(x) = +\infty$ and $\lim_{x \to x_0} g(x) = -\infty$:

    (a) $\lim_{x \to x_0} [f(x) + g(x)] = 0$ (b) $\lim_{x \to x_0} [f(x) + g(x)] = +\infty$

    (c) $\lim_{x \to x_0} [f(x) + g(x)] - \infty$ (d) $\lim_{x \to x_0} [f(x) + g(x)] = L \neq 0$.

14. Show by examples that the form $(+\infty) \cdot 0$ is indeterminate. That is, in each of the following, find functions f and g satisfying the given condition, and such that $\lim_{x \to x_0} f(x) = +\infty$ and $\lim_{x \to x_0} g(x) = 0$:

    (a) $\lim_{x \to x_0} f(x)g(x) = 0$ (b) $\lim_{x \to x_0} f(x)g(x) = +\infty$

    (c) $\lim_{x \to x_0} f(x)g(x) = -\infty$ (d) $\lim_{x \to x_0} f(x)g(x) = L \neq 0$.

15. In each of the following, a function f and a number x_0 are given. Use the approach of Example 4.4.4 to investigate $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$, and use the results to investigate $\lim_{x \to x_0} f(x)$.

    (a) $f(x) = \dfrac{x}{(x-1)^2}$; $x_0 = 1$ (b) $f(x) = \dfrac{3}{(x+2)^2}$; $x_0 = -2$

    (c) $f(x) = \dfrac{x}{x-1}$; $x_0 = 1$ (d) $f(x) = \dfrac{x+3}{x+5}$; $x_0 = -5$

    (e) $f(x) = \dfrac{x^2-1}{x-3}$; $x_0 = 3$ (f) $f(x) = \dfrac{x^2+x-2}{x^2-3x+2}$; $x_0 = 2$

    (g) $f(x) = \dfrac{3x-9}{x^2-9}$; $x_0 = -3$ (h) $f(x) = \dfrac{x^2+2x+1}{x^2-3x}$; $x_0 = 0$

16. Investigate each of the following (use the familiar algebraic properties of $\cos x$ and $\sin x$):

    (a) $\lim_{x \to 0} \cos \dfrac{1}{x}$ (b) $\lim_{x \to 0} x \cos \dfrac{1}{x}$

    (c) $\lim_{x \to 0^-} \dfrac{1}{x} \cos x$ (d) $\lim_{x \to 0^+} \dfrac{1}{x} \cos \dfrac{1}{x}$

17. **Vertical Asymptotes:** The graph of a function f is said to have the vertical line $x = x_0$ as a **vertical asymptote** if the domain of f contains an interval of the form $(x_0 - \delta, x_0)$ or $(x_0, x_0 + \delta)$ for some $\delta > 0$, and $f(x) \to +\infty$ (or $-\infty$) as $x \to x_0^-$ or $x \to x_0^+$.

(a) Prove that if $R(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, then the graph of $R(x)$ has a vertical asymptote $x = x_0$ at any point x_0 where $q(x_0) = 0$ but $p(x_0) \neq 0$.

(b) Find all vertical asymptotes of the graph for each function given in Exercise 15.

LIMITS AT INFINITY

We now consider limits as $x \to +\infty$ or $-\infty$. Six definitions are needed. We supply three of them and leave the other three as exercises.

Definition 4.4.11 $\lim\limits_{x \to +\infty} f(x) = L \Leftrightarrow \mathcal{D}(f)$ is unbounded above, and

$$\forall \varepsilon > 0, \exists N > 0 \ni \forall x \in \mathcal{D}(f),\ x > N \Rightarrow |f(x) - L| < \varepsilon.$$

Definition 4.4.12 $\lim\limits_{x \to +\infty} f(x) = +\infty \Leftrightarrow \ldots$ (Exercise 1.)

Definition 4.4.13 $\lim\limits_{x \to +\infty} f(x) = -\infty \Leftrightarrow \mathcal{D}(f)$ is unbounded above, and

$$\forall M > 0, \exists N > 0 \ni \forall x \in \mathcal{D}(f),\ x > N \Rightarrow f(x) < -M.$$

Definition 4.4.14 $\lim\limits_{x \to -\infty} f(x) = L \Leftrightarrow \ldots$ (Exercise 2.)

Definition 4.4.15 $\lim\limits_{x \to -\infty} f(x) = +\infty \Leftrightarrow \mathcal{D}(f)$ is unbounded below, and

$$\forall M > 0, \exists N > 0 \ni \forall x \in \mathcal{D}(f),\ x < -N \Rightarrow f(x) > M.$$

Definition 4.4.16 $\lim\limits_{x \to -\infty} f(x) = -\infty \Leftrightarrow \ldots$ (Exercise 3.)

Example 4.4.17 Prove that $\lim\limits_{x \to +\infty} (5 - 4x) = -\infty$.

Solution: Let $M > 0$. Choose $N = \dfrac{M+5}{4}$. Then

$$\begin{aligned} x > N &\Rightarrow x > \frac{M+5}{4} \\ &\Rightarrow 4x > M + 5 \\ &\Rightarrow 4x - 5 > M \\ &\Rightarrow 5 - 4x < -M. \end{aligned}$$

Therefore, by Definition 4.4.13, $\lim\limits_{x \to +\infty} (5 - 4x) = -\infty$. $\square$

Theorem 4.4.18 (***Fundamental Limits***)

(a) $\forall n \in \mathbb{N}$, $\displaystyle\lim_{x \to +\infty} x^n = +\infty$;

(b) $\forall n \in \mathbb{N}$, *if n is even, then* $\displaystyle\lim_{x \to -\infty} x^n = +\infty$;

(c) $\forall n \in \mathbb{N}$, *if n is odd, then* $\displaystyle\lim_{x \to -\infty} x^n = -\infty$.

Proof. (a) Let n be a fixed natural number, and $M > 0$. Choose $N = M + 1$. Then

$$\begin{aligned} x > N &\Rightarrow x > 1 \text{ and } x > M \\ &\Rightarrow x^n > x \text{ and } x > M \\ &\Rightarrow x^n > M. \end{aligned}$$

Therefore, by Definition 4.4.12, $\displaystyle\lim_{x \to +\infty} x^n = +\infty$.

(b) Let n be a fixed **even** natural number, and $M > 0$. Then $\exists\, k \in \mathbb{N} \ni n = 2k$. Choose $N = M + 1$. Then

$$\begin{aligned} x < -N &\Rightarrow -x > N \\ &\Rightarrow |x| = -x > N > 1 \text{ and } |x| = -x > M \\ &\Rightarrow x^{2k} = |x|^{2k} > |x| > M \\ &\Rightarrow x^{2k} > M \\ &\Rightarrow x^n > M. \end{aligned}$$

Therefore, by Definition 4.4.15, $\displaystyle\lim_{x \to -\infty} x^n = +\infty$.

(c) Exercise 5. ■

The following theorem shows relationships between $\displaystyle\lim_{x \to 0} f(x)$ and $\displaystyle\lim_{x \to \infty} f\left(\frac{1}{x}\right)$ that are often useful.

Theorem 4.4.19 (a) $\displaystyle\lim_{x \to 0^+} f(x) = L$ *(finite) if and only if* $\displaystyle\lim_{x \to +\infty} f\left(\frac{1}{x}\right) = L$;

(b) $\displaystyle\lim_{x \to 0^-} f(x) = L$ *(finite) if and only if* $\displaystyle\lim_{x \to -\infty} f\left(\frac{1}{x}\right) = L$.

Proof. (a) Part 1 ($\Rightarrow$): Suppose $\lim\limits_{x\to 0^+} f(x) = L$. Let $\varepsilon > 0$. Then $\exists\,\delta > 0 \ni 0 < x < \delta \Rightarrow |f(x) - L| < \varepsilon$. Let $M = \dfrac{1}{\delta}$. Then

$$x > M \Rightarrow 0 < \frac{1}{x} < \frac{1}{M} = \delta \Rightarrow \left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$$

Therefore, $\lim\limits_{x\to+\infty} f\left(\dfrac{1}{x}\right) = L$.

Part 2 ($\Leftarrow$): Suppose $\lim\limits_{x\to+\infty} f\left(\dfrac{1}{x}\right) = L$. Then $\exists\,M > 0 \ni x > M \Rightarrow \left|f\left(\dfrac{1}{x}\right) - L\right| < \varepsilon$. Let $\delta = \dfrac{1}{M}$. Then $\delta > 0$ and

$$0 < x < \delta \Rightarrow x < \frac{1}{M} \Rightarrow \frac{1}{x} > M \Rightarrow |f(x) - L| < \varepsilon.$$

Therefore, $\lim\limits_{x\to 0^+} f(x) = L$.

To prove (b), modify the proof of (a) in the obvious ways. ■

Note: From Theorem 4.4.19 it follows that $\lim\limits_{x\to+\infty} \dfrac{1}{x} = 0$, because $\lim\limits_{x\to 0^+} x = 0$. Similarly, $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$, because $\lim\limits_{x\to 0^-} x = 0$.

Many other results like these follow from another, similar theorem. First, however, we make a useful definition.

Definition 4.4.20 (a) A **neighborhood of** $+\infty$ is any open interval of the form $(a, +\infty)$.

(b) A **neighborhood of** $-\infty$ is any open interval of the form $(-\infty, a)$.

Theorem 4.4.21 (a) *Suppose $f(x) > 0$ for all x in some neighborhood of $+\infty$. Then*

$$\lim_{x\to+\infty} f(x) = +\infty \Leftrightarrow \lim_{x\to+\infty} \frac{1}{f(x)} = 0.$$

(b) *Suppose $f(x) < 0$ for all x in some neighborhood of $+\infty$. Then*

$$\lim_{x\to+\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to+\infty} \frac{1}{f(x)} = 0.$$

(c) *Suppose $f(x) > 0$ for all x in some neighborhood of $-\infty$. Then*

$$\lim_{x\to-\infty} f(x) = +\infty \Leftrightarrow \lim_{x\to-\infty} \frac{1}{f(x)} = 0.$$

(d) *Suppose $f(x) < 0$ for all x in some neighborhood of $-\infty$. Then*

$$\lim_{x \to -\infty} f(x) = -\infty \Leftrightarrow \lim_{x \to -\infty} \frac{1}{f(x)} = 0.$$

Proof. Exercise 6. ■

Example 4.4.22 Prove that $\forall n \in \mathbb{N}$, $\lim\limits_{x \to \pm\infty} \dfrac{1}{x^n} = 0$.

Proof. In Theorem 4.4.18 (a) we proved that $\lim\limits_{x \to +\infty} x^n = +\infty$. Thus, $\lim\limits_{x \to +\infty} \dfrac{1}{x^n} = 0$ by Theorem 4.4.21 (a). The proof of $\lim\limits_{x \to -\infty} \dfrac{1}{x^n} = 0$ requires two cases: when n is even, and when n is odd. Use Theorem 4.4.18 (b), (c), and Theorem 4.4.21 (c), (d). (Exercise 7.) □

ALGEBRA OF LIMITS AT INFINITY

Theorem 4.4.23 *Suppose $\lim\limits_{x \to +\infty} f(x) = L$, $\lim\limits_{x \to +\infty} g(x) = M$, and $c \in \mathbb{R}$. Then*

(a) $\lim\limits_{x \to +\infty} (cf(x)) = cL$.

(b) $\lim\limits_{x \to +\infty} (f(x) \pm g(x)) = L \pm M$;

(c) $\lim\limits_{x \to +\infty} (f(x)g(x)) = LM$;

(d) $\lim\limits_{x \to +\infty} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$ *(provided $M \neq 0$).*

(e) ***Squeeze Principle:*** *if $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} g(x) = L$, and $\forall x$ in some neighborhood of $+\infty$, $f(x) \leq h(x) \leq g(x)$, then $\lim\limits_{x \to +\infty} h(x) = L$.*

(f) ***Limits Preserve Inequalities:*** *if $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to +\infty} g(x)$ exist and $f(x) \leq g(x)$ in some neighborhood of $+\infty$, then $\lim\limits_{x \to +\infty} f(x) \leq \lim\limits_{x \to +\infty} g(x)$.*

The above results remain true if $+\infty$ is replaced by $-\infty$. They also remain true if L and M are replaced by $+\infty$ or $-\infty$, in the sense described by Tables 4.1, 4.2, and 4.3 of this section.

Proof. Exercise 17 (Project). ■

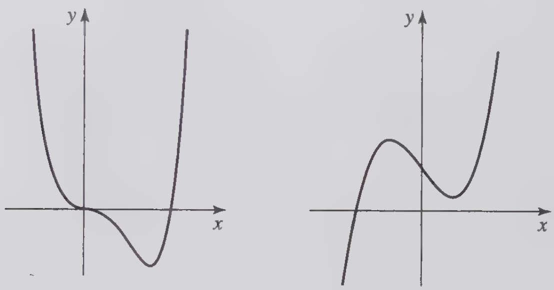
Theorem 4.4.24 (***Limits of Polynomials at*** $\pm\infty$) *Let* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *denote a polynomial. Then*

$$\lim_{x\to+\infty} p(x) = \begin{cases} +\infty \text{ if } a_n > 0; \\ -\infty \text{ if } a_n < 0. \end{cases}$$

$$\text{If } n \text{ is even,} \quad \lim_{x\to-\infty} p(x) = \begin{cases} +\infty \text{ if } a_n > 0; \\ -\infty \text{ if } a_n < 0. \end{cases}$$

$$\text{If } n \text{ is odd,} \quad \lim_{x\to-\infty} p(x) = \begin{cases} -\infty \text{ if } a_n > 0; \\ +\infty \text{ if } a_n < 0. \end{cases}$$

Proof. Exercise 18. ■



n even
$f(x) = x^4 - 2x^3$

n odd
$f(x) = x^3 - 3x + 3$

Figure 4.9

RATIONAL FUNCTIONS AND HORIZONTAL ASYMPTOTES

Definition 4.4.25 The graph of a function f has a horizontal line $y = m$ as a **horizontal asymptote** if $\lim_{x\to+\infty} f(x) = m$ or $\lim_{x\to-\infty} f(x) = m$.

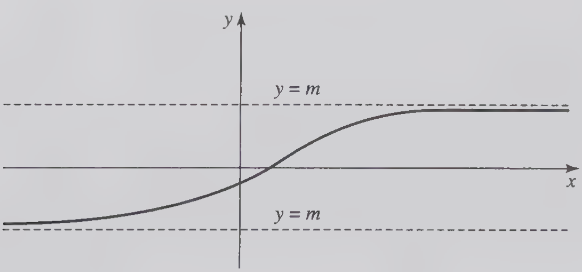


Figure 4.10

Theorem 4.4.26 (***Horizontal Asymptotes of Rational Functions***) *Consider the rational function*

$$R(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}.$$

(a) *If $n > m$, then the graph of $R(x)$ has no horizontal asymptotes;*

(b) *If $n = m$, then the line $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote;*

(c) *If $n < m$, then the x-axis is a horizontal asymptote.*

Proof. Exercise 19. ■

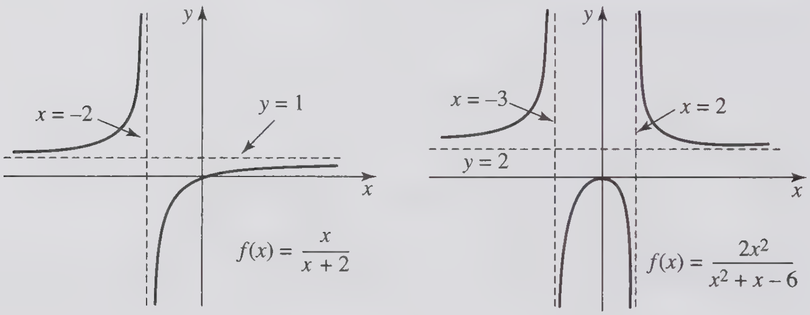


Figure 4.11

EXERCISE SET 4.4-B

1. Complete Definition 4.4.12.

2. Complete Definition 4.4.14.

3. Complete Definition 4.4.16

4. Use Definitions 4.4.11–4.4.16 to prove the following limit statements:

(a) $\displaystyle\lim_{x\to+\infty} \frac{3x-1}{6x+5} = \frac{1}{2}$ (b) $\displaystyle\lim_{x\to+\infty} \frac{3x^2+2x-1}{x+4} = +\infty$

(c) $\displaystyle\lim_{x\to-\infty} \frac{1-x^2}{x+2} = +\infty$ (d) $\displaystyle\lim_{x\to-\infty} \frac{x-1}{x+1} = 1$

(e) $\displaystyle\lim_{x\to+\infty} \frac{1-x^2}{1+x} = -\infty$ (f) $\displaystyle\lim_{x\to-\infty} \frac{x^2+x-2}{x+2} = -\infty$

5. Prove Theorem 4.4.18 (c).

6. Prove Theorem 4.4.21.

7. Complete the proof of Example 4.4.22.

8. State and prove a **sequential criterion** for $\lim_{x\to+\infty} f(x) = L$ (or $\pm\infty$) and a **sequential criterion** for $\lim_{x\to-\infty} f(x) = L$ (or $\pm\infty$).

9. Investigate each of the following:

   (a) $\lim_{x\to\infty} x\sin x$ (b) $\lim_{x\to\infty} \frac{1}{x}\sin x$ (c) $\lim_{x\to\infty} \frac{1}{x}\sin\frac{1}{x}$

10. Suppose f is defined on a neighborhood of $+\infty$ and $\lim_{x\to\infty} xf(x) = L \in \mathbb{R}$. Prove that $\lim_{x\to\infty} f(x) = 0$.

11. Suppose f and g are defined on a neighborhood of $+\infty$, g is positive on this interval, and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \in \mathbb{R}$. Prove:

    (a) if $L > 0$, then $\lim_{x\to\infty} f(x) = +\infty \Leftrightarrow \lim_{x\to\infty} g(x) = +\infty$;

    (b) if $L < 0$, then $\lim_{x\to\infty} f(x) = -\infty \Leftrightarrow \lim_{x\to\infty} g(x) = +\infty$.

12. Apply Theorem 4.4.24 to find each of the following:

    (a) $\lim_{x\to+\infty} \left(5x^6 - 12x^5 + 2x^3 - 87\right)$ $\lim_{x\to-\infty} \left(5x^6 - 12x^5 + 2x^3 - 87\right)$

    (b) $\lim_{x\to+\infty} \left(13x^7 + 8x^4 - 7x^3 + 35\right)$ $\lim_{x\to-\infty} \left(13x^7 + 8x^4 - 7x^3 + 35\right)$

    (c) $\lim_{x\to+\infty} \left(9 - x^2 + 4x^3 - 7x^{11}\right)$ $\lim_{x\to-\infty} \left(9 - x^2 + 4x^3 - 7x^{11}\right)$

13. Apply Theorem 4.4.26 to find the horizontal asymptote(s) for the graph of each of the following rational functions:

    (a) $f(x) = \dfrac{x+2}{3x-1}$ (b) $f(x) = \dfrac{x^2 - 3x + 1}{x+8}$

    (c) $f(x) = \dfrac{7x-5}{4x^2+3x-7}$ (d) $f(x) = \dfrac{1-9x}{x^2+4}$

    (e) $f(x) = \dfrac{x^3-5x}{4x^2+1}$ (f) $f(x) = \dfrac{6x^4+13x^2}{11-x^4}$

14. Prove that Theorem 4.4.19 remains true if L is $+\infty$ or $-\infty$.

15. Prove the following **monotone convergence theorem**: if f is monotone increasing on a neighborhood of $+\infty$, say (a,∞), then $\lim_{x\to\infty} f(x)$ exists $\Leftrightarrow$ f is bounded above on (a,∞); and in this case, $\lim_{x\to\infty} f(x) = \sup f(a,\infty)$. State similar results for monotone decreasing functions, and for $x \to -\infty$.

16. **Cauchy Criterion for Limits of Functions at Infinity:** Suppose $\mathcal{D}(f)$ is unbounded above. Prove that $\lim_{x\to\infty} f(x)$ exists iff $\forall\varepsilon > 0$, $\exists N > 0 \ni \forall x, y \in \mathcal{D}(f)$, $x, y > N \Rightarrow |f(x) - f(y)| < \varepsilon$.

    State a similar theorem for $\lim_{x\to-\infty} f(x)$.

(Project) The Algebra of Limits at Infinity:

17. Prove Theorem 4.4.23.

18. Prove Theorem 4.4.24.

19. Prove Theorem 4.4.26.

Chapter 5

Continuous Functions

> Sections 5.1–5.3 are among the most important in the entire book. The ideas discussed here, especially in Section 5.3, are quite powerful. Section 5.4 through Theorem 5.4.7, is also important, but in a one-semester course can be delayed until it is needed in Chapter 7. Section 5.5 is optional, but it does contain a complete treatment of the Cantor function. Cover Section 5.6 only if you want a rigorous "early" treatment of exponential and logarithm functions. Section 5.7 requires advanced mathematical maturity.

The concept of continuous functions is introduced in a typical freshman calculus course, but is not investigated in depth there because applied concepts are regarded as more important in that course. However, the concept of continuity is of great importance in analysis. Briefly, you will recall the *intuitive* notion of continuity: A function is "continuous" if you can draw its graph without lifting your pencil from the paper. This definition is obviously too vague for rigorous mathematical purposes. In beginning calculus, an attempt is made at being more rigorous: there, a function is defined to be **continuous at a point** x_0 if three conditions hold:

(a) $f(x_0)$ exists;

(b) $\lim\limits_{x \to x_0} f(x)$ exists; and

(c) $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

The definition we are going to use is equivalent to these three conditions only when x_0 is a cluster point of the domain of f. It is stated as an ε-δ criterion.

5.1 Continuity of a Function at a Point

Definition 5.1.1 (Continuous Function at a Point) Suppose $f : \mathcal{D}(f) \to \mathbb{R}$, and $x_0 \in \mathcal{D}(f)$. Then f is **continuous at** x_0 if

$$\boxed{\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in \mathcal{D}(f), |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.}$$

If f is not continuous at x_0 then we say that f is **discontinuous** at x_0.

Notes: (1) In Definition 5.1.1, x_0 must be in $\mathcal{D}(f)$ but need not be a cluster point of $\mathcal{D}(f)$. Thus, $\lim\limits_{x \to x_0} f(x)$ need not exist, even when f is continuos at x_0. (See Exercises 1 and 3.)

(2) In case x_0 is a cluster point of $\mathcal{D}(f)$, Definition 5.1.1 is equivalent to conditions (a)–(c) given above (See Exercise 2). In fact, we usually combine (a)–(c) into one statement, and say:

If x_0 is a cluster point of $\mathcal{D}$(f), then f is continuous at x_0 iff $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

(3) Suppose x_0 is a cluster point of $\mathcal{D}(f)$. Calculating $\lim\limits_{x \to x_0} f(x)$ is trivial if f is continuous at x_0. Continuity of f at x_0 means that to calculate $\lim\limits_{x \to x_0} f(x)$ we merely "plug in" $x = x_0$. In a sense, this suggests that limits are not of much interest in studying functions that are known to be continuous at a point; limits will often be of more interest at points where a function is *discontinuous.* But this definition also suggests that we can use limits to determine whether a function is continuous at a point.

Students frequently ask for an example that shows how to use Definition 5.1.1 to prove that a function is continuous at a point. For them we include the following example. You will see right away that there is nothing more to this example than showing that $\lim\limits_{x \to 2} f(x) = f(2)$. In truth, however, this example serves another purpose: to show the difference between ordinary continuity and another type of continuity known as *uniform* continuity, to be introduced in Section 5.4. The details will be seen in Example 5.4.2.

Example 5.1.2 Use Definition 5.1.1 to prove that the function $f(x) = 3x^2 - 2x - 1$ is continuous at $x_0 = 2$.

Solution. Let $f(x) = 3x^2 - 2x - 1$. Note that $f(2) = 7$.
(a) **Scratchwork**: Let $\varepsilon > 0$.
We must find $\delta > 0 \ni |x - 2| < \delta \Rightarrow |(3x^2 - 2x - 1) - 7| < \varepsilon$. i.e.,

$$|3x^2 - 2x - 8| < \varepsilon, \text{ or } |3x + 4||x - 2| < \varepsilon.$$

If $|x-2|<1$, then $-1<x-2<1$, so $1<x<3$, and so

$$\begin{aligned} 3 &< 3x < 9 \\ 7 &< 3x+4 < 13 \\ |3x+4| &< 13. \end{aligned}$$

Thus, we want to make sure that $\delta \leq 1$ and $\delta \leq \dfrac{\varepsilon}{13}$.

(b) **Proof**: Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \dfrac{\varepsilon}{13}\right\}$. Then

$\forall \varepsilon > 0, \exists \delta > 0$
$\Rightarrow |f($

$$\begin{aligned} |x-2|<\delta &\Rightarrow |x-2|<1 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow -1<x-2<1 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow 1<x<3 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow 3<3x<9 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow 7<3x+4<13 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow |3x+4|<13 \text{ and } |x-2|<\frac{\varepsilon}{13} \\ &\Rightarrow |3x+4||x-2|<13\cdot\frac{\varepsilon}{13} \\ &\Rightarrow |3x^2-2x-8|<\varepsilon \\ &\Rightarrow |(3x^2-2x-1)-7|<\varepsilon. \end{aligned}$$

δ ε

Therefore, the function $f(x) = 3x^2 - 2x - 1$ is continuous at $x_0 = 2$. □

As we have already suggested, sequences play a significant role in virtually all areas of real analysis. The topic of continuity is no exception.

Theorem 5.1.3 (*Sequential Criterion for Continuity of f at x_0*)
A function $f : \mathcal{D}(f) \to \mathbb{R}$ is continuous at a point $x_0 \in \mathcal{D}(f)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) \ni x_n \to x_0$, $f(x_n) \to f(x_0)$.

Proof. Exercise 6. (Compare with Theorem 4.1.9.) ■

ε

Corollary 5.1.4 (*Sequential Criterion for Discontinuity of f at x_0*)
A function $f : \mathcal{D}(f) \to \mathbb{R}$, is ***discontinuous*** *at a point $x_0 \in \mathcal{D}(f)$ iff $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) \ni x_n \to x_0$, but $\{f(x_n)\}$ does not converge to $f(x_0)$.*

Example 5.1.5 The **signum function**, $\textbf{sgn}(x) = \left\{\begin{array}{ll} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array}\right\}$, is discontinuous at $x = 0$.

Proof. Consider the sequence $\{\frac{1}{n}\}$. Observe that $\frac{1}{n} \to 0$ and $sgn\left(\frac{1}{n}\right) = 1 \to 1 \neq sgn(0)$. That is, $sgn\left(\frac{1}{n}\right) \not\to sgn(0)$. Thus, by Corollary 5.1.4 above, sgn is discontinuous at 0. □

Definition 5.1.6 A function $f : \mathcal{D}(f) \to \mathbb{R}$ is **continuous on a set** $A \subseteq \mathcal{D}(f)$ if it is continuous at every point of A. If f is continuous on $\mathcal{D}(f)$ we say that f is **continuous everywhere on its domain**, or simply, $f : \mathcal{D}(f) \to \mathbb{R}$ **is continuous**. If $f : \mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$, we say that f is **continuous everywhere.**

Theorem 5.1.7 ***Polynomial functions*** *are continuous everywhere.*

Proof. Exercise 7. ■

Theorem 5.1.8 *A* ***rational function*** $R(x) = \dfrac{p(x)}{q(x)}$, *where* $p(x)$ *and* $q(x)$ *are polynomials, is continuous everywhere on its domain [i.e., at every real number* x_0 *for which* $q(x_0) \neq 0$].

Proof. Exercise 8. ■

Examples 5.1.9 (a) The absolute value function $f(x) = |x|$ is continuous everywhere.

(b) The square root function $f(x) = \sqrt{x}$ is continuous everywhere on its domain $[0, +\infty)$.

Proof. Exercise 9. □

Examples 5.1.10 (a) The function $f(x) = \dfrac{2x^2 - 18}{x - 3}$ is **not** continuous at $x = 3$ but

(b) The function $g(x) = \left\{ \begin{array}{ll} \dfrac{2x^2 - 18}{x - 3} & \text{if } x \neq 3 \\ 12 & \text{if } x = 3 \end{array} \right\}$ **is** continuous at $x = 3$.

Proof. (a) Since $f(3)$ does not exist, f is not continuous at $x = 3$.

(b) In Example 4.1.4, we saw that $\lim\limits_{x\to 3} g(x) = 12$. By definition, $g(3) = 12$. Thus, $\lim\limits_{x\to 3} g(x) = g(3)$, and therefore, g is continuous at $x = 3$. □

Example 5.1.11 (A Function That Is Continuous Nowhere)

The **Dirichlet function** $f(x) = \left\{ \begin{array}{l} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{array} \right\}$ is discontinuous everywhere.

Proof. Let $x_0 \in \mathbb{R}$. For contradiction, suppose f is continuous at x_0. Since the rational numbers are dense[1] in $\mathbb{R}$, $\exists$ sequence $\{x_n\}$ of rational numbers such that $x_n \to x_0$. Since the irrational numbers are also dense in $\mathbb{R}$, $\exists$ sequence $\{y_n\}$ of irrational numbers such that $y_n \to x_0$. But f is continuous at x_0. Thus, by the sequential criterion,

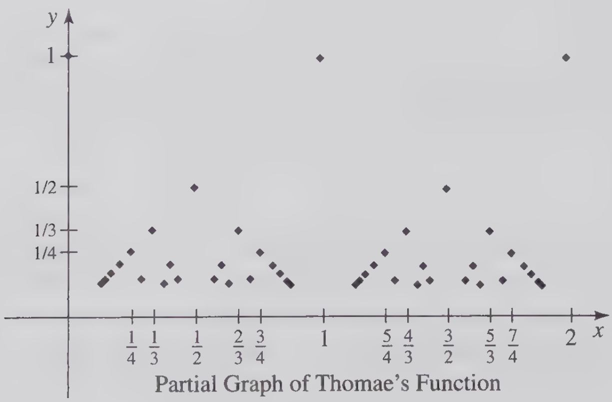
$$f(x_n) \to f(x_0), \text{ and}$$
$$f(y_n) \to f(x_0).$$

Now, $\forall n \in \mathbb{N}$, $f(x_n) = 1$ and $f(y_n) = 0$. Thus, $f(x_0) = 1$ and $f(x_0) = 0$. Contradiction. Therefore, f is not continuous at x_0. Since x_0 is a perfectly general real number, we have proved that f is discontinuous at every real number.
$\square$

Example 5.1.12 Thomae's Function (A function that is continuous at every irrational number and discontinuous at every rational number) The following function, known as **Thomae's function**, is extremely interesting:

$$T(x) = \left\{ \begin{array}{ll} \frac{1}{n} & \text{if } x = \frac{m}{n} \neq 0, \text{ where } m \in \mathbb{Z}, n \in \mathbb{N} \text{ and} \\ & m \text{ and } n \text{ have no common prime factors} \\ 1 & \text{if } x = 0 \\ 0 & \text{if } x \text{ is irrational.} \end{array} \right\}$$

Thomae's function $T(x)$ is continuous at every irrational number, but is discontinuous at every rational number.



Partial Graph of Thomae's Function

Figure 5.1

1. See Theorems 2.3.6 and 3.2.21.

Proof. (a) Suppose x is an irrational number. Then $T(x) = 0$.

Let $n \in \mathbb{N}$. Since x is not rational, $\not\exists m \in \mathbb{Z} \ni x = \dfrac{m}{n}$. For each $n \in \mathbb{N}$, let $\delta_n =$ the distance from x to the nearest rational number of the form $\dfrac{m}{n}$. Then $N_{\delta_n}(x)$ contains no rational number with denominator n.

Let $\varepsilon > 0$. By the Archimedean property, $\exists n_0 \in \mathbb{N} \ni \dfrac{1}{n_0} < \varepsilon$. Choose $\delta = \min\{\delta_1, \delta_2, \cdots, \delta_{n_0}\}$. Then $N_\delta(x)$ contains no rational number with denominator $\leq n_0$.

Consider arbitrary $y \in N_\delta(x)$.

(i) Suppose y is rational, say $y = \dfrac{m}{n} \neq 0$ (where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and m and n have no common prime factors). Then $T(y) = \dfrac{1}{n}$, where $n > n_0$, so $|T(y)| = \dfrac{1}{n} < \dfrac{1}{n_0} < \varepsilon$.

(ii) If y is irrational, then $T(y) = 0$.

In either case, (i) or (ii),

$$\begin{aligned} y \in N_\delta(x) &\Rightarrow |T(y) - 0| < \varepsilon \\ &\Rightarrow |T(y) - T(x)| < \varepsilon. \end{aligned}$$

Therefore, T is continuous at x, an arbitrary irrational number.

(b) Let x be a rational number. Then $T(x) \neq 0$. However, since the irrationals are dense in $\mathbb{R}$, there exists a sequence $\{z_n\}$ of irrational numbers such that $z_n \to x$. By the sequential criterion for continuity at x, if T were continuous at x, then $T(z_n) \to T(x)$. But $\forall n \in \mathbb{N}$, $T(z_n) = 0$. This would imply that $T(x) = 0$, which would contradict $T(x) \neq 0$. Therefore, T cannot be continuous at x. ■

Theorem 5.1.13 (***Algebra of Continuous Functions***) *Suppose f and g are continuous at a point x_0 and $c \in \mathbb{R}$. Then,*

(a) *cf is continuous at x_0;*

(b) *$f + g$ is continuous at x_0;*

(c) *$f - g$ is continuous at x_0;*

(d) *$f \cdot g$ is continuous at x_0;*

(e) *$\dfrac{1}{g}$ is continuous at x_0, if $g(x_0) \neq 0$;*

(f) $\dfrac{f}{g}$ *is continuous at* x_0, *if* $g(x_0) \neq 0$.

Proof. Exercise 14. ∎

Theorem 5.1.14 (*Composite Functions*)

(a) *Suppose* f *is continuous at* x_0 *and* g *is continuous at* $f(x_0)$. *Then the composite function* $g \circ f$ *is continuous at* x_0.

(b) *Suppose* $\lim\limits_{x \to x_0} f(x) = y_0 \in \mathcal{D}(g)$, *and* g *is continuous at* y_0. *Then*

$$\lim_{x \to x_0} g(f(x)) = g\left(\lim_{x \to x_0} f(x)\right) = g(y_0).$$

(c) *Same as (b), with* $x \to x_0$ *replaced by* $x \to +\infty$ *(or* $-\infty$*).*

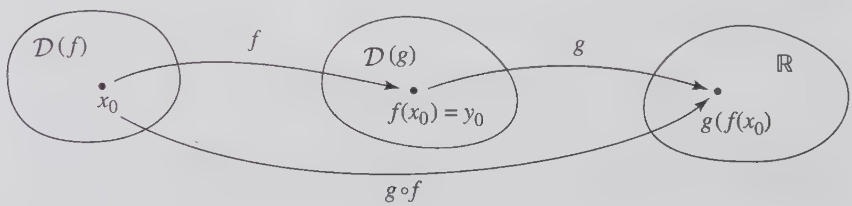


Figure 5.2

Proof. (a) Suppose f is continuous at x_0 and g is continuous at $f(x_0)$. Let $\varepsilon > 0$. Since g is continuous at $f(x_0)$, $\exists\, \delta > 0 \ni \forall y \in \mathcal{D}(g)$,

$$|y - f(x_0)| < \delta \Rightarrow |g(y) - g(f(x_0))| < \varepsilon. \qquad (1)$$

Since f is continuous at x_0, $\exists\, \delta' > 0 \ni \forall x \in \mathcal{D}(g \circ f)$,

$$\begin{aligned} |x - x_0| < \delta' &\Rightarrow |f(x) - f(x_0)| < \delta \\ &\Rightarrow |g(f(x)) - g(f(x_0))| < \varepsilon \quad \text{by (1)} \\ &\Rightarrow |(g \circ f)(x) - (g \circ f)(x_0)| < \varepsilon. \end{aligned}$$

Therefore, $g \circ f$ is continuous at x_0.

(b) Suppose $\lim\limits_{x \to x_0} f(x) = y_0 \in \mathcal{D}(g)$, and g is continuous at y_0. Let $\varepsilon > 0$. Since g is continuous at y_0, $\exists\, \delta > 0 \ni \forall y \in \mathcal{D}(g)$, $|y - y_0| < \delta \Rightarrow |g(y) - g(y_0)| < \varepsilon$. Since $\lim\limits_{x \to x_0} f(x) = y_0$, $\exists\, \delta' > 0 \ni \forall x \in \mathcal{D}(g \circ f)$,

$$\begin{aligned} 0 < |x - x_0| < \delta' &\Rightarrow |f(x) - y_0| < \delta \\ &\Rightarrow |g(f(x)) - g(y_0)| < \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to x_0} g(f(x)) = g(y_0) = g\left(\lim_{x \to x_0} f(x)\right)$.

(c) Exercise 15. ■

Corollary 5.1.15 *Suppose f and g are continuous at a point $x_0 \in \mathbb{R}$. Then*

(a) $\sqrt{f}$ *is continuous at* x_0 *(assuming $f(x) \geq 0$ in some nbd. of x_0);*

(b) $|f|$ *is continuous at* x_0;

(c) $\max\{f, g\}$[2] *is continuous at* x_0;

(d) $\min\{f, g\}$ *is continuous at* x_0.

Proof. Exercises 16 and 17. ■

TRIGONOMETRIC FUNCTIONS

We shall not define $\sin x$ and $\cos x$ here. That is a task for later chapters.[3] After the definition is given it will be routine to prove that $\forall x \in \mathbb{R}$,

$$|\sin x| \leq |x| \text{ and } |\cos x| \leq 1.$$

Also, it will be possible to establish the identity

$$\forall x, y \in \mathbb{R}, \quad \sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right).$$

Combining the above inequalities and identity, we have the identity

$$\begin{aligned} |\sin x - \sin y| &= 2\left|\sin\left(\frac{x-y}{2}\right)\right|\left|\cos\left(\frac{x+y}{2}\right)\right| \\ &\leq 2 \cdot \left|\frac{x-y}{2}\right| \cdot 1 \\ &= |x - y|. \end{aligned}$$

2. For definitions of $\max\{f, g\}$ and $\min\{f, g\}$ see Definition B.3.1 in Appendix B.

3. See Sections 7.7 and 8.8.

Thus (see Exercise 18) we conclude that

$$\boxed{\text{The sine function is continuous everywhere.}}$$

Similarly, after $\sin x$ and $\cos x$ are defined it will be possible to prove that $\forall x \in \mathbb{R}$, $|\sin x| \leq 1$, and that

$$\forall x, y \in \mathbb{R}, \quad \cos x - \cos y = 2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{y-x}{2}\right).$$

Reasoning as we did above, we use these identities to show that

$$|\cos x - \cos y| \leq 2 \cdot 1 \cdot \left|\frac{x-y}{2}\right| = |x-y|, \text{ and thus (see Exercise 18),}$$

$$\boxed{\text{The cosine function is continuous everywhere.}}$$

Now the remaining trigonometric functions are defined by

$$\boxed{\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}.}$$

By the "algebra of continuous functions" theorem, these functions are continuous wherever they exist. The following theorem summarizes these results.

Theorem 5.1.16 *The six trigonometric functions are continuous everywhere on their domains.*

Proof. For $\tan x$, $\cot x$, $\sec x$, and $\csc x$, see Exercise 20. ∎

EXERCISE SET 5.1

1. Prove that if x_0 is an isolated point of $\mathcal{D}(f)$, then f is continuous at x_0 but $\lim_{x \to x_0} f(x)$ does not exist. What does this tell you about the function $f(x) = \sqrt{x^3 - x^2}$?

2. Prove that if x_0 is a cluster point of $\mathcal{D}(f)$, then f is continuous at x_0 iff
$$\begin{cases} \text{(a) } f(x_0) \text{ exists;} \\ \text{(b) } \lim_{x \to x_0} f(x) \text{ exists; and} \\ \text{(c) } \lim_{x \to x_0} f(x) = f(x_0). \end{cases}$$

3. Sketch the graph of $f(x) = \sqrt{x^3 + 2x^2 + x}$ on the interval $[-2, 2]$. Is f continuous at -1? Is $\lim_{x \to -1} f(x) = f(-1)$? Justify your answers, and reconcile them with the claims made in Exercises 1 and 2.

4. Use Definition 5.1.1 to prove that the function $f(x) = 2x^2 + 3x + 5$ is continuous at $x_0 = -1$.

5. Use Definition 5.1.1 to prove that the function $f(x) = 4x^2 - 5x - 3$ is continuous at $x_0 = 2$.

6. Prove Theorem 5.1.3.

7. Prove Theorem 5.1.7.

8. Prove Theorem 5.1.8.

9. Prove the claims made in Example 5.1.9 (a). [See Theorems 4.2.1 and 4.2.11.]

10. Prove that the **signum function** defined in Example 5.1.5 is continuous everywhere except at $x = 0$.

11. Let $f(x) = \left\{ \begin{array}{ll} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ and $g(x) = \left\{ \begin{array}{ll} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$. Which one or more of these functions is (or are) continuous at $x = 0$? Justify your answer. [See Examples 4.1.12 and 4.2.21.]

12. Consider the function $f(x) = \left\{ \begin{array}{ll} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational} \end{array} \right\}$. Prove that f is continuous *only* at 0. [See Exercise 4.2.18.]

13. Use Theorems 5.1.13–5.1.16 to determine the intervals over which the following functions are continuous:

    (a) $f(x) = \sqrt{x^2 + 5}$ (b) $f(x) = \sqrt{3x - 2}$

    (c) $f(x) = \dfrac{x - 1}{x^2 + 2x - 3}$ (d) $f(x) = \dfrac{x}{x^2 + 1}$

    (e) $f(x) = \dfrac{3x + 5}{x^2 - 4}$ (f) $f(x) = \sin\left(\dfrac{x + 2}{x - 2}\right)$

    (g) $f(x) = \cos\sqrt{x}$ (h) $f(x) = \sqrt{\cos x}$

    (i) $f(x) = \tan x$ (j) $f(x) = \tan(\sin x)$

14. Prove Theorem 5.1.13. [Hint: See how Theorem 4.2.11 was proved, or use the sequential criterion.]

15. Prove Theorem 5.1.14 (c).

16. Prove Corollary 5.1.15, Parts (a) and (b). [Hint: use Example 5.1.9 and Theorem 5.1.14.]

17. Prove Corollary 5.1.15, Parts (c) and (d). [See Exercise 1.2-B.6.]

18. Suppose that $\exists K > 0 \ni \forall x, y \in \mathcal{D}(f)$, $|f(x) - f(y)| \leq K|x - y|$. Prove that f is continuous everywhere on its domain.

19. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point of $\mathbb{R}$, but such that $|f|$ is continuous everywhere on $\mathbb{R}$.

20. Use the "algebra of continuous functions" theorem to prove Theorem 5.1.16: $\tan x$, $\cot x$, $\sec x$, and $\csc x$ are continuous everywhere on their domains.

21. Give examples for each of the following:

    (a) functions $f, g : \mathbb{R} \to \mathbb{R}$, which are discontinuous at every point of $\mathbb{R}$, but such that $f + g$ is continuous everywhere on $\mathbb{R}$.

    (b) functions $f, g : \mathbb{R} \to \mathbb{R}$, which are discontinuous at every point of $\mathbb{R}$, but such that fg is continuous everywhere on $\mathbb{R}$.

22. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $a < b$. Prove that $f^{-1}(a, b)$ must be open and $f^{-1}[a, b]$ must be closed, but $f(a, b)$ need not be open. (In Theorem 5.3.6 we shall prove that $f[a, b]$ must be closed.)

23. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$. Prove that $\forall a \in \mathbb{R}$,

    (a) the sets $f^{-1}(-\infty, a) = \{x : f(x) < a\}$ and $f^{-1}(a, +\infty)$ are open;

    (b) the sets $f^{-1}(-\infty, a] = \{x : f(x) \leq a\}$ and $f^{-1}[a, +\infty)$ are closed;

    (c) the set $f^{-1}(a) = \{x : f(x) = a\}$ is closed.

24. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$. Prove that

    (a) the set $\{x : f(x) = g(x)\}$ is closed;

    (b) the set $\{x : f(x) \leq g(x)\}$ is closed;

    (c) the set $\{x : f(x) < g(x)\}$ is open.

25. **Local Boundedness Property:** Prove that if f is continuous at x_0 then f is **locally bounded at** x_0 (i.e., bounded on some neighborhood of x_0). That is, $\exists \varepsilon > 0$ and $\exists M > 0 \ni \forall x \in \mathcal{D}(f) \cap N_\varepsilon(x_0)$, $|f(x)| \leq M$.

26. **Neighborhood Inequality Property of Continuous Functions:** Prove that if f is continuous and positive at x_0, then f is positive in some neighborhood of x_0. In fact, if f is continuous at x_0 and $f(x_0) > c$, then $\exists$ neighborhood $N_\delta(x_0) \ni \forall x \in \mathcal{D}(f) \cap N_\delta(x_0)$, $f(x) > c$. State and prove an analogous result for f continuous at x_0 and $f(x_0) < c$.

27. **Open Set Definition of Continuous:** Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous everywhere on $\mathbb{R}$ iff for every open set U, $f^{-1}(U)$ is open. That is, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous (everywhere) iff the inverse image of every open set is open.

28. Suppose $f : A \to \mathbb{R}$ is continuous on A and $f(x) = 0$ for all x in a dense subset of A. Prove that $f(x) = 0$ for *all* x in A. [Hint: use sequences.]

    For example, if $f : A \to \mathbb{R}$ is continuous on A and $f(x) = 0$ for all rational numbers in A, then $f(x) = 0$ *everywhere* on A.

29. Suppose f and g are continuous on a set A and $f(x) = g(x)$ for all x in a dense subset of A. Prove that $f(x) = g(x)$ for *all* x in A.

    For example, if f and g are continuous on a set A and $f(x) = g(x)$ for all rational numbers in A, then $f(x) = g(x)$ *everywhere* on A.

30. Revise the proof given in Example 5.1.12 to prove that **Thomae's function** has limit 0 at every real number.

31. Find a function f and sets $A, B \subseteq \mathbb{R}$ such that $f : A \to \mathbb{R}$ is continuous and $f : B \to \mathbb{R}$ is continuous, but $f : A \cup B \to \mathbb{R}$ is not continuous.

32. **(Project) "Additive" Functions:** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **additive** if $\forall x, y \in \mathbb{R}$, $f(x + y) = f(x) + f(y)$. Suppose f is additive.

    (a) Prove that $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}$, $f(nx) = nf(x)$.

    (b) Prove that $\forall n \in \mathbb{Z}, \forall x \in \mathbb{R}$, $f(nx) = nf(x)$.

    (c) Prove that $\forall r \in \mathbb{Q}, \forall x \in \mathbb{R}$, $f(rx) = rf(x)$.

    (d) Prove that if an additive function $f : \mathbb{R} \to \mathbb{R}$ is continuous at one point $x_0 \in \mathbb{R}$, then it must be continuous at every $x \in \mathbb{R}$.

    (e) Prove that if f is continuous on $\mathbb{R}$, then $\forall c \in \mathbb{R}, \forall x \in \mathbb{R}$, $f(cx) = cf(x)$. [This means that any continuous, additive function must be "linear" in the sense in which that word is used in a linear algebra course.]

    (f) Caution: An additive function need not be continuous, but a non-continuous additive function must be wildly pathological. The construction of such a function is beyond the scope of this book, but can be found in Boas [16]. The graph of such a function must be "dense" in the plane.

33. **(Project)** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\forall x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$, and that f is not the "zero" function. Prove that

    (a) $f(0) = 1$ but there is no x such that $f(x) = 0$;

    (b) $\forall x, y \in \mathbb{R}$, $f(x - y) = f(x)/f(y)$;

(c) $\forall r \in \mathbb{Q}, \forall x \in \mathbb{R}$, $f(rx) = [f(x)]^r$.

(d) If f is continuous at 0, then it is continuous everywhere.

5.2 Discontinuities and Monotone Functions

Definition 5.2.1 (Continuity from the Left at a Point) Suppose $f : \mathcal{D}(f) \to \mathbb{R}$, and $x_0 \in \mathcal{D}(f)$. Then f is **continuous from the left at** x_0 if $\forall \varepsilon > 0\ \exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $x_0 - \delta < x < x_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Definition 5.2.2 (Continuity from the Right at a Point) Suppose $f : \mathcal{D}(f) \to \mathbb{R}$, and $x_0 \in \mathcal{D}(f)$. Then f is **continuous from the right at** x_0 if $\forall \varepsilon > 0\ \exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $x_0 < x < x_0 + \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Notes: (1) In Definitions 5.2.1 and 5.2.2, x_0 need not be a cluster point of $\mathcal{D}(f)$ but must be in $\mathcal{D}(f)$.

(2) In case x_0 is a cluster point of $\mathcal{D}(f) \cap (-\infty, x_0)$, then Definition 5.2.1 is equivalent to saying that f is **continuous from the left** at x_0 if

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = f(x_0).$$

If x_0 is a cluster point of $\mathcal{D}(f) \cap (x_0, +\infty)$, then Definition 5.2.2 is equivalent to saying that f is **continuous from the right** at x_0 if

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

(3) f is said to have **one-sided continuity** at x_0 if it is either continuous from the left at x_0 or continuous from the right at x_0.

Example 5.2.3 Consider the function $f(x) = \left\{ \begin{array}{ll} -1 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{array} \right\}$. (See Figure 5.3.)

This function is continuous from the left at 2, since $\lim_{x \to 2^-} f(x) = -1 = f(2)$. However, this function is not continuous from the right at 2, since $\lim_{x \to 2^+} f(x) = 1 \neq f(2)$. $\square$

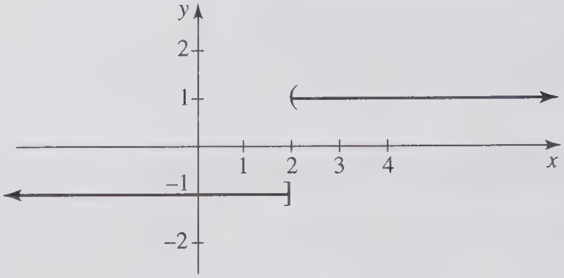


Figure 5.3

Theorem 5.2.4 (***Sequential Criterion for One-Sided Continuity***)

(a) *A function* $f : \mathcal{D}(f) \to \mathbb{R}$, *is continuous from the left at a point* $x_0 \in \mathcal{D}(f)$ *iff* $\forall$ *sequences* $\{x_n\}$ *in* $\mathcal{D}(f) \cap (-\infty, x_0) \ni x_n \to x_0$, $f(x_n) \to f(x_0)$.

(b) *A function* $f : \mathcal{D}(f) \to \mathbb{R}$, *is continuous from the right at a point* $x_0 \in \mathcal{D}(f)$ *iff* $\forall$ *sequences* $\{x_n\}$ *in* $\mathcal{D}(f) \cap (x_0, +\infty) \ni x_n \to x_0$, $f(x_n) \to f(x_0)$.

Proof. Exercise 1. (Compare with Theorem 5.1.3) ∎

Theorem 5.2.5 *Suppose* $f : \mathcal{D}(f) \to \mathbb{R}$, *and* $x_0 \in \mathcal{D}(f)$. *Then* f *is* ***continuous at*** x_0 *iff* f *is continuous from the left at* x_0 *and continuous from the right at* x_0.

Proof. Exercise 2. ∎

Example 5.2.6 The function $f(x) = \left\{ \begin{array}{ll} -1 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{array} \right\}$ described in Example 5.2.3 is not continuous at 2, since it is not continuous from the right at 2.

SOME TYPES OF DISCONTINUITIES

Definition 5.2.7 If $\lim\limits_{x \to x_0} f(x)$ exists but either $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$ or $f(x_0)$ does not exist, then we say that f has a **removable discontinuity** at x_0.

What is "removable" about a "removable discontinuity"? We shall see. Suppose f has a removable discontinuity at x_0. Define a new function g by defining

$$g(x) = \left\{ \begin{array}{ll} f(x) & \text{if } x \neq x_0 \\ \lim\limits_{x \to x_0} f(x) & \text{if } x = x_0 \end{array} \right\}.$$

Then g is continuous at x_0 since $\lim\limits_{x \to x_0} g(x) = \lim\limits_{x \to x_0} f(x) = g(x_0)$. In a word, g "removes" the discontinuity of f.

Examples 5.2.8 (a) The function $f(x) = \dfrac{x^2 - 4}{x - 2}$ has a removable discontinuity at 2, since $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ exists but $f(2)$ does not. The function $g(x) = x + 2$ "removes" the discontinuity since it is continuous at 2 and agrees with f everywhere except at 2.

(b) The function $f(x) = \dfrac{\sin x}{x}$ has a removable discontinuity at 0. As shown in calculus, $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, but $f(0)$ does not exist. Thus, the function $g(x) = \left\{ \begin{array}{ll} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{array} \right\}$ "removes" the discontinuity.

Definition 5.2.9 If $\lim\limits_{x \to x_0^-} f(x)$ and $\lim\limits_{x \to x_0^+} f(x)$ both exist but $\lim\limits_{x \to x_0^-} f(x) \neq \lim\limits_{x \to x_0^+} f(x)$, then we say that f has a **jump discontinuity** at x_0.

Example 5.2.10 The function $f(x) = \left\{ \begin{array}{ll} -1 & \text{if } x \leq 2 \\ 1 & \text{if } x > 2 \end{array} \right\}$ described in Examples 5.2.3 and 5.2.6 has a jump discontinuity at 2, since $\lim\limits_{x \to 2^+} f(x) = -1$, while $\lim\limits_{x \to 2^-} f(x) = 1$. [See Figure 5.3.]

Definition 5.2.11 A function f is said to have a **simple discontinuity**[4] (or a **discontinuity of the first kind**) at x_0 if f has either a removable discontinuity or a jump discontinuity at x_0. Any other discontinuity of f at x_0 is called an **essential discontinuity** (or a **discontinuity of the second kind**).

---

4. To see that a "simple" discontinuity need not look especially simple, see Exercise 12.

Definition 5.2.12 (a) A function f is said to have an **infinite discontinuity** at x_0 if either $\lim_{x\to x_0^-} f(x)$ or $\lim_{x\to x_0^+} f(x)$ is infinite.

(b) Any other discontinuity of the second kind is called an **oscillating discontinuity**.

Example 5.2.13 The function $f(x) = \dfrac{1}{x}$ has an infinite discontinuity at 0, since $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = +\infty$.

Example 5.2.14 (a) The function $f(x) = \sin\frac{1}{x}$ has an oscillating discontinuity at 0;

(b) Dirichlet's function (5.1.11) has an oscillating discontinuity at every real number.

The examples above show that the term "oscillating discontinuity" covers a multitude of different cases, which may not seem similar at all. It is not always an adequate description of a particular discontinuity. Perhaps "wild discontinuity" would be a better term.

MONOTONE FUNCTIONS

Definition 5.2.15 A function f is

(a) **monotone increasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2$ in A,

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2);$$

(b) **monotone decreasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2$ in A,

$$x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2);$$

(c) **strictly increasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2$ in A,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2);$$

(d) **strictly decreasing** on a set $A \subseteq \mathcal{D}(f)$ if $\forall x_1, x_2$ in A,

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

(e) **monotone on** $A \subseteq \mathcal{D}(f)$ if it satisfies (a) or (b), and **strictly monotone on** $A \subseteq \mathcal{D}(f)$ if it satisfies (c) or (d).

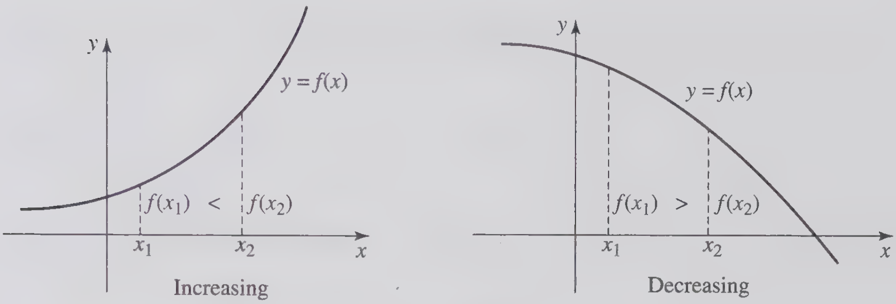


Figure 5.4

Examples 5.2.16 (a) The **greatest integer function**[5] is defined by $\lfloor x \rfloor =$ the greatest integer $\leq x$. (See Figure 5.5 (a).) It is monotone increasing on $(-\infty, +\infty)$, but is not strictly increasing there.

(b) The function $f(x) = x^2$ is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.

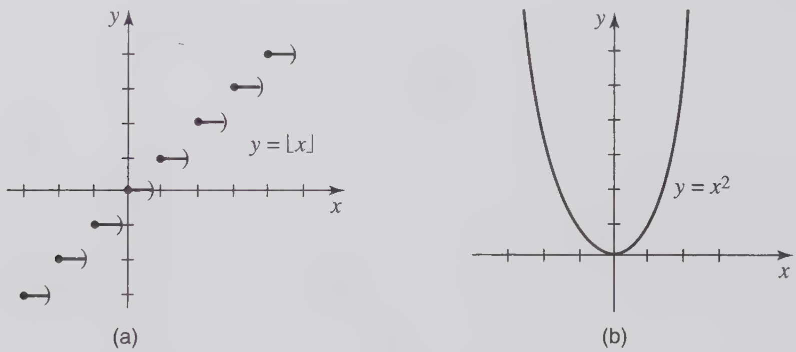


Figure 5.5

□

An interesting fact about monotone functions is that they can have only one type of discontinuity, which we establish in the following two theorems and a corollary.

5. This function is sometimes called the "bracket function" or the "integer floor function."

Theorem 5.2.17 *Suppose f is monotone increasing and bounded on an open interval $I = (a, b)$, where $a < b$. Then*

(a) $\forall c \in (a, b]$, $\lim\limits_{x \to c^-} f(x)$ *exists and equals* $\sup\{f(x) : a < x < c\}$;

(b) $\forall c \in [a, b)$, $\lim\limits_{x \to c^+} f(x)$ *exists and equals* $\inf\{f(x) : c < x < b\}$;

(c) $\forall c \in (a, b)$, $\lim\limits_{x \to c^-} f(x) \le f(c) \le \lim\limits_{x \to c^+} f(x)$; *i.e.*, $f(c^-) \le f(c) \le f(c^+)$;

(d) $\forall c < d$ *in* (a, b), $\lim\limits_{x \to c^+} f(x) \le \lim\limits_{x \to d^-} f(x)$; *i.e.*, $f(c^+) \le f(d^-)$.

Proof. Suppose f is monotone increasing and bounded on the open interval $I = (a, b)$, where $a < b$.

(a) Suppose $c \in (a, b]$, and let $A = \{f(x) : a < x < c\}$. Then A is nonempty since $f(x_1) \in A$ for any $a < x_1 < c$. Also, A is bounded above since f is bounded on I. Thus, by the completeness property, $\exists\, u = \sup A$. We shall prove that $\lim\limits_{x \to c^-} f(x) = u$.

Let $\varepsilon > 0$. By the ε criterion for supremum (Theorem 1.6.6), $\exists\, y \in A \ni u - \varepsilon < y$. Since $y \in A$, $y = f(x_0)$ for some $x_0 \in (a, c)$. So we have $u - \varepsilon < f(x_0)$.

Let $\delta = c - x_0$. Then $\delta > 0$ and

$$\begin{aligned} c - \delta < x < c &\Rightarrow x_0 < x < c \text{ by definition of } \delta \\ &\Rightarrow f(x_0) \le f(x) \le u \text{ since } f \text{ is monotone on } (a, b) \\ &\Rightarrow u - \varepsilon < f(x) < u + \varepsilon \\ &\Rightarrow |f(x) - u| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x \to c^-} f(x) = u$.

(b) Exercise 8.

(c) Exercise 9.

(d) Exercise 10. ■

Theorem 5.2.18 *Suppose f is monotone decreasing and bounded on an open interval $I = (a, b)$, where $a < b$. Then*

(a) $\forall c \in (a, b]$, $\lim\limits_{x \to c^-} f(x)$ *exists and equals* $\inf\{f(x) : a < x < c\}$;

(b) $\forall c \in [a, b)$, $\lim\limits_{x \to c^+} f(x)$ *exists and equals* $\sup\{f(x) : c < x < b\}$;

(c) $\forall c \in (a, b)$, $\lim\limits_{x \to c^-} f(x) \ge f(c) \ge \lim\limits_{x \to c^+} f(x)$; *i.e.*, $f(c^-) \ge f(c) \ge f(c^+)$;

(d) $\forall c < d$ *in* (a, b), $\lim\limits_{x \to c^+} f(x) \ge \lim\limits_{x \to d^-} f(x)$; *i.e.*, $f(c^+) \ge f(d^-)$.

Proof. Exercise 11. ■

Corollary 5.2.19 *If a function f is monotone on an interval I, then the only discontinuities that f can have in the interior of I are jump discontinuities.*

*__Theorem 5.2.20__ *For a function f that is monotone on an interval I, the set of discontinuities of f in I must be a countable set.*[6]

Proof. We prove the theorem for monotone *increasing* functions, and leave the proof for monotone decreasing functions to Exercise 13.

Suppose f is monotone increasing on I. By Theorem 5.2.17 (c), the set of points of discontinuities of f in the interior of I is the set

$$S = \{c \in I : f(c^-) < f(c^+)\}.$$

We must prove that the set S is countable. To do that it suffices to exhibit a function $f : S \overset{1-1}{\to} \mathbb{Q}$, since $\mathbb{Q}$ is countable.

For each $c \in S$, the density of $\mathbb{Q}$ in $\mathbb{R}$ allows us to choose a rational number r_c in the open interval $I_c = (f(c^-),\ f(c^+))$. We define the function f by $\forall$ $c \in S$, $f(c) = r_c$.

By Theorem 5.2.17 (d), $c < c' \Rightarrow I_c \cap I_{c'} = \varnothing$, so $r_c \neq r_{c'}$. Thus, f is $1-1$, as desired. ■

EXERCISE SET 5.2

1. Prove Theorem 5.2.4.

2. Prove Theorem 5.2.5.

3. The **greatest integer function**[7] $\lfloor x \rfloor$ was defined in Example 5.2.16 above. Prove that this function is

   (a) continuous at every $x_0 \notin \mathbb{Z}$;

   (b) continuous from the right at every $x_0 \in \mathbb{Z}$;

   (c) *not* continuous from the left at x if $x_0 \in \mathbb{Z}$.

4. By sketching their graphs, find where each of the following functions is continuous; continuous from the left; continuous from the right:

   (a) $x - \lfloor x \rfloor$ (b) $x \lfloor x \rfloor$ (c) $\lfloor \sin x \rfloor$

6. Understanding this theorem requires the concepts of Section 2.8.

7. or "bracket function" or "integer floor function."

5. The **characteristic function** of a set A of real numbers defined by $\chi_A(x) = \left\{ \begin{array}{l} 1 \text{ if } x \in A; \\ 0 \text{ if } x \notin A. \end{array} \right\}$. For each of the following intervals I, determine the points where $\chi_I(x)$ is *not* continuous from the left, and the points at which $\chi_I(x)$ is *not* continuous from the right:

   (a) $I = (a, b)$ (b) $I = [a, b]$
   (c) $I = [a, b)$ (d) $I = (a, b]$
   (e) $I = (-\infty, a)$ (f) $I = (-\infty, a]$
   (g) $I = (a, +\infty)$ (h) $I = [a, +\infty)$

6. Give an example of

   (a) a function that has a removable discontinuity at 0;

   (b) a function that has removable discontinuities at 0, 1, and 2;

   (c) a function that has a removable discontinuity at every natural number.

7. Prove that if f has a removable discontinuity at an interior point x_0 of its domain, then f is continuous neither from the left nor from the right at x_0

8. Prove Theorem 5.2.17 (b).

9. Prove Theorem 5.2.17 (c).

10. Prove Theorem 5.2.17 (d).

11. Prove Theorem 5.2.18.

12. Prove that Thomae's function defined in 5.1.12 has only removable discontinuities (at every rational number), and that the zero function "removes" them. (See Exercise 5.1.30.)

13. Prove Theorem 5.2.20 for the monotone decreasing case.

14. Prove that if a function $f : \mathcal{D}(f) \to \mathbb{R}$ is monotone on $A \subseteq \mathcal{D}(f)$ and $\lim_{x \to x_0} f(x)$ exists at an interior point x_0 of A, then f is continuous at x_0.

15. Prove that if $f : (a, b) \to \mathbb{R}$ is continuous, monotone, and bounded on (a, b), where $a < b$, then f can be defined at a and b in such a way that the extended $f : [a, b] \to \mathbb{R}$ is continuous and monotone on $[a, b]$.

16. Prove that if a function $f : I \to \mathbb{R}$ is monotone on an interval I, then f can be redefined at the points of I where it is discontinuous in such a way that the redefined function is continuous from the left everywhere on I. [The same is true if "left" is replaced by "right."]

17. Prove that Theorem 5.2.17 remains true if $\{f(x) : a < x < c\}$ and $\{f(x) : c < x < b\}$ are replaced by $\{f(r) : r \in \mathbb{Q}$ and $a < r < c\}$ and $\{f(r) : r \in \mathbb{Q}$ and $c < r < b\}$, respectively. State a similar revision of Theorem 5.2.18.

18. Suppose $a < b$. Prove that if f is monotone on $[a, b)$, and $\lim_{x \to b^-} f(x)$ exists, then f is bounded on $[a, b)$. State and prove a similar result for $(a, b]$ and $\lim_{x \to a^+} f(x)$.

19. Explain and justify the claim that the function $f(x) = \left\{ \begin{array}{cc} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{array} \right\}$ is "oscillating" at 0 and is discontinuous at 0 but does not have an oscillating discontinuity at 0. What kind of discontinuity does it have there?

20. Prove that Theorem 5.2.17 can be extended to infinite intervals:

    (a) If f is monotone increasing and bounded on some $[a, +\infty)$, then $\lim_{x \to +\infty} f(x)$ exists and equals $\sup\{f(x) : x \geq a\}$.

    (b) If f is monotone decreasing and bounded on some $[a, +\infty)$ then $\lim_{x \to -\infty} f(x)$ exists and equals $\inf\{f(x) : x \geq a\}$.

    State corresponding results for f monotone and bounded on $(-\infty, a]$.

5.3 Continuity on Compact Sets and Intervals

We begin this section with a subtle point that may at first hardly seem worth mentioning, but which can lead to subtle errors of thought if ignored. When discussing continuity of a function *on a set* it is often important to understand the role played by the declared domain of the function.

RESTRICTING THE DOMAIN OF A FUNCTION

Suppose $f : \mathcal{D}(f) \to \mathbb{R}$. Sometimes we are interested in the behavior of the function only on a subset A of $\mathcal{D}(f)$. For a given $A \subseteq \mathcal{D}(f)$, there is a subtle difference between the following two statements:[8]

Statement #1 $f : \mathcal{D}(f) \to \mathbb{R}$ is continuous on A.

Statement #2 $f : A \to \mathbb{R}$ is continuous (on A).

8. Recall that a function consists of two sets and a "rule" f that associates to each member of the first set (its domain) a member of the second set (its codomain). See Appendix B.2. We often use the same symbol, f, to denote the function when we restrict its domain to A.

Statement #1 is stronger than Statement #2; that is, #1 $\Rightarrow$ #2 but #2 $\not\Rightarrow$ #1. We give an example in which #2 is true but #1 is false (see also Exercise 1).

Example 5.3.1 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \chi_{[0,1]}(x)$, the "characteristic function"[9] of the interval $[0, 1]$.

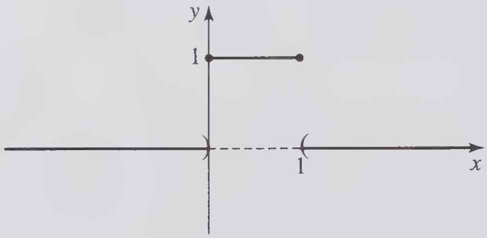


Figure 5.6

In this example,

- $f : \mathbb{R} \to \mathbb{R}$ is *not* continuous on $[0, 1]$; it is not continuous at 0 since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$.
- $f : [0, 1] \to \mathbb{R}$ *is* continuous on $[0, 1]$, since in determining its continuity, no x outside $[0, 1]$ may be used. f is continuous at every $a \in [0, 1]$ since
$$\forall \varepsilon > 0, \exists \delta > 0 \ni \forall x \in [0, 1],\ 0 < |x - a| < \delta \Rightarrow |f(x) - 1| = 0 < \varepsilon. \quad \blacksquare$$

Because Statement #1 is stronger than Statement #2, we prefer to use #1 in conclusions of theorems and #2 in hypotheses, whenever possible. Remember, a theorem is strongest when its hypotheses are as weak as we can make them and its conclusion is as strong as we can make it.

So far, the notation we are using does not distinguish between the function symbol f used for $f : A \to \mathbb{R}$ and the symbol f used for $f : \mathcal{D}(f) \to \mathbb{R}$. Occasionally, it is useful to have two different symbols to distinguish these two different functions.

Definition 5.3.2 Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and $A \subseteq \mathcal{D}(f)$. We define the function $f|_A$ (called "f **restricted to** A") as follows:

(a) The domain of $f|_A$ is A;

(b) $\forall x \in A,\ f|_A(x) = f(x)$.

---

9. Characteristic functions are defined in Exercise 5.2.5.

That is, the function $f|_A$ is the same as the function f, except that its domain has been "restricted to" A. Thus the expressions $f : A \to \mathbb{R}$ and $f|_A : A \to \mathbb{R}$ mean the same thing.

With this subtlety behind us, we are ready to discuss continuity of functions on compact sets and intervals.

CONTINUITY ON COMPACT SETS (EXTREME VALUE THEOREM)

Compact sets were introduced and discussed more fully in Section 3.3. For our purposes, and for anyone who omitted Section 3.3, the following definition is sufficient.

Definition 5.3.3 A set $A \subseteq \mathbb{R}$ is said to be a **compact set** if it is closed and bounded.

The following are examples of compact sets:

(a) finite sets;
(b) closed intervals of the form $[a, b]$;
(c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$;
(d) $\{x_n : n \in \mathbb{N}\} \cup \{L\}$, where $x_n \to L$;
(e) unions of finitely many of the above.

Theorem 5.3.4 *Every nonempty compact set has a maximum and a minimum.*

Proof. Exercise 3. ■

Theorem 5.3.5 (***Sequential Criterion for Compactness***)[10] *A set A of real numbers is compact if and only if every sequence of points of A has a subsequence that converges to a point of A.*

Proof. Let A be a set of real numbers.

Part 1 ($\Rightarrow$): Suppose A is compact. Let $\{a_n\}$ be a sequence of points of A. Now A is a bounded set, so $\{a_n\}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem for sequences, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Let $L = \lim\limits_{k\to\infty} a_{n_k}$. Now A is closed, since it is compact. So, by the sequential criterion for closed sets (3.2.19), $L \in A$. Thus, $\{a_n\}$ has a subsequence converging to a point of A.

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10. For readers who did not study Section 3.3, this theorem and its proof are repeated verbatim from 3.3.13.

Part 2 ($\Leftarrow$): Suppose every sequence of points of A has a subsequence converging to a point of A. We want to prove that A is compact; i.e., closed and bounded.

Suppose A is not bounded. Then $\forall n \in \mathbb{N}$, $\exists\, a_n \in A \ni |a_n| > n$. By our hypothesis, the sequence $\{a_n\}$ has a convergent subsequence, $\{a_{n_k}\}$. Now, $\forall k$, $|a_{n_k}| > n_k \geq k$. This means $\{a_{n_k}\}$ is unbounded. But every convergent sequence is bounded. Contradiction. Therefore, A is bounded.

We shall prove that A is closed using the sequential criterion for closed sets. Suppose $\{b_n\}$ is a sequence of points of A that converges; say, $b_n \to M$. Then $\{b_n\}$ is bounded. So, by our hypothesis, $\{b_n\}$ must have a convergent subsequence $\{b_{n_k}\}$ whose limit is in A. By Theorem 2.6.8, this limit must be M. Therefore, $M \in A$. So, by the sequential criterion for closed sets, A is closed.

Therefore, A is closed and bounded, and hence is compact. ■

Compactness is a very special property in relation to continuity. The following theorem says, in brief, that *continuous functions "preserve" compactness.* This leads to the corollary that *continuous functions have the "extreme value property" on compact sets.* In less formal words, a continuous function on a nonempty compact set must have both a maximum and a minimum value there—a fact of great importance in calculus. See also Exercise 27.

Theorem 5.3.6 ***The Continuous Image of a Compact Set Is Compact.*** *That is, if A is a compact set and $f : A \to \mathbb{R}$ is continuous,*[11] *then $f(A)$ is compact.*

Proof. Suppose A is a compact set and $f : A \to \mathbb{R}$ is continuous. Let $\{y_n\}$ be any sequence of points of $f(A)$. Then $\forall n \in \mathbb{N}$, $y_n \in f(A)$, so $\exists\, a_n \in A \ni f(a_n) = y_n$. Consider the sequence $\{a_n\}$. Since A is compact, Theorem 5.3.5 guarantees that $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ whose limit is in A; i.e.,

$$a_{n_k} \to L \in A.$$

Since f is continuous on A, f is continuous at L, so

$$f(a_{n_k}) \to f(L)$$
$$\text{i.e., } y_{n_k} \to f(L).$$

Thus, $\{y_n\}$ has a convergent subsequence whose limit is in $f(A)$. Therefore, $f(A)$ is compact, by Theorem 5.3.5. ■

11. Notice that our theorem is strengthened by using the weaker statement #2 in the hypotheses. The following theorem is also true, but weaker: If $f : \mathcal{D}(f) \to \mathbb{R}$ is continuous on a compact set A, then $f\ (A)$ is compact.

Caution: Theorem 5.3.6 says that the continuous image of a closed, bounded set is closed and bounded. It does **not** say that the continuous image of a bounded set must be bounded,[12] or that the continuous image of a closed set must be closed. Indeed, neither of these is true! (See Exercise 6.)

Corollary 5.3.7 (*Extreme Value Theorem*): *If A is a nonempty compact set and $f : A \to \mathbb{R}$ is continuous, then f has the* ***extreme value property on*** A:

(a) $\exists\, u = \min f(A) = \min\{f(x) : x \in A\}$, *and*

(b) $\exists\, v = \max f(A) = \max\{f(x) : x \in A\}$.

That is, a continuous function assumes a maximum and a minimum value on any nonempty compact set.

Proof. Exercise 5. ■

CONTINUITY ON INTERVALS (INTERMEDIATE VALUE THEOREM)

Before discussing continuous functions on intervals, we remind ourselves of the definition of an "interval," given in Section 1.6. There, we first defined a *closed* interval $[a, b]$ and then defined an interval[13] in general as any set I that satisfies the condition

$$\forall x < y \text{ in } I, [x, y] \subseteq I.$$

The following theorem looks innocent enough, but it has far-reaching consequences in calculus and analysis. It leads to the "intermediate value theorem" and other important consequences.

Theorem 5.3.8 *Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous. Then $f(I)$ is an interval.*[14]

Proof. Suppose $f : I \to \mathbb{R}$ is continuous on an interval I. To prove that $f(I)$ is an interval, let $u, v \in f(I)$, where $u < v$. We must show that every $w \in (u, v)$ is also in $f(I)$. So, suppose $u < w < v$. Since $u, v \in f(I)$, $\exists\, a, b \in I \ni u = f(a)$ and $v = f(b)$. Then either $a < b$ or $b < a$.

---

12. But, see Theorem 5.4.6.

13. See Definition 1.2.16.

14. In the general setting of topology, this theorem would state that the continuous image of a "connected" set is "connected." See references such as [4], [45], [94], or [122].

Case 1 ($a < b$): Since I is an interval, $[a, b] \subseteq I$. Let

$$A = \{x \in [a, b] : f(x) < w\}.$$

Then A is nonempty, since $a \in A$. Also, A is bounded above, since $A \subseteq [a, b]$. Thus, by the completeness property of $\mathbb{R}$,

$$\exists\, c = \sup A.$$

To prove: $f(c) = w$.

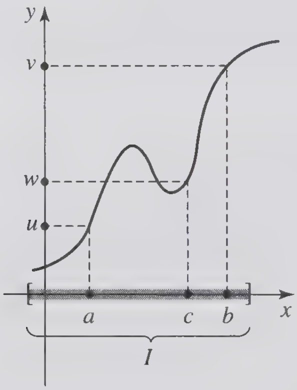

Figure 5.7

By Theorem 2.3.5, since $c = \sup A$, $\exists$ sequence $\{a_n\}$ of points of A such that

$$a_n \to c. \qquad (2)$$

Note that $c \in [a, b]$ since $A \subseteq [a, b]$. Since $f : I \to \mathbb{R}$ is continuous and $c \in [a, b] \subseteq I$, f must be continuous at c. Thus, by the sequential criterion for continuity of f at c applied to (2),

$$f(a_n) \to f(c). \qquad (3)$$

Now $\forall n$, $a_n \in A$, so $f(a_n) < w$. Thus, since limits preserve inequalities (Theorem 2.3.12), $\lim\limits_{n\to\infty} f(a_n) \leq w$. That is, by (3),

$$\boxed{f(c) \leq w.} \qquad (4)$$

We shall now complete the proof by proving that $f(c) \geq w$. First, note that $c \neq b$, since $f(c) \leq w < v = f(b)$. But, $c \in [a, b]$. Thus, $c < b$.

For all $n \in \mathbb{N}$, we define $t_n = \min\left\{c + \frac{1}{n}, b\right\}$. Thus, $\forall n \in \mathbb{N}$, $c < t_n \le c + \frac{1}{n}$. Hence, by the squeeze principle, $t_n \to c$. (5)

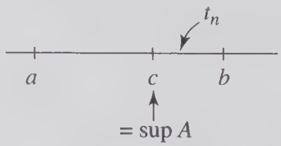

Figure 5.8

Now f is continuous at c and $\forall n$, $t_n \in [a, b]$. Thus, by (5) and the sequential criterion for continuity of f at c,

$$f(t_n) \to f(c). \tag{6}$$

But, $\forall n$, $t_n > c = \sup A$, so $t_n \notin A$, which by definition of A means that $f(t_n) \ge w$. Since limits preserve inequalities, $\lim\limits_{n\to\infty} f(t_n) \ge w$. That is, by (6),

$$\boxed{f(c) \ge w.} \tag{7}$$

Putting (4) and (7) together, we have $f(c) = w$. That is, $w \in f(I)$. Therefore, $f(I)$ is an interval.

Case 2 ($b < a$): The proof of this case is just like that of Case 1, with the roles of a and b interchanged. ∎

Corollary 5.3.9 (***Intermediate Value Theorem***) *Suppose $a < b$. Any continuous $f : [a, b] \to \mathbb{R}$ must satisfy the* ***intermediate value property*** *on $[a, b]$:*

$$\forall y \text{ between } f(a) \text{ and } f(b),\ \exists c \in [a, b] \ni f(c) = y.$$

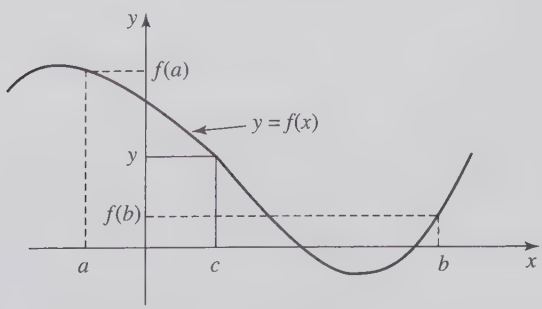

Figure 5.9

Proof. Exercise 11. ■

The intermediate value theorem furnishes one of the principal tools used in finding the roots of an equation $f(x) = 0$, for a continuous function f. The following corollary expresses the principle behind this method.

Corollary 5.3.10 (***Location of Roots Principle***): *If $f : I \to \mathbb{R}$ is a continuous function on an interval I containing a and b, and if $f(a)$ and $f(b)$ have opposite signs, then $\exists\, c$ between a and b such that $f(c) = 0$.*

Implementing the Location of Roots Principle: Given a continuous function f, to find a real number c such that $f(c) = 0$ we find numbers a_n and b_n successively closer to each other, for which $f(a_n)$ and $f(b_n)$ have opposite signs, say $f(a_n) < 0$ and $f(b_n) > 0$. When $|a_n - b_n|$ is satisfactorily small, any number in the interval between a_n and b_n is regarded as a good approximate root of $f(x) = 0$. (Complications can occur, but we ignore them here.)

Example 5.3.11 Find a solution of the equation $x^3 - 2x^2 + 4x - 1 = 0$ correct to 2 decimal places.

Solution: Let $f(x) = x^3 - 2x^2 + 4x - 1$. We see that $f(0) = -1$ and $f(1) = 2$. Thus, by the location of roots principle, $\exists\, c$ between 0 and 1 such that $f(c) = 0$. To find the tenth's digit in the expansion of c we calculate $f(.1)$, $f(.2), \cdots, f(.9)$. Using a calculator, we find $f(.2) = -.272$ and $f(.3) = .047$. Thus, the root is between .2 and .3. To find the hundred's digit in the expansion of c we calculate $f(.21), f(.22), \cdots, f(.29)$. By calculation, we find $f(.28) = -.014848$ and $f(.29) = .016189$. Thus, the root is between .28 and .29. To decide whether the root rounds off to .28 or .29, we calculate $f(.285) = .000699$. Since $f(.280) = -.014848$, the root must be between .280 and .285. Therefore, we can be confident that the correct answer to two decimal places is $x = .28$. □

Of all intervals, the **compact intervals** $[a, b]$ enjoy special status. The following two results about continuous functions on compact intervals are immediate consequences of Theorems 5.3.6 and 5.3.8. We shall see more later.

Corollary 5.3.12 *If $f : I \to \mathbb{R}$ is continuous on a compact interval I, then $f(I)$ is a compact interval.*

Corollary 5.3.13 (***Fixed Point Theorem***): *Suppose $a \le b$, and $f : [a, b] \to [a, b]$ is continuous. Then $\exists\, c \in [a, b] \ni f(c) = c$.*

Proof. Suppose $a \leq b$, and $f : [a, b] \to [a, b]$ is continuous. Define the function h on $[a, b]$ by $h(x) = f(x) - x$. By the algebra of continuous functions, h is continuous on $[a, b]$. Moreover,

$$h(a) = f(a) - a \geq 0 \text{ since } f(a) \in [a, b];$$
$$h(b) = f(b) - b \leq 0 \text{ since } f(b) \in [a, b].$$

If $h(a) = 0$ or $h(b) = 0$, then $f(a) = a$ or $f(b) = b$, and we have found c as desired. Thus, we suppose that $h(a) \neq 0$ and $h(b) \neq 0$. Then, 0 is between $h(a)$ and $h(b)$, so by the intermediate válue theorem, $\exists\, c \in [a, b]$ such that $h(c) = 0$. That is, $f(c) - c = 0$, i.e., $f(c) = c$. ■

Corollary 5.3.14 *Suppose $f : I \to \mathbb{R}$ is continuous, strictly monotone, and bounded on $I = (a, b)$, where $a < b$. Then $f(I)$ is a bounded open interval. In fact, $f(I) = (c, d)$ where $c = \inf f(I)$ and $d = \sup f(I)$. Further, we can extend f to a continuous, strictly monotone function $f : [a, b] \to \mathbb{R}$ as follows:*

(a) *if f is strictly increasing on (a, b), define $f(a) = c$ and $f(b) = d$.*

(b) *if f is strictly decreasing on (a, b), define $f(a) = d$ and $f(b) = c$.*

In either case, f is continuous and strictly monotone on $[a, b]$, and $f([a, b]) = [c, d]$.

Proof. Exercise 21. ■

In Section 1.6 we proved that the completeness property guarantees that in $\mathbb{R}$, every positive element has a square root. With the help of the intermediate value theorem we can now go further and prove that $\forall n \geq 2$ in $\mathbb{N}$, every positive real number has a unique positive n^{th} root. First, we prove the following lemma.

Lemma 5.3.15 $\forall n \in \mathbb{N}$, *the function $f(x) = x^n$ is* 1-1 *on the interval* $(0, +\infty)$.

Proof. Since the case $n = 1$ is trivial, we assume $n \geq 2$ in $\mathbb{N}$, and consider the function $f(x) = x^n$ on $(0, +\infty)$. Suppose $a, b \in (0, +\infty) \ni f(a) = f(b)$. Then $a^n = b^n$. By factoring, this means

$$(a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = 0.$$

Since $a, b > 0$, the second factor cannot equal 0. Thus, $a - b = 0$, and so $a = b$. That is, f is 1-1 on $(0, +\infty)$. ■

Theorem 5.3.16 (***Existence of Unique Positive n^{th} Roots***) $\forall n \in \mathbb{N}$, *and $\forall x_0 > 0$ in $\mathbb{R}$, $\exists$ unique $y > 0$ such that $y^n = x_0$. That is, every positive real number x_0 has a unique positive n^{th} root, $y = \sqrt[n]{x_0}$.*

Proof. If $n = 1$ we merely take $y = x_0$.

Suppose $n \geq 2$ in $\mathbb{N}$, and $x_0 > 0$ in $\mathbb{R}$. Consider the (polynomial) function $p(x) = x^n$. We know that $p(0) = 0$. By Theorem 4.4.18, $\lim_{x \to +\infty} p(x) = +\infty$, so $\exists\, b \in (0, +\infty) \ni p(b) > x_0$. Thus, by the intermediate value theorem, $\exists\, y \in (0, b) \ni p(y) = x_0$. Uniqueness follows from Lemma 5.3.15. ■

EXERCISE SET 5.3

1. Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and $A \subseteq \mathcal{D}(f)$, and consider Statements #1 and #2 in the introductory paragraphs of this section. Prove that these two statements are equivalent if and only if A is open. [Hint: If A is **not** open, try the characteristic function χ_A.]

2. Prove that $f : [a, b] \to \mathbb{R}$ is continuous $\Leftrightarrow$ f is continuous on (a, b) and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

3. Prove Theorem 5.3.4.

4. Give an example for each of the following: (Draw a graph to explain your reasoning.)

   (a) A function that exists on $[0, 2]$ and has a maximum value on $[0, 2]$, but does not have a minimum value there.

   (b) A function that exists on $[0, 2]$ and has a minimum value on $[0, 2]$, but does not have a maximum value there.

   (c) A function that exists on $[0, 2]$ and has neither a minimum value on $[0, 2]$ nor a maximum value there.

   (d) A function that is continuous on $[1, +\infty)$ and has a maximum value there, but no minimum value there.

   (e) A function that is continuous on $[1, +\infty)$ and has a minimum value there, but no maximum value there.

   (f) A function that is continuous on $[1, +\infty)$ and has neither a minimum value nor a maximum value there.

   (g) A function that is continuous on $(-1, 1)$ and has neither a minimum value nor a maximum value there.

   (h) A function that is continuous on $(-1, 1)$ and has a minimum value on $(-1, 1)$, but does not have a maximum value there.

   (i) A function that is continuous on $(-1, 1)$ and has a maximum value on $(-1, 1)$, but does not have a minimum value there.

5. Prove the extreme value theorem (Corollary 5.3.7).

6. Prove by example that

   (a) the continuous image of a bounded set need not be bounded.

   (b) the continuous image of a closed set need not be closed.

7. Suppose $f : A \to \mathbb{R}$ is continuous on a closed set A. Prove that $\forall\, c \in \mathbb{R}$, the set $\{x \in A : f(x) = c\}$ is closed.

8. Suppose $f, g : A \to \mathbb{R}$ are continuous on a closed set A. Prove that the set $\{x \in A : f(x) = g(x)\}$ is closed.

9. Suppose f is positive and continuous on some compact set A. Prove that f is bounded away from 0[15] on A.

10. Prove the intermediate value theorem (Corollary 5.3.9).

11. Suppose f is continuous on $[a, b]$ and $\forall x \in [a, b]$, $\exists\, y \in [a, b] \ni |f(y)| \leq \frac{1}{2}|f(x)|$. Prove that $\exists\, x_0 \in [a, b] \ni f(x_0) = 0$. [Hint: use sequences.]

12. Suppose $f : I \to \mathbb{R}$ is continuous on an interval I and $f(a)f(b) < 0$ for some $a, b \in I$. Prove that $f(x) = 0$ for some x between a and b.

13. Use the location of roots principle to find a root of the equation $4x^3 - 5x^2 + x - 7 = 0$, correct to three decimal places.

14. Use the location of roots principle to find two roots of the equation $x^4 - x^3 - 10 = 0$, correct to three decimal places.

15. Prove that a polynomial function of odd degree has at least one real root. [Hint: Use Theorem 4.4.24 and the intermediate value theorem.]

16. Prove that a polynomial function $p(x) = a_0 + a_1x + \cdots + a_nx^n$ of even degree n, in which $a_0a_n < 0$, must have at least two real roots, one positive and one negative. [See hint for Exercise 15.]

17. Prove that $\cos x = x$ for some $x \in \left(0, \frac{\pi}{2}\right)$.

18. Prove that $x2^x = 1$ for some $x \in (0, 1)$. [Assume 2^x is continuous everywhere.]

19. Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous, $f(a) < g(a)$ and $f(b) > g(b)$. Prove that $f(c) = g(c)$ for at least one $c \in (a, b)$.

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15. See Definition 4.2.8.

20. Prove that if $f : I \to \mathbb{R}$ is continuous on an interval I, and $f(I)$ contains only rational numbers, then f is constant on I. Can the same statement be proved if the rational numbers are replaced by irrational numbers? What if the rational numbers are replaced by any set whose complement is dense in $\mathbb{R}$?

21. Prove Corollary 5.3.14.

22. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to+\infty} f(x) = 0$. Prove that f is bounded on $\mathbb{R}$, and attains either a maximum value or a minimum value on $\mathbb{R}$, but not necessarily both.

23. Prove that if $f : A \to \mathbb{R}$ is continuous and 1-1 on a compact set A, then $f^{-1} : f(A) \to A$ is continuous. [Hint: Use sequences and Exercise 2.6.21.]

24. Suppose $f : [0, 2\pi] \to \mathbb{R}$ is continuous, and $f(0) = f(2\pi)$. Prove that there exists at least one point $c \in [0, \pi]$ such that $f(c) = f(c+\pi)$. [Hint: Consider $g(x) = f(x) - f(x+\pi)$.] Explain how from this result you can conclude that on any great circle around the earth, there are at least two diametrically opposite points at which the temperature is exactly the same.

25. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) = f(b)$. Prove that

    (a) $\exists x, y \in [a, b] \ni y - x = \frac{1}{2}(b - a)$ and $f(x) = f(y)$, and hence,

    (b) $\forall \varepsilon > 0$, $\exists x, y \in [a, b] \ni |y - x| < \varepsilon$ and $f(x) = f(y)$.

26. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $a \le s < t \le b$. Prove that
$$\forall m, n > 0,\ \exists c \in (s, t) \ni f(c) = \frac{mf(s) + nf(t)}{m + n}.$$

27. **Characterization of Compact Sets:** Prove that a nonempty set A is compact if and only if every continuous $f : A \to \mathbb{R}$ has the extreme value property on A. [Hint: Somewhere in the proof a function of the form $1/|x - x_0|$ may be helpful.]

28. **Characterization of Intervals:** We say that a function $f : A \to \mathbb{R}$ has the **intermediate value property** if for all $a < b$ in A, $\forall y$ between $f(a)$ and $f(b)$, $\exists c \in A \ni f(c) = y$. Prove that a set A is an interval if and only if every continuous $f : A \to \mathbb{R}$ has the intermediate value property.

5.4 Uniform Continuity

> In a one-semester course, this section may be postponed until Section 7.2, when Theorem 5.4.7 is needed to prove Theorem 7.2.17.

We now consider a little more closely what it means for a function to be continuous on a set. We shall refine this concept and define a slightly stronger form of continuity on a set, called "uniform" continuity. This will have special significance in Chapter 7, when we study the Riemann integral. First recall what it means for a function to be continuous on a set.

Recall: A function $f : \mathcal{D}(f) \to \mathbb{R}$ is **continuous** on a set $A \subseteq \mathcal{D}(f)$ if it is continuous at each point of A; that is

$$\forall x \in A,\ \forall \varepsilon > 0,\ \exists\, \delta > 0 \ni \forall y \in \mathcal{D}(f),\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

We are now ready to define the stronger form of continuity. You will have to look very carefully to see the difference.

Definition 5.4.1 A function $f : \mathcal{D}(f) \to \mathbb{R}$ is **uniformly continuous** on a set $A \subseteq \mathcal{D}(f)$ if

$$\forall \varepsilon > 0,\ \exists\, \delta > 0 \ni \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

The difference between these two types of continuity on a set A is quite subtle. It lies in the order of quantification. In the definition of "continuous on A," for every choice of x and ε there exists a $\delta = \delta(\varepsilon, x)$, dependent on both ε and x. In the definition of "uniform continuity" on A, for every choice of ε there exists a $\delta = \delta(\varepsilon)$ that works *for all x in A* (independent of the choice of x, and dependent only on ε).

Although the difference is subtle, it is very significant. It is perhaps easier to see the difference if we outline the differing strategies we would use to prove each:

> To prove that f is continuous on A: Let $x \in A$, and let $\varepsilon > 0$.
> Find $\delta > 0 \ni \forall y \in \mathcal{D}(f),\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.
>
> To prove that f is uniformly continuous on A: Let $\varepsilon > 0$.
> Find $\delta > 0 \ni \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

To prove that f is *continuous* on A, for each $x \in A$ and each $\varepsilon > 0$ we must find a $\delta > 0$, *depending on x and ε*, that makes a certain implication true for all $y \in \mathcal{D}(f)$. To prove that f is *uniformly continuous* on A, for each $\varepsilon > 0$ we must find a $\delta > 0$, *depending only on ε*, that makes the implication true for all $x, y \in A$. Thus, it seems that uniform continuity is *stronger* than continuity. Indeed, Theorem 5.4.3 below says that for $f : A \to \mathbb{R}$, uniform continuity implies continuity.

Example 5.4.2 Prove that the function $f(x) = 3x^2 - 2x - 1$ is uniformly continuous on the interval $[-1, 5]$.

Solution. Let $f(x) = 3x^2 - 2x - 1$. This is the same function used in Example 5.1.2. You are encouraged to compare this solution with the solution of that example, to observe the difference between proving continuity at a point and proving uniform continuity on a set.

(a) **Scratchwork:** Let $\varepsilon > 0$. We want to find $\delta > 0$ such that

$\forall x, y \in [-1, 5],\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Thus, we want

$$|x - y| < \delta \Rightarrow \quad |(3x^2 - 2x - 1) - (3y^2 - 2y - 1)| < \varepsilon$$
$$\text{i.e., } \left|3(x^2 - y^2) - 2(x - y)\right| < \varepsilon$$
$$\text{i.e., } |x - y||3(x + y) - 2| < \varepsilon.$$

Note that for $-1 \le x, y \le 5$,

$$\begin{gathered} -2 \le x + y \le 10 \\ -6 < 3(x + y) < 30, \\ -8 < 3(x + y) - 2 < 28, \\ |3(x + y) - 2| < 28. \end{gathered}$$

Thus, we want to make sure that $|x - y| < \varepsilon/28$.

(b) **Proof:** Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{28}$. Then $\forall x, y \in [-1, 5]$,

$$\begin{aligned} |x - y| < \delta &\Rightarrow -2 \le x + y \le 10 \ \text{ and } \ |x - y| < \frac{\varepsilon}{28} \\ &\Rightarrow -6 < 3(x + y) < 30 \ \text{ and } |x - y| < \frac{\varepsilon}{28} \\ &\Rightarrow |3(x + y) - 2| < 28 \ \text{ and } \ |x - y| < \frac{\varepsilon}{28} \\ &\Rightarrow -8 < 3(x + y) - 2 < 28 \ \text{ and } \ |x - y| < \frac{\varepsilon}{28} \\ &\Rightarrow |3(x + y) - 2| < 28 \ \text{ and } |x - y| < \frac{\varepsilon}{28} \\ &\Rightarrow |x - y||3(x + y) - 2| < \frac{\varepsilon}{28} \cdot 28 \\ &\Rightarrow \left|3(x^2 - y^2) - 2(x - y)\right| < \varepsilon \\ &\Rightarrow |(3x^2 - 2x - 8) - (3y^2 - 2y - 8)| < \varepsilon. \end{aligned}$$

Therefore, the function $f(x) = 3x^2 - 2x - 1$ is uniformly continuous on the interval $[0, 5]$. □

Theorem 5.4.3 *If $f : \mathcal{D}(f) \to \mathbb{R}$ is uniformly continuous on a set $A \subseteq \mathcal{D}(f)$, then $f : A \to \mathbb{R}$ is continuous (on A).*

Proof. Exercise 3. ■

Notes:

(1) The conclusion of Theorem 5.4.3 cannot be strengthened to say that $f : \mathcal{D}(f) \to \mathbb{R}$ is continuous on A. The Dirichlet function (Example 5.1.10) is uniformly continuous on $\mathbb{Q}$ but is not continuous on $\mathbb{Q}$.

(2) There is no difference between the statements "$f : \mathcal{D}(f) \to \mathbb{R}$ is uniformly continuous on a set $A \subseteq \mathcal{D}(f)$" and "$f : A \to \mathbb{R}$ is uniformly continuous." The statement "f is uniformly continuous on A" will be used to cover both.

(3) The converse of Theorem 5.4.3 is *not* true. Continuity on A does not imply uniform continuity on A. The following example demonstrates that.

Example 5.4.4 The function $f(x) = \dfrac{1}{x}$ is continuous on $(0, 1)$, but is not uniformly continuous there. The proof is contained in the following discussion.

We already know that this function is continuous on $(0, 1)$, by Theorem 5.1.13. What does it mean to say that it is *not* uniformly continuous there? In Figure 5.10 below, imagine that a fixed $\varepsilon > 0$ is given. The figure on the left shows that when $x = 2$, a fairly large δ is satisfactory. The figure on the right shows that as x approaches 0, the value of δ must get quite small; and in fact δ must $\to 0$ as $x \to 0$. It is apparent that no single $\delta > 0$ is small enough to work for *all* $x > 0$. (This is a convincing argument, but is not a proof.)

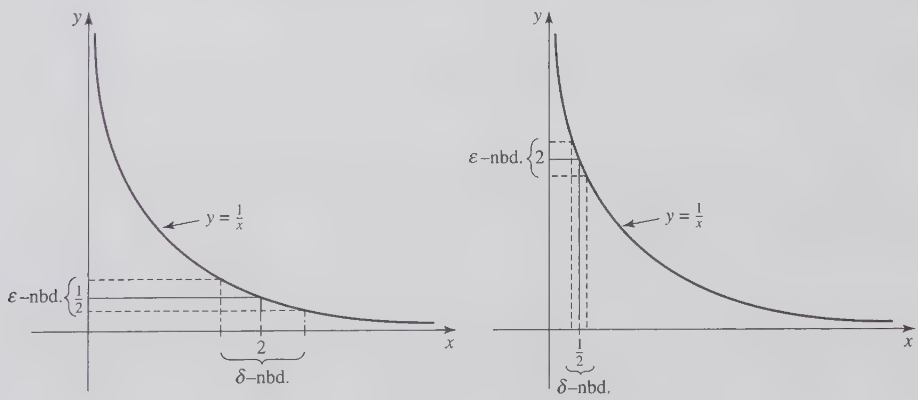

Figure 5.10

In order to prove that a function is *not* uniformly continuous on a set A, we need a criterion to use. If we negate Definition 5.4.1, we get the following criterion:

Lemma 5.4.5 (*Negation of Uniform Continuity*) *A function $f : \mathcal{D}(f) \to \mathbb{R}$ is **not uniformly continuous** on a set $A \subseteq \mathcal{D}(f)$ iff $\exists\, \varepsilon > 0 \ni \forall \delta > 0$, $\exists\, x, y \in A \ni |x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.*

Proof. Exercise 4. ■

We now give two proofs that the function $f(x) = \dfrac{1}{x}$ is *not* uniformly continuous on $(0, 1)$.

First proof that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$:

Choose $\varepsilon = 1$. Let $\delta > 0$ and choose $\lambda = \min\{2, \delta\}$, $x = \frac{\lambda}{3}$ and $y = \frac{\lambda}{6}$.

$$|x - y| = \frac{\lambda}{3} - \frac{\lambda}{6} = \frac{\lambda}{6} \leq \frac{\delta}{6} < \delta, \text{ and}$$

$$|f(x) - f(y)| = \left|\frac{3}{\lambda} - \frac{6}{\lambda}\right| = \left|-\frac{3}{\lambda}\right| = \frac{3}{\lambda} \geq \frac{3}{2} > \varepsilon.$$

Thus, $\exists\, \varepsilon > 0 \ni \forall \delta > 0$, $\exists\, x, y \in (0, 1) \ni |x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. That is, by Lemma 5.4.5, $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. ■

Despite the fact that the "First Proof" is technically correct, it doesn't seem very satisfying. It seems to be more technical than necessary; there must be an easier way to see that f is not uniformly continuous on $(0, 1)$. The next theorem will allow us to give an (easier) Second Proof.

Theorem 5.4.6 *If f is uniformly continuous on a bounded set A, then f is bounded*[16] *on A.*

Proof. Suppose f is uniformly continuous on a bounded set A. Since A is a bounded set, $\exists\, a, b \in \mathbb{R} \ni A \subseteq [a, b]$.

Letting $\varepsilon = 1$ in Definition 5.4.1,

$$\exists\, \delta > 0 \ni \forall x, y \in A,\ |x - y| < \delta \Rightarrow |f(x) - f(y)| < 1. \tag{8}$$

Keep this δ fixed in what follows. By the Archimedean property, $\exists\, n \in \mathbb{N} \ni \frac{b-a}{n} < \delta$. For $i = 0, 1, 2, \cdots, n$, let $x_i = a + i\frac{b-a}{n}$.

Figure 5.11

16. But, recall Exercise 5.3.6.

For each i, $|x_i - x_{i-1}| = x_i - x_{i-1} = \left[a + i\frac{b-a}{n}\right] - \left[a + (i-1)\frac{b-a}{n}\right] = \frac{b-a}{n} < \delta$. To see that f is bounded on $[x_{i-1}, x_i]$, for each i, first select a point $x_i^* \in \mathcal{D}(f)$ in $[x_{i-1}, x_i]$ if such a point exists. Thus, when $[x_{i-1}, x_i] \cap \mathcal{D}(f) \neq \varnothing$,

$$\begin{aligned} y \in [x_{i-1}, x_i] \cap \mathcal{D}(f) &\Rightarrow |y - x_i^*| \leq |x_i - x_{i-1}| \\ &\Rightarrow |y - x_i^*| < \delta \\ &\Rightarrow |f(y) - f(x_i^*)| < 1 \quad \text{by (8)} \\ &\Rightarrow f(x_i^*) - 1 < f(y) < f(x_i^*) + 1 \\ &\Rightarrow |f(y)| < |f(x_i^*)| + 1. \end{aligned}$$

Now, $A \subseteq [a, b] = \displaystyle\bigcup_{i=1}^{n} [x_{i-1}, x_i]$. Thus, $\forall y \in A$, $|f(y)| < \max\{|f(x_i^*)| + 1 : [x_{i-1}, x_i] \cap \mathcal{D}(f) \neq \varnothing\}$. Therefore, f is bounded on A. ■

Second proof that the function $f(x) = \dfrac{1}{x}$ **is not uniformly continuous on** $(0, 1)$**:**

This function is unbounded on $(0, 1)$. Hence, the conclusion follows easily from Theorem 5.4.6. ■

The next result shows another special property of compact sets. It is often very useful, as in proving the Riemann integrability of continuous functions, in Chapter 7.

Theorem 5.4.7 *If $f : A \to \mathbb{R}$ is continuous on a compact set A, then f is uniformly continuous on A.*

Proof. Suppose A is compact and $f : A \to \mathbb{R}$ is continuous. For contradiction, suppose f is not uniformly continuous on A. By Lemma 5.4.5, this means

$$\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x, y \in A \ni |x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon.$$

Keep this $\varepsilon > 0$ fixed. Then, $\forall n \in \mathbb{N}$,

$$\exists x_n, y_n \in A \ni |x_n - y_n| < \frac{1}{n} \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon.$$

By the squeeze principle, $x_n - y_n \to 0$. Also, the sequence $\{x_n\}$ is "in" the compact set A, so by the Bolzano-Weierstrass Theorem, it has a convergent subsequence, $\{x_{n_k}\}$; say $x_{n_k} \to L$.

Now, $\forall k \in \mathbb{N}$, $y_{n_k} = (y_{n_k} - x_{n_k}) + x_{n_k} \to 0 + L$. Hence, $y_{n_k} \to L$.

Since A is closed, Theorem 3.2.19 says that $L \in A$. Thus, f is continuous at L. Hence, by the sequential criterion for continuity,

$$f(y_{n_k}) \to f(L).$$

Thus, $\lim_{k\to\infty} [f(x_{n_k}) - f(y_{n_k})] = \lim_{k\to\infty} f(x_{n_k}) - \lim_{k\to\infty} f(y_{n_k}) = 0$. However, $\forall n \in \mathbb{N}$, $|f(x_n) - f(y_n)| \geq \varepsilon$. Contradiction.

Therefore, f is uniformly continuous on A. ■

*SEQUENTIAL CRITERIA FOR UNIFORM CONTINUITY

***Theorem 5.4.8** *If f is uniformly continuous on A then, for all Cauchy sequences $\{x_n\}$ in A, $\{f(x_n)\}$ is a Cauchy sequence.*

Proof. Suppose f is uniformly continuous on A and $\{x_n\}$ is a Cauchy sequence in A. Let $\varepsilon > 0$. Since f is uniformly continuous on A, $\exists\, \delta > 0 \ni \forall x, y \in A$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon. \tag{9}$$

Since $\{x_n\}$ is a Cauchy sequence, $\exists\, n_0 \in A \ni$

$$\begin{aligned} m, n > n_0 &\Rightarrow |x_m - x_n| < \delta \\ &\Rightarrow |f(x_m) - f(x_n)| < \varepsilon \text{ by (9).} \end{aligned}$$

Therefore, $\{f(x_n)\}$ is a Cauchy sequence. ■

The following example is an application of this theorem.

Example 5.4.9 The function $f(x) = \sin\left(\frac{1}{x}\right)$ is **not** uniformly continuous on $(0,1)$.

Proof. $\forall n \in \mathbb{N}$, let $x_n = \dfrac{2}{\pi n}$. Then $\{x_n\}$ is a Cauchy sequence in $(0,1)$ since it converges. But $\{f(x_n)\}$ is the sequence

$$1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, \cdots,$$

which does not converge, so $\{f(x_n)\}$ is not a Cauchy sequence. Therefore, by Theorem 5.4.8, f is not uniformly continuous on $(0,1)$. □

***Theorem 5.4.10** *If A is a bounded set, then a function $f : A \to \mathbb{R}$ is uniformly continuous on $A \Leftrightarrow$ for all Cauchy sequences $\{x_n\}$ in A, $\{f(x_n)\}$ is a Cauchy sequence.*

*An asterisk with a theorem, proof, or other item in this chapter inidcates that the item is optional and can be omitted, especially in a one-semester course.

Proof. Part 1 ($\Rightarrow$): See Theorem 5.4.8 above.

Part 2 ($\Leftarrow$): We shall prove the contrapositive. Suppose A is a bounded set and the function $f : A \to \mathbb{R}$ is *not* uniformly continuous on A. Then as noted in Lemma 5.4.5, $\exists \varepsilon > 0 \ni \forall \delta > 0, \exists x, y \in A \ni |x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Fix this value of ε throughout the remainder of this proof. Then,

$$\forall n \in \mathbb{N}, \exists x_n, y_n \in A \ni |x_n - y_n| < \tfrac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon. \qquad (10)$$

Consider the two sequences $\{x_n\}$ and $\{y_n\}$:

(1) The sequence $\{x_n\}$ is a bounded sequence, so it has a convergent subsequence, which we shall denote $\{x_{i_n}\}$. Say $x_{i_n} \to L$.

(2) The sequence $\{y_{i_n}\}$ is a bounded sequence, so it has a convergent subsequence, which we shall denote $\{y_{j_n}\}$.

(3) Consider the sequences $\{x_{j_n}\}$ and $\{y_{j_n}\}$. Since $\{x_{j_n}\}$ is a subsequence of $\{x_{i_n}\}$, we must have $x_{j_n} \to L$.

(4) We shall prove that $y_{j_n} \to L$.

Let $\varepsilon > 0$. Then $\exists n_1 \in \mathbb{N} \ni \dfrac{1}{n_1} < \dfrac{\varepsilon}{2}$. Since $x_{j_n} \to L$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |x_{j_n} - L| < \dfrac{\varepsilon}{2}$. Choose $n_0 = \max\{n_1, n_2\}$. Then

$$\begin{aligned} n \geq n_0 &\Rightarrow \frac{1}{j_n} \leq \frac{1}{n} < \frac{\varepsilon}{2} \text{ and } |x_{j_n} - L| < \frac{\varepsilon}{2} \\ &\Rightarrow |x_{j_n} - y_{j_n}| < \frac{1}{j_n} < \frac{\varepsilon}{2} \text{ and } |x_{j_n} - L| < \frac{\varepsilon}{2} \\ &\qquad \text{(from (10) above)} \\ &\Rightarrow |y_{j_n} - L| \leq |y_{j_n} - x_{j_n}| + |x_{j_n} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\Rightarrow |y_{j_n} - L| < \varepsilon. \end{aligned}$$

Therefore, $y_{j_n} \to L$.

(5) Consider the sequence

$$\{z_n\} = \{x_{j_1}, y_{j_1}, x_{j_2}, y_{j_2}, x_{j_3}, y_{j_3}, \cdots, x_{j_n}, y_{j_n}, \cdots\}.$$

We have $z_n \to L$, by Exercise 2.6.6. Thus, $\{z_n\}$ is a Cauchy sequence in A.

(6) However, $\forall n \in \mathbb{N}$, $|f(x_{j_n}) - f(y_{j_n})| \geq \varepsilon$ from (10) above. Therefore, $\{f(z_n)\}$ is not a Cauchy sequence.

Thus, if $f : A \to \mathbb{R}$ is not uniformly continuous on A, then there exists a Cauchy sequence $\{z_n\}$ in $A \ni \{f(z_n)\}$ is not a Cauchy sequence. ■

*CONTINUOUS EXTENSIONS

A useful property of uniformly continuous functions $f : (a, b) \to \mathbb{R}$ is that they can be "extended continuously" to $[a, b]$. That is because, for such functions, $\lim\limits_{x \to a^+} f(x)$ and $\lim\limits_{x \to b^-} f(x)$ always exist. The following definition and theorems will make these ideas precise.

***Definition 5.4.11** Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and $\mathcal{D}(f) \subseteq A$. Then a function $g : A \to \mathbb{R}$ is said to be an **extension of** f **to** A if $\forall x \in \mathcal{D}(f)$, $g(x) = f(x)$. That is, the two functions agree on $\mathcal{D}(f)$; i.e., $g|_{\mathcal{D}(f)} = f$.

***Theorem 5.4.12** *Suppose* $a < b$. *A function* $f : (a,b) \to \mathbb{R}$ *is uniformly continuous on* $(a,b) \Leftrightarrow f$ *has a continuous extension* $g : [a,b] \to \mathbb{R}$.

Proof. Suppose $f : (a,b) \to \mathbb{R}$ where $a < b$.

Part 1 ($\Leftarrow$): This direction is trivial. Suppose f has a continuous extension $g : [a,b] \to \mathbb{R}$. Then, by Theorem 5.4.7, g is uniformly continuous on $[a,b]$, and hence on (a,b). But $\forall x \in (a,b)$, $f(x) = g(x)$. Thus, f is uniformly continuous on (a,b).

Part 2 ($\Rightarrow$): Suppose f is uniformly continuous on (a,b). Then f is continuous at every point of (a,b), by Theorem 5.4.3. We claim that both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist.

Claim #1: $\lim_{x \to a^+} f(x)$ exists.

Proof: Suppose that $\{x_n\}$ is any sequence in $(a,b) \ni x_n \to a$. Then $\{x_n\}$ is a Cauchy sequence in (a,b), so by Theorem 5.4.8, $\{f(x_n)\}$ is a Cauchy sequence. Hence, $\{f(x_n)\}$ converges. Let $L = \lim_{n \to \infty} f(x_n)$.

Now, suppose $\{y_n\}$ is any other sequence in $(a,b) \ni y_n \to a$. Consider the sequence $\{z_n\}$ defined by

$$\{z_n\} = \{x_1, y_1, x_2, y_2, \cdots, x_n, y_n \cdots\}.$$

Then $z_n \to a$ (see Exercises 2.6.6). Thus, $\{z_n\}$ is a Cauchy sequence, so by Theorem 5.4.8, $\{f(z_n)\}$ is a Cauchy sequence, hence converges. But every subsequence of a convergent sequence must have the same limit. Thus,

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n) = L.$$

Therefore, $\forall$ sequences $\{x_n\}$ in (a,b) converging to a, $\{f(x_n)\}$ converges to the same limit, L. By the sequential criterion for one-sided limits (Theorem 4.3.5) $\lim_{x \to a^+} f(x) = L$.

Claim #2: $\lim_{x \to b^-} f(x)$ exists. (The proof is just like that of Claim #1.)

Now, we are ready to define the extension $g : [a,b] \to \mathbb{R}$ of f by

$$g(x) = \left\{ \begin{array}{ll} \lim_{x \to a^+} f(x) & \text{if } x = a; \\ f(x) & \text{if } a < x < b; \\ \lim_{x \to a^-} f(x) & \text{if } x = b \end{array} \right\}.$$

By Exercise 5.3.2, g is continuous on $[a,b]$. ■

***Corollary 5.4.13** *A continuous* $f : (a, b) \to \mathbb{R}$ *is uniformly continuous on* $(a, b) \Leftrightarrow$ *both* $\lim_{x \to a^+} f(x)$ *and* $\lim_{x \to b^-} f(x)$ *exist.*

***Corollary 5.4.14** *If* $f : (a, b) \to \mathbb{R}$ *is continuous, monotone, and bounded on* (a, b)*, then it is uniformly continuous on* (a, b).

Proof. Theorems 5.2.17 and 5.2.18, and Corollary 5.4.13. ■

EXERCISE SET 5.4

1. Prove directly from Definition 5.4.1 that the function $f(x) = 2x^2 - 5x + 2$ is uniformly continuous on $[-3, 1]$. Also, on $(-2, 2)$. (See Example 5.4.2.)

2. Prove directly from Definition 5.4.1 that the function $f(x) = x^3$ is uniformly continuous on $[0, 3]$. Also, on $(-2, 1)$. (See Example 5.4.2.)

3. Prove Theorem 5.4.3.

4. Prove Lemma 5.4.5.

5. Prove directly from the ε-δ definition that the function $f(x) = 7x - 8$ is uniformly continuous on $\mathbb{R}$. Is this function bounded on $\mathbb{R}$? Does that contradict Theorem 5.4.6?

6. Prove directly from the ε-δ definition (or its negation) that the function $g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

7. Prove that the function $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$.

8. Prove that the function $f(x) = 1/x^2$ is not uniformly continuous on $(0, 1)$, first directly from the ε-δ definition or its negation, and then as a simple corollary of a theorem of this section.

9. Prove that the function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$. [Use an inequality established in the material leading to Theorem 5.1.16.]

10. Prove that if f is defined on an interval I and $\exists M > 0 \ni \forall x, y \in I$, $|f(x) - f(y)| \le M|x - y|$, then f is uniformly continuous on I. [When this happens, we say that f **satisfies a Lipschitz condition** on I.]

11. Determine which of the following functions f are uniformly continuous on the given set. [Use the theorems of this section where helpful.]

    (a) $f(x) = 5x^2 - 3x + 7$ on $[1, 3]$ (b) $f(x) = 5x^2 - 3x + 7$ on $(1, 3)$
    (c) $f(x) = 1/x^2$ on $(1, 5)$ (d) $f(x) = x \sin x$ on $\left(0, \frac{\pi}{2}\right)$
    (e) $f(x) = \tan x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (f) $f(x) = \tan x$ on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

12. Prove that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$. [Hint: first prove that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$.]

13. Prove that the sum of two functions that are uniformly continuous on a set A is uniformly continuous on A.

14. Prove by counterexample that the product of two functions uniformly continuous on a set A need not be uniformly continuous on A.

15. Use the theorems of this and previous sections to prove that the functions $\tan x$ and $\sec x$ are continuous, but not uniformly, on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, while $\cot x$ and $\csc x$ are continuous, but not uniformly, on $(0, \pi)$.

16. Find an example of an interval (a, b) and a function $f : (a, b) \to \mathbb{R}$ that is continuous and bounded, but not uniformly continuous.

17. Prove that if $f : [a, \infty) \to \mathbb{R}$ is continuous and $\lim_{x \to \infty} f(x)$ is finite, then f is uniformly continuous on $[a, \infty)$.

18. Suppose f and g are uniformly continuous on a set A. Prove that

    (a) if f and g are bounded on A, then fg is uniformly continuous on A;

    (b) if A is bounded, then fg is uniformly continuous on A.

19. Prove that if $f : \mathcal{D}(f) \to \mathbb{R}$ is uniformly continuous on A, and g is uniformly continuous on $f(A)$, then $g \circ f$ is uniformly continuous on A.

20. Show by counterexamples that Theorem 5.4.10 is not true if "uniformly continuous" is replaced by "continuous," or if A is not a bounded set.

21. Prove that the converse of Exercise 10 is not true, by showing that the function $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ but does not satisfy a Lipschitz condition there. [Thus, a Lipschitz condition is strictly stronger than uniform continuity.]

22. Suppose $f : \mathbb{R} \to \mathbb{R}$ is **periodic with period** $p > 0$. That is, $\forall x \in \mathbb{R}$, $f(x+p) = f(x)$. Prove that if f is continuous on any compact interval of the form $[a, a+p]$, it must be bounded and uniformly continuous on $\mathbb{R}$.

23. Suppose $a < b < c < d$. Prove that if f is uniformly continuous on (a, b) and on (c, d) then f is uniformly continuous on $(a, b) \cup (c, d)$. Prove that the same is true if the intervals are closed, even when $b = c$.

24. Based on the result of Exercise 22, one might make the following conjecture: If f is uniformly continuous on disjoint sets A and B, then f is uniformly continuous on $A \cup B$.

    (a) Find a function f and two bounded open intervals that prove this conjecture false.

(b) Prove that the conjecture is true if A and B are bounded and $\sup A < \inf B$.

(c) Prove that the conjecture is true if A and B are compact sets.

5.5 *Monotonicity, Continuity, and Inverses

Monotone functions were defined in Definition 5.2.15. Since strictly monotone functions are 1-1, they must have inverses.[17] We shall now establish some connections between continuity, monotonicity, and invertibility of functions. First, we make note of a useful lemma.

Lemma 5.5.1 (a) *If $f : \mathcal{D}(f) \to \mathbb{R}$ is monotone increasing on a set A, then $\forall x_1, x_2 \in A$, $f(x_1) < f(x_2) \Rightarrow x_1 < x_2$.*

(b) *If $f : \mathcal{D}(f) \to \mathbb{R}$ is monotone decreasing on a set A, then $\forall x_1, x_2 \in A$, $f(x_1) < f(x_2) \Rightarrow x_1 > x_2$.*

Proof. Exercise 1. ∎

The next theorem is a partial converse of the intermediate value theorem. Its proof is a good example of the usefulness of the concept of restricted domain.

Theorem 5.5.2 *If $f{:}I \to \mathbb{R}$ is monotone on an interval I, and $f(I)$ is an interval, then f is continuous on I. [That is, a monotone function with the intermediate value property on an interval must be continuous on that interval.]*

Proof. We shall prove the monotone increasing case and leave the monotone decreasing case to Exercise 7. Suppose $f{:}I \to \mathbb{R}$ is monotone increasing on a nonempty interval I, and $f(I)$ is an interval with nonempty interior (the case in which $f(I)^\circ = \varnothing$ is trivial; explain). For notational simplicity, let $J = f(I)$. Let $x_0 \in I$. We shall prove that f is continuous at x_0.

Case 1: $f(x_0)$ is an interior point of J. Then $\exists\, \varepsilon_0 > 0 \ni N_{\varepsilon_0}(f(x_0)) \subseteq J$.

Let $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < \varepsilon_0$. Then $N_\varepsilon(f(x_0)) \subseteq J = f(I)$, so $\exists\ x_1, x_2 \in I \ni$

$$f(x_1) = f(x_0) - \varepsilon/2;$$
$$f(x_2) = f(x_0) + \varepsilon/2.$$

17. See Appendix B.3, especially Definition B.3.11 and Theorem B.3.12.

Since f is monotone increasing on I, and $f(x_1) < f(x_0) < f(x_2)$, we must have $x_1 < x_0 < x_2$ (by Lemma 5.5.1). Then $\forall x \in I$,

$$\begin{aligned} x_1 < x < x_2 \Rightarrow\ & x \in I \quad \text{(since } I \text{ is an interval)} \\ & \text{and } f(x_1) \le f(x) \le f(x_2) \quad \text{(by monotonicity of } f\text{)} \\ \Rightarrow\ & f(x_0) - \varepsilon/2 \le f(x) \le f(x_0) + \varepsilon/2 \\ \Rightarrow\ & |f(x) - f(x_0)| < \varepsilon. \end{aligned} \tag{11}$$

Choose $\delta = \min\{x_2 - x_0, x_0 - x_1\}$. Then $\delta > 0$ and

$$\begin{aligned} |x - x_0| < \delta \Rightarrow\ & -\delta < x - x_0 < \delta \\ \Rightarrow\ & x_1 - x_0 < x - x_0 < x_2 - x_0 \\ \Rightarrow\ & x_1 < x < x_2 \\ \Rightarrow\ & |f(x) - f(x_0)| < \varepsilon \text{ by (11) above.} \end{aligned}$$

Therefore, f is continuous at x_0.

Case 2: $f(x_0)$ is the left endpoint of J. Then, $\forall x \in I$, $f(x) \ge f(x_0)$, which implies $x \ge x_0$ by Lemma 5.5.1. So x_0 is the left endpoint of I. Since J is an interval containing more than one point, $\exists\, \varepsilon_0 > 0 \ni (f(x_0),\ f(x_0) + \varepsilon_0) \subseteq J$.

Let $\varepsilon > 0$. Without loss of generality, assume $\varepsilon < \varepsilon_0$. Then $(f(x_0), f(x_0) + \varepsilon) \subseteq J = f(I)$, so $\exists\, x_1 \in I \ni$

$$f(x_1) = f(x_0) + \varepsilon/2.$$

Since f is monotone increasing on I and $f(x_0) < f(x_1)$, we must have $x_0 < x_1$. Thus, $\forall x \in I$,

$$\begin{aligned} x_0 \le x < x_1 \Rightarrow\ & f(x_0) \le f(x) \le f(x_1) \quad \text{(by monotonicity of } f \text{ on } I\text{)} \\ \Rightarrow\ & 0 \le f(x) - f(x_0) \le f(x_1) - f(x_0) = \varepsilon/2 \\ \Rightarrow\ & |f(x) - f(x_0)| < \varepsilon. \end{aligned} \tag{12}$$

Choose $\delta = x_1 - x_0$. Then $\delta > 0$ and

$$\begin{aligned} |x - x_0| < \delta \text{ and } x \in I \Rightarrow\ & -\delta < x - x_0 < \delta \text{ and } x \ge x_0 \\ \Rightarrow\ & 0 < x - x_0 < x_1 - x_0 \\ \Rightarrow\ & x_0 \le x < x_1 \\ \Rightarrow\ & |f(x) - f(x_0)| < \varepsilon \text{ by (12) above.} \end{aligned}$$

Therefore, f is continuous at x_0.

Case 3: $f(x_0)$ is the right endpoint of J. Exercise 6. ■

Corollary 5.5.3 (***Inverse Function Theorem for Continuous Monotone Functions***) *Suppose I is a nonempty interval, and $f : I \to \mathbb{R}$ is continuous and strictly monotone. Then $f^{-1} : f(I) \to I$ is continuous and strictly monotone in the same sense. ($f(I)$ is an interval, by Theorem 5.3.8.)*

Proof. We prove the strictly increasing case, and leave the strictly decreasing case as Exercise 8.

Suppose I is a nonempty interval, and $f : I \to \mathbb{R}$ is continuous and strictly increasing. Then $f : I \to f(I)$ is a 1-1 correspondence, and so has an inverse,[18] $f^{-1} : f(I) \to I$, which is also 1-1 and onto.

By Lemma 5.5.1, f^{-1} is strictly increasing on $f(I)$. Thus, by Theorem 5.5.2, $f^{-1} : f(I) \to I$ is continuous on $f(I)$. ■

The following theorem is intuitively plausible, but its proof is a little tricky.

Theorem 5.5.4 *If $f : I \to \mathbb{R}$ is 1-1 and continuous on an interval I, then f is strictly monotone on I.*

Proof. Suppose I is an interval and $f : I \to \mathbb{R}$ is 1-1 and continuous.

Claim #1: $\forall a < b < c$ in I, $f(b)$ is between $f(a)$ and $f(c)$.

Proof: Let $a < b < c$ in I. For contradiction, suppose $f(b)$ is not between $f(a)$ and $f(c)$. Then

$$\begin{array}{ll} \text{either} & f(b) > \max\{f(a),\ f(c)\} \\ \text{or} & f(b) < \min\{f(a),\ f(c)\} \end{array}$$

We consider the former case, and leave the latter to Exercise 9. In the former case, $\exists$ real number y such that

$$f(b) > y > \max\{f(a), f(c)\}.$$

Then $f(b) > y > f(a)$, so by the intermediate value theorem, $\exists\, x_1 \in (a,b) \ni f(x_1) = y$. Similarly, $f(b) > y > f(c)$, so $\exists\, x_2 \in (b,c) \ni f(x_2) = y$. But then we have $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, contradicting the hypothesis that f is 1-1 on I. Therefore, in this case, $f(b)$ is between $f(a)$ and $f(c)$.

Claim #2: f is either strictly increasing or strictly decreasing on I.

Proof: It suffices to prove[19] that if f is not strictly increasing on I, then f is strictly decreasing on I. Suppose f is not strictly increasing on I. Then $\exists$ $c < d$ in $I \ni f(c) \geq f(d)$. Since f is 1-1, we must have $f(c) > f(d)$. We shall prove that f must be strictly decreasing on I.

First, we note that $\forall x \in I$, one of the following must hold: $x < c$, $c < x < d$, or $x > d$. Applying Claim #1, we see that

$$\begin{aligned} x < c &\Rightarrow x < c < d \\ &\Rightarrow f(c) \text{ lies between } f(x) \text{ and } f(d) \\ &\Rightarrow f(d) < f(c) < f(x) \end{aligned}$$

18. See Theorem B.3.12 in Appendix B.

19. To prove P or Q, it suffices to assume $\sim P$ and prove Q. (See Proof Strategy "PS-4" in Appendix A.3.)

while

$$\begin{aligned} c < x < d &\Rightarrow f(x) \text{ lies between } f(c) \text{ and } f(d) \\ &\Rightarrow f(d) < f(x) < f(c) \end{aligned}$$

and

$$\begin{aligned} x > d &\Rightarrow x > d > c \\ &\Rightarrow f(d) \text{ lies between } f(x) \text{ and } f(c) \\ &\Rightarrow f(x) < f(d) < f(c). \end{aligned}$$

From these inequalities we see that $\forall x \in I$,

$$\left\{ \begin{array}{l} x < c \Rightarrow f(x) > f(c) \\ x > c \Rightarrow f(x) < f(c) \end{array} \right\}. \tag{13}$$

Now, consider any $x_1 < x_2$ in I. We have the following cases to consider:

Case 1 ($x_1 \leq c \leq x_2$): Then by (13) and the 1-1 property of f, we have $f(x_1) > f(x_2)$.

Case 2 ($x_1 < x_2 < c$): Then by (13) we have $f(x_1) > f(c)$ and by Claim #1, $f(x_2)$ must lie between $f(x_1)$ and $f(c)$, so $f(x_2) < f(x_1)$.

Case 3 ($c < x_1 < x_2$): Then by (13) we have $f(x_2) < f(c)$ and by Claim #1, $f(x_1)$ must lie between $f(x_2)$ and $f(c)$, so $f(x_1) > f(x_2)$.

In all cases $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$. That is, f is strictly decreasing on I. Therefore, f is either strictly increasing or strictly decreasing on I. ■

THE CANTOR FUNCTION

The Cantor set described in Section 3.4 allows us to define a curious function, called the "Cantor function," $\varphi : [0,1] \to [0,1]$, which is continuous and monotone increasing; $f(0) = 0$, $f(1) = 1$, and yet f is not strictly increasing on any nonempty open interval. This function will be defined in stages, first on the Cantor set $\mathbf{C}$, and then extended to $[0,1]$.

Definition 5.5.5 Defining the function φ, first on the Cantor set.

Recall that the Cantor set $\mathbf{C}$ consists of all those real numbers x in $[0,1]$ with a ternary (base-3) decimal expansion consisting of only 0's and 2's:

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i}. \qquad (\text{where } x_i = 0 \text{ or } 2)$$

Define $\varphi_{\mathbf{c}} : \mathbf{C} \to [0,1]$ by

$$\varphi_{\mathbf{c}}(x) = \sum_{i=1}^{\infty} \frac{x_i/2}{2^i}, \text{ regarded as a binary (base-two) decimal.}$$

REMARKS:

(1) $\varphi_{\mathbf{c}}(x)$ is the binary expansion of a unique real number in $[0,1]$.

(2) $\varphi_{\mathbf{c}}$ is onto $[0,1]$. This is because $\mathbf{C}$ consists of all possible ternary expansions of real numbers in $[0,1]$ that consist of only 0's and 2's, implying that the range of $\varphi_{\mathbf{c}}$ consists of all possible binary expansions of real numbers in $[0,1]$.

(3) $\varphi_{\mathbf{c}}\left(\frac{1}{3}\right) = \varphi_{\mathbf{c}}\left(\frac{2}{3}\right) = \frac{1}{2}$. This is because, in base-3,

$$\frac{1}{3} = 0.02222222\cdots, \text{ and}$$
$$\frac{2}{3} = 0.20000000\cdots,$$

while in base-2, $0.011111111\cdots = 0.100000000 = \frac{1}{2}$.

(4) $\varphi_{\mathbf{c}}\left(\frac{1}{9}\right) = \varphi_{\mathbf{c}}\left(\frac{2}{9}\right) = \frac{1}{4}$, because, in base-3,

$$\frac{1}{9} = 0.002222222\cdots, \text{ and}$$
$$\frac{2}{9} = 0.020000000\cdots,$$

while in base-2, $0.0011111111\cdots = 0.0100000000 = \frac{1}{4}$. Also,

$$\varphi_{\mathbf{c}}\left(\tfrac{7}{9}\right) = \varphi_{\mathbf{c}}\left(\tfrac{8}{9}\right) = \tfrac{3}{4}, \text{ because, in base-3,}$$
$$\frac{7}{9} = \frac{2}{3} + \frac{1}{3} = 0.202222222\cdots, \text{ and}$$
$$\frac{8}{9} = \frac{2}{3} + \frac{2}{9} = 0.220000000\cdots,$$

while in base-2, $0.1011111111\cdots = 0.1100000000 = \frac{3}{4}$.

(5) By remark (3), $\varphi_{\mathbf{c}}$ takes on the same value at both end points of the interval "removed" to create $\mathbf{C}_1$ (see Definition 3.4.1). By remark (4), $\varphi_{\mathbf{c}}$ takes on the same value at both endpoints of each interval "removed" to create $\mathbf{C}_2$.

(6) Continuing in this way, we can show that $\varphi_{\mathbf{c}}$ takes on the same value at both endpoints of every interval "removed" to create the Cantor set $\mathbf{C}$. (We omit the details.)

Theorem 5.5.6 $\varphi_{\mathbf{c}} : \mathbf{C} \to [0,1]$ *is monotone increasing on* $\mathbf{C}$.

Proof. Let $x < y$ in $\mathbf{C}$. Expressing x and y in ternary decimal form,

$$x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \text{ and } y = \sum_{i=1}^{\infty} \frac{y_i}{3^i},$$

where $x_i, y_i \in \{0,2\}$. Let n denote the first (smallest) natural number such that $x_n \neq y_n$. (These are the first ternary digits of x and y that are not equal.) Then $x_n = 0$ and $y_n = 2$. Thus,

$$\begin{aligned}
\varphi_{\mathbf{c}}(y) - \varphi_{\mathbf{c}}(x) &= \sum_{i=1}^{\infty} \frac{y_i/2}{2^i} - \sum_{i=1}^{\infty} \frac{x_i/2}{2^i} \\
&= \frac{1}{2}\left[\sum_{i=1}^{n} \frac{y_i - x_i}{2^i} + \sum_{i=n+1}^{\infty} \frac{y_i - x_i}{2^i}\right] \\
&= \frac{1}{2}\left[\frac{2}{2^n} + \sum_{i=n+1}^{\infty} \frac{y_i - x_i}{2^i}\right] \\
&\geq \frac{1}{2^n} + \frac{1}{2} \sum_{i=n+1}^{\infty} \frac{0-2}{2^i} \\
&= \frac{1}{2^n} - \frac{1}{2} \cdot \frac{2}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} \\
&= \frac{1}{2^n} - \frac{1}{2^{n+1}}(2) \\
&= 0.
\end{aligned}$$

Thus $x < y \Rightarrow \varphi_{\mathbf{c}}(y) - \varphi_{\mathbf{c}}(x) \geq 0 \Rightarrow \varphi_{\mathbf{c}}(x) \leq \varphi_{\mathbf{c}}(y)$. Therefore, $\varphi_{\mathbf{c}}$ is monotone increasing on **C**. ■

Definition 5.5.7 We extend $\varphi_{\mathbf{c}} : \mathbf{C} \to [0,1]$ to a function $\varphi : [0,1] \to [0,1]$ as follows:

For all $x \in [0,1]$, if $x \in \mathbf{C}$ we define $\varphi(x) = \varphi_{\mathbf{c}}(x)$. If $x \notin \mathbf{C}$, then x is a member of exactly one of the open intervals (a,b) deleted from $[0,1]$ to create **C**; we define $\varphi(x) = \varphi_{\mathbf{c}}(a) = \varphi_{\mathbf{c}}(b)$. In summary,

$$\varphi(x) = \left\{ \begin{array}{l} \varphi_{\mathbf{c}}(x) \text{ if } x \in \mathbf{C}; \\ \\ \varphi_{\mathbf{c}}(a) \text{ if } x \in (a,b), \text{ where } (a,b) \text{ is one of the} \\ \qquad \text{intervals removed from } [0,1] \text{ to create } \mathbf{C} \end{array} \right\}. \quad \square$$

This function is called **the Cantor function on** $[0,1]$. It is constant on each of the open intervals removed from $[0,1]$ in creating the Cantor set. A rough idea of its graph is sketched at the top of page 273.

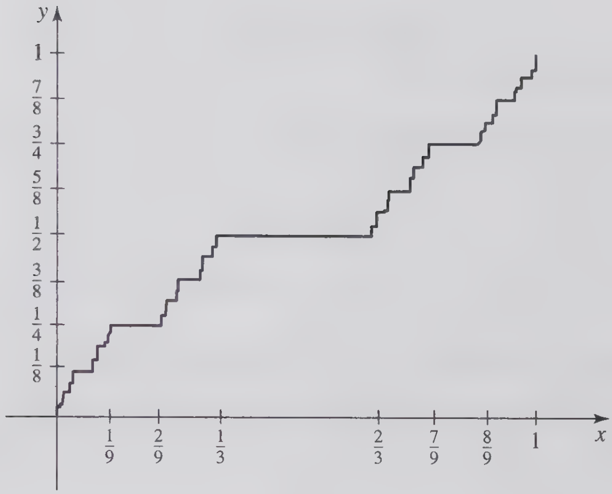

Figure 5.12

Theorem 5.5.8 *The Cantor function* $\varphi : [0,1] \overset{onto}{\rightarrow} [0,1]$ *is continuous and monotone increasing, yet is not strictly increasing on any nonempty open interval.*

Proof. We have seen by construction that φ is monotone increasing, and that $\varphi[0,1] = [0,1]$. Thus, Theorem 5.5.2 tells us that φ is continuous. Also, by construction φ is constant on each of the disjoint open intervals comprising the complement of the Cantor set.

Let I denote *any* nonempty open subinterval of $[0,1]$. Since $\mathbf{C}$ contains no nonempty open intervals (Theorem 3.4.3) I must contain a point $x \notin \mathbf{C}$. Then, using de Morgan's law,

$$x \in [0,1] - \bigcap_{n=1}^{\infty} \mathbf{C}_n = \bigcup_{n=1}^{\infty} ([0,1] - \mathbf{C}_n).$$

Thus, x belongs to one of the open intervals J that was removed to create $\mathbf{C}$. Since I and J are open, and $x \in I \cap J$, $\exists\, \varepsilon > 0 \ni$

$$N_\varepsilon(x) \subseteq I \cap J.$$

Then φ is constant on $N_\varepsilon(x)$ since it is constant on J. Thus, φ is not strictly increasing on $N_\varepsilon(x)$. Therefore, φ is not strictly increasing on I. ∎

EXERCISE SET 5.5

1. Prove Lemma 5.5.1.

2. Prove that a function f is monotone (or strictly) increasing on an interval I if and only if $-f$ is monotone (or strictly) decreasing on I.

3. Suppose f and g are monotone increasing on an interval I.

   (a) Prove that $f + g$ is monotone increasing on I. [Strictly increasing if f and g are strictly increasing.]

   (b) Show by example that fg need not be monotone on I.

   (c) Prove that if f, g are nonnegative on I, then fg is monotone increasing on I. [Strictly increasing if f, g are positive and strictly increasing on I.]

4. Suppose f is continuous on an interval I and x_0 is an interior point of I such that $f(x_0) = \max\{f(x) : x \in I\}$. Prove that $f|_I$ cannot be 1-1. [Of course, the same conclusion holds if $f(x_0) = \min\{f(x) : x \in I\}$.]

5. Find an example of a function $f : [0,1] \to [0,1]$ that is 1-1 and onto but not monotone on any $(a, b) \subseteq I$ where $a < b$.

6. Complete Case 3 of the proof of Theorem 5.5.2.

7. Prove Theorem 5.5.2 in the case where f is monotone decreasing on I.

8. Prove Corollary 5.5.3 for the case in which f is strictly decreasing.

9. Complete the proof of Claim #1 in the proof of Theorem 5.5.4 by considering the case in which $f(b) < \min\{f(a), f(c)\}$.

10. **Cantor Function:** Find the numbers $x_1, x_2, \cdots, x_{14}$ that are the endpoints of the open intervals removed from $\mathbf{C}_3$ to create $\mathbf{C}_4$. Calculate $\varphi_{\mathbf{c}}(x_i)$ for $i = 1, 2, \cdots 14$ and verify that $\varphi_{\mathbf{c}}$ takes on the same value at both endpoints of each of these removed intervals.

11. Prove that the Cantor function $\varphi : [0,1] \to [0,1]$ can be extended to a function $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ that is continuous and monotone increasing, but not strictly increasing on any nonempty open interval.

12. **(Project) Positive Integral Power Functions:** For a given $n \in \mathbb{N}$, the function $f(x) = x^n$ is continuous on $\mathbb{R}$ [see Theorem 5.1.7]. Prove that

    (a) f is positive and strictly increasing on $(0, +\infty)$. [See Exercise 1.3.19.]

(b) f is $\left\{ \begin{array}{l} \text{positive and strictly decreasing on } (-\infty, 0) \text{ if } n \text{ is even;} \\ \text{negative and strictly increasing on } (-\infty, 0) \text{ if } n \text{ is odd} \end{array} \right\}$.

(c) $f(\mathbb{R}) = \left\{ \begin{array}{l} [0, +\infty) \text{ if } n \text{ is even;} \\ (-\infty, +\infty) \text{ if } n \text{ is odd} \end{array} \right\}$. [See proof of Theorem 5.3.16.]

(d) if n is even, then $f : [0, +\infty) \to [0, +\infty)$ is invertible, and f^{-1} is continuous and strictly increasing.

(e) if n is odd, then $f : \mathbb{R} \to \mathbb{R}$ is invertible, and f^{-1} is continuous and strictly increasing.

13. **(Project) Negative Integral Power Functions:** For a given $n \in \mathbb{N}$, the function $f(x) = x^{-n}$ is continuous on $\mathbb{R} - \{0\}$ [see Theorem 5.1.8]. State and prove properties analogous to (a)–(e) in Exercise 12 above.

14. **(Project) The nth Root Function:** Suppose $n \in \mathbb{N}$. Using the inverse function f^{-1} of the function $f(x) = x^n$ described in Exercise 12, we define the n^{th} root function g by

$$g(x) = \sqrt[n]{x} = x^{\frac{1}{n}} = \left\{ \begin{array}{l} f^{-1}(x) \text{ if } x \geq 0; \\ -f^{-1}(-x) \text{ if } x < 0 \text{ and } n \text{ is odd} \end{array} \right\}.$$

Prove that:

(a) this definition of $\sqrt[n]{x}$ is consistent with that given in Theorem 5.3.16.

(b) if $x < 0$ and n is odd, $\sqrt[n]{x} = -\sqrt[n]{|x|}$ (e.g., $\sqrt[5]{-3} = -\sqrt[5]{3}$).

(c) if n is even, then $g : [0, +\infty) \overset{onto}{\to} [0, +\infty)$; and if n is odd, then $g : \mathbb{R} \overset{onto}{\to} \mathbb{R}$. In both cases, g is strictly increasing (hence 1-1) and continuous.

15. **(Project) Rational Exponents:** First, if $x \neq 0$, we define $x^0 = 1$. If $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and m and n have no common prime factor, then whenever $\sqrt[n]{x}$ is defined, we define $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$.

(a) Prove that if $m \in \mathbb{Z}$, $n \in \mathbb{N}$, and m and n have no common prime factor, then whenever $\sqrt[n]{x}$ is defined,

$$(1)\ x^{\frac{m}{n}} = \sqrt[n]{x^m} \text{ and } (2)\ x^{\frac{mk}{nk}} = x^{\frac{m}{n}}, \forall k \in \mathbb{N};$$

Thus, $\forall x > 0$, $\forall r \in \mathbb{Q}$, we define x^r using $r = \frac{m}{n}$ as above. Note that $x^r > 0$.

(b) Prove that the following "laws of exponents" hold: $\forall r, s \in \mathbb{Q}$ and whenever x^r and x^s are defined,

$$\begin{array}{lll} (1)\ (xy)^r = x^r y^r & (3)\ x^r x^s = x^{r+s} & (5)\ 1^r = 1 \\ (2)\ (x^r)^s = x^{rs} & (4)\ x^r / x^s = x^{r-s} & (6)\ x^{-r} = \dfrac{1}{x^r} \end{array}$$

[Use the definition of x^r given above; also see Exercise 1.4.13.]

(c) Let $r \in \mathbb{Q}$. Prove that if $r > 0$, the function $f(x) = x^r$ is positive, continuous, and strictly increasing on $(0, +\infty)$; also if $r < 0$, the function $f(x) = x^r$ is positive, continuous, and strictly decreasing on $(0, +\infty)$.

5.6 *Exponentials, Powers, and Logarithms

> If "early" definition and use of exponential and logarithm functions are not of concern in your course, this section should be omitted. These functions will be reintroduced in Chapter 7.
>
> This section is another demonstration of the power of the monotone convergence theorem. It can be covered as a class project, assigning portions of the material to small groups. The proofs may seem a bit tedious.

The trend in recent years has been toward introducing exponential and logarithmic functions earlier in elementary calculus courses than had been customary in preceding decades. The motivation for this comes from wanting to make these highly useful functions available for use in examples and applications as soon as possible. This trend has resulted in a slightly embarrassing situation: these functions are introduced and used before they have been rigorously defined. Students are asked to believe claims about existence and continuity of these functions, and related limits, on the basis of plausibility arguments. While acceptable in an elementary calculus course, such an argument does not meet the standards of rigor required by a real analysis course.

This section is included here to make rigorous the notions of exponential, power, and logarithm functions, in recognition of the trend to introduce these functions as early as feasible.

The traditional approach taken in elementary calculus courses has been to first define the natural logarithm, $\ln x = \int_1^x \frac{1}{t} dt$. (Indeed, we shall take this approach in Section 7.7.) While this definition seems contrived and bears little resemblance to the approach to logarithms taken in elementary algebra courses, the function so defined is shown to have the usual algebraic properties associated with logarithms. Strangely, these properties are derived using the derivative of this function and properties of antiderivatives (strange, since these properties seem to have nothing to do with derivatives). Since $\ln x$ is a strictly increasing function, it has an inverse, which is denoted e^x. This, too, is shown to have the algebraic properties associated with exponential functions. When this approach is taken, the derivative of $\ln x$ is obvious from its definition, while the derivative of e^x is found as the derivative of an inverse function. Finally, the general exponential function a^x (for $a > 0$) is defined by $a^x = e^{x \ln a}$ and the general logarithm function is defined as the inverse of a^x. It is easy to

show that $\log_a x = \ln x / \ln a$. In a later chapter of elementary calculus, power series expansions of these functions are easily derived from these definitions. A serious disadvantage of this approach is that it requires us to delay defining these functions until the theory of Riemann integration has been developed, later in the course.

Another approach often taken in real analysis textbooks is to delay the actual definition of e^x until power series have been studied. Indeed, we take this approach in our Chapter 8. In a straightforward manner in Section 8.8 we define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Then, using the machinery of power series, we prove that this function has all the algebraic properties expected of the exponential function. We obtain its derivative as a power series, and easily see that it is e^x itself. We then obtain its inverse function, $\ln x$, and show it to have the expected algebraic properties and derivative.

The power series approach provides quick and direct definitions, even though it leaves significant algebraic details to be worked out using power series methods. One advantage of this approach is that it makes obvious a beautiful connection between the exponential function and the sine and cosine functions. When complex numbers are allowed, it leads quickly to the elegant formula $e^{i\theta} = \cos\theta + i\sin\theta$, from which Euler's famous formula, $e^{i\pi} = -1$, is an immediate consequence. A big disadvantage of this approach, however, is that it requires us to postpone defining the exponential and logarithm functions until after the theory of power series has been developed.

The approach we take here has an elegance of its own. While foregoing the sophisticated power of either the Riemann integral or power series, it demonstrates the power of the concepts already developed in this course, particularly the ideas of supremum and infimum, limits of sequences and functions, continuity, monotonicity, and even the denseness of the rational numbers in $\mathbb{R}$.

The problem of defining a^x, where $a > 0$ and x is an arbitrary real number, is more subtle than one might expect. In the exercises at the end of Section 5.5, we indicated how we can use the inverse function theorem for continuous functions to define $a^{1/n}$ $(n \in \mathbb{N})$ and consequently a^r for all rational numbers r. It is straightforward to show that the function $f(r) = a^r$, defined in this manner for rational numbers r, has all the expected algebraic properties. [See Exercise 5.5.15.] In the present section we take up the problem of extending the domain of this function in a natural way to the set of all real numbers.

A related function is $g(x) = x^c$, where c is a constant real number. In elementary calculus, we usually presume the existence of this function, and we intuitively accept the claim that its derivative is $g'(x) = cx^{c-1}$. Without a rigorous definition of this function, however, we cannot even prove that it is continuous, let alone differentiable. In this section we place this function on a firm foundation. It turns out that we use this function $g(x) = x^c$ (constant

exponent) to derive an important property of the function $f(x) = a^x$ (constant base). Thus, the developments of these two functions are interrelated.

The disadvantage of this "early" approach to exponential and logarithm functions is that it requires a rather difficult and tedious, "brute-force" approach. On the other hand, it puts to use many of the concepts and techniques we have developed so far. You will see.

EXPONENTIAL FUNCTIONS

Let a be any positive real number. In Exercise 5.5.15(b), we showed that $\forall r \in \mathbb{Q}$, a^r is defined and positive. We now extend this definition to obtain a function $f(x) = a^x$ defined for all $x \in \mathbb{R}$. First, we need a few technical lemmas.

Lemma 5.6.1 *For each $a > 1$ the exponential function $f : \mathbb{Q} \to \mathbb{R}$ given by $f(r) = a^r$ (as defined in Exercise 5.5.15) is positive and strictly increasing on $\mathbb{Q}$. For each $0 < a < 1$ the exponential function $f : \mathbb{Q} \to \mathbb{R}$ given by $f(r) = a^r$ (as defined in Exercise 5.5.15) is positive and strictly decreasing on $\mathbb{Q}$.*

Proof. Suppose $a > 1$. Let $r < s$ in $\mathbb{Q}$. Then $\exists\, m, n \in \mathbb{Z}$ and $\exists p \in \mathbb{N} \ni r = m/p$, $s = n/p$ and $m < n$. Then $a^{n-m} \geq a > 1$. Thus,

$$a^n = a^m \cdot a^{n-m} > a^m.$$

By Exercise 5.5.14, $x^{1/p}$ is strictly increasing on $[0, +\infty)$, so

$$(a^m)^{1/p} < (a^n)^{1/p}$$
$$a^r = a^{m/p} < a^{n/p} = a^s.$$

Thus, f is strictly increasing on $\mathbb{Q}$.

For $0 < a < 1$, $f(r) = a^r = \dfrac{1}{\left(\frac{1}{a}\right)^r}$, and the desired result follows from the above argument, since $\frac{1}{a} > 1$. ■

Lemma 5.6.2 *Given any $x \in \mathbb{R}$, there exists a monotone increasing sequence $\{r_n\}$ of rational numbers converging to x.*

Proof. Exercise 1. ■

Lemma 5.6.3 *Let $a \geq 1$ and $x \in \mathbb{R}$. If $\{r_n\}$ is any monotone increasing sequence of rational numbers converging to x, then $\{a^{r_n}\}$ converges.*

Proof. Let $a \geq 1$ and $x \in \mathbb{R}$. Suppose $\{r_n\}$ is any monotone increasing sequence of rational numbers converging to x. Since the function $f : \mathbb{Q} \to \mathbb{R}$ given by $f(r) = a^r$ is strictly increasing, $\{a^{r_n}\}$ is monotone increasing. If r is

any rational number greater than x, then $\forall n \in \mathbb{N}$, $r_n \le x < r$, so $a^{r_n} < a^r$. Hence $\{a^{r_n}\}$ is bounded above. Thus, by the monotone convergence theorem, $\{a^{r_n}\}$ converges. ■

Lemma 5.6.4 *Let $a \ge 1$ and $x \in \mathbb{R}$. If $\{r_n\}$ and $\{s_n\}$ are monotone increasing sequences of rational numbers converging to x, then $\lim_{n\to\infty} a^{r_n} = \lim_{n\to\infty} a^{s_n}$.*

Proof. Suppose a, x, $\{r_n\}$ and $\{s_n\}$ satisfy the hypotheses. Since the theorem is trivially true when $a = 1$, we shall assume $a > 1$. By Lemma 5.6.3, $\exists\, L = \lim_{n\to\infty} a^{r_n}$ and $\exists\, M = \lim_{n\to\infty} a^{s_n}$. We shall prove $L = M$.

Since $\{r_n\}$ and $\{s_n\}$ are monotone increasing, and the function $f(r) = a^r$ is strictly increasing on $\mathbb{Q}$, $\{a^{r_n}\}$ and $\{a^{s_n}\}$ are monotone increasing, so

$$L = \sup\{a^{r_n} : n \in \mathbb{N}\} \text{ and } M = \sup\{a^{s_n} : n \in \mathbb{N}\}.$$

Define a new sequence $\{S_n\}$ by $S_n = s_n - \frac{1}{n}$. Then $\{S_n\}$ is a strictly increasing sequence of rational numbers with

$$\lim_{n\to\infty} a^{S_n} = \lim_{n\to\infty} a^{s_n - \frac{1}{n}} = \lim_{n\to\infty} \frac{a^{s_n}}{a^{\frac{1}{n}}} = \frac{M}{1} = M. \text{ (See Example 2.3.9.)}$$

Let $\varepsilon > 0$. Then $\exists\, n_1 \in \mathbb{N} \ni$

$$M - \varepsilon < a^{S_{n_1}} \le M. \qquad (14)$$

Now $\{S_n\}$ is a strictly increasing sequence converging to x, so

$$S_{n_1} < x.$$

Since $\{r_n\}$ is a monotone increasing sequence converging to x, $\exists\, n_2 \in \mathbb{N} \ni$

$$n \ge n_2 \Rightarrow S_{n_1} < r_n \le x,$$

and, since the function $f(r) = a^r$ is strictly increasing on $\mathbb{Q}$,

$$n \ge n_2 \Rightarrow a^{S_{n_1}} < a^{r_n}. \qquad (15)$$

Putting together (14) and (15) we have, since limits preserve inequalities,

$$M - \varepsilon \le \lim_{n\to\infty} a^{r_n} = L$$
$$M - L \le \varepsilon.$$

Since this holds for all $\varepsilon > 0$, the forcing principle implies

$$M - L \le 0; \text{ i.e.,}$$
$$M \le L.$$

Reversing the roles of $\{r_n\}$ and $\{s_n\}$ will allow us to prove $L \leq M$. Therefore, $L = M$. ■

Because of Lemmas 5.6.2–5.6.4 we can make the following definition:

Definition 5.6.5 (Exponential Functions a^x, where $a > 0$)
Let $a \geq 1$. Then $\forall x \in \mathbb{R}$, we define $a^x = \lim_{n\to\infty} a^{r_n}$, where $\{r_n\}$ is any monotone increasing sequence of rational numbers converging to x.

If $0 < a < 1$, then $a^{-1} > 1$, so $\forall x \in \mathbb{R}$, we define $a^x = \dfrac{1}{(a^{-1})^x}$.

Note that these two definitions are consistent since, when $0 < a < 1$ and $\{r_n\}$ is any monotone increasing sequence of rational numbers converging to x, $\lim_{n\to\infty} a^{r_n} = \lim_{n\to\infty} \dfrac{1}{(a^{-1})^{r_n}} = \dfrac{1}{\lim_{n\to\infty}(a^{-1})^{r_n}} = \dfrac{1}{(a^{-1})^x} = a^x$. Thus, $\forall a \geq 0$ and $\forall x \in \mathbb{R}$, $a^x = \lim_{n\to\infty} a^{r_n}$.

Remark: When x is a rational number, we now have two definitions of a^r, the definition just given and the one given in Exercise 5.5.15. To see that these two definitions agree when x is a rational number, just use the constant sequence $\{x_n\} = \{x\}$ in Definition 5.6.5, and apply Lemma 5.6.4.

Theorem 5.6.6 *Let $a > 1$. Then the exponential function $f(x) = a^x$ defined in Definition 5.6.5 is a strictly increasing positive-valued function with domain $\mathbb{R}$ whose range has no upper bound.*

Proof. Let $a > 1$. Consider the exponential $f(x) = a^x$ defined in Definition 5.6.5.

(1) $\mathcal{D}(f) = \mathbb{R}$.

(2) Let $x < y$ in $\mathbb{R}$. Since the rationals are dense in $\mathbb{R}$, $\exists$ rational number q such that $x < q < y$. Let $\{r_n\}$ and $\{s_n\}$ be monotone increasing sequences of rational numbers such that $r_n \to x$ and $s_n \to y$. Since the terms of $\{s_n\}$ must eventually be greater than q, we may assume, without loss of generality, that $\forall n \in \mathbb{N}$,

$$r_n \leq x < q < s_n \leq y.$$

Then, by Lemma 5.6.1,

$$a^{r_n} < a^q < a^{s_n}.$$

Thus, since limits preserve inequalities, and since $\{a^{s_n}\}$ is monotone increasing,

$$a^x = \lim_{n\to\infty} a^{r_n} \leq a^q < \lim_{n\to\infty} a^{s_n} = a^y.$$

Thus, $a^x < a^y$; that is, f is strictly increasing on $\mathbb{R}$.

(3) Let $x \in \mathbb{R}$. In Definition 5.6.5, the sequence $\{a^{r_n}\}$ is a monotone increasing sequence of positive numbers, so $a^x = \lim_{n\to\infty} a^{r_n} > 0$. Thus, $f(x) = a^x$ is positive-valued.

(4) We know that $\lim_{n\to\infty} a^n = +\infty$ (Example 2.4.6). Thus, the range of $f(x) = a^x$ has no upper bound. ■

Corollary 5.6.7 (a) *If $a > 1$, then $\lim_{x\to+\infty} a^x = +\infty$ and $\lim_{x\to-\infty} a^x = 0$.*

(b) *If $0 < a < 1$, then $\lim_{x\to+\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = +\infty$.*

Proof. Exercise 2. ■

We would also like to prove that the exponential function $f(x) = a^x$ is continuous everywhere on $\mathbb{R}$. The proof of that, however, must wait until we establish a few more properties of this function.

Theorem 5.6.8 (***Algebraic Properties of Exponents***) *Let $a, b > 0$. The exponential function $f(x) = a^x$ satisfies the following algebraic properties:*

(a) $a^0 = 1$	(d) $(ab)^x = a^x b^x$
(b) $a^x a^y = a^{x+y}$	(e) $a^{-x} = (a^x)^{-1} = (a^{-1})^x$
(c) $a^x / a^y = a^{x-y}$	(f) $(a/b)^x = a^x / b^x$

Proof. (a) Use the constant sequence $\{r_n\} = \{0\}$ in Definition 5.6.5.

(b) Let $x, y \in \mathbb{R}$. Then $\exists$ monotone increasing sequences $\{r_n\}$, $\{s_n\}$ of rational numbers $\ni r_n \to x$ and $s_n \to y$. Then $\{r_n + s_n\}$ is a monotone increasing sequence of rational numbers converging to $x + y$, so

$$\begin{aligned} a^x a^y &= \left(\lim_{n\to\infty} a^{r_n}\right)\left(\lim_{n\to\infty} a^{s_n}\right) = \lim_{n\to\infty} (a^{r_n} a^{s_n}) \\ &= \lim_{n\to\infty} (a^{r_n+s_n}) \qquad \text{(by Exercise 5.5.15)} \\ &= a^{x+y}. \end{aligned}$$

(c) Exercise 3.

(d) Let $a, b \in \mathbb{R}$. Let $\{r_n\}$ be a monotone increasing sequence of rational numbers converging to x. Then

$$\begin{aligned} (ab)^x &= \lim_{n\to\infty} (ab)^{r_n} = \lim_{n\to\infty} (a^{r_n} b^{r_n}) \qquad \text{(by Exercise 5.5.15)} \\ &= \left(\lim_{n\to\infty} a^{r_n}\right)\left(\lim_{n\to\infty} b^{r_n}\right) \\ &= a^x b^x. \end{aligned}$$

(e) Exercise 4.

(f) Exercise 5. ■

Lemma 5.6.9 *Let $a \geq 1$ and $x \in \mathbb{R}$. If $\{t_n\}$ is any monotone decreasing sequence of rational numbers converging to x, then* $\lim\limits_{n\to\infty} a^{t_n} = a^x$.

Proof. Let a, x, and $\{t_n\}$ satisfy the hypotheses. Define the sequence $\{r_n\}$ by $r_n = 2x - t_n$. Then, $\{r_n\}$ is a monotone increasing sequence of rational numbers and $r_n \to x$, So, by Definition 5.6.5, $a^{r_n} \to a^x$. Using the properties of a^x obtained in Theorem 5.6.8,

$$\begin{aligned}\lim_{n\to\infty} a^{t_n} &= \lim_{n\to\infty} a^{2x-r_n} = \lim_{n\to\infty} \frac{a^{2x}}{a^{r_n}} \\ &= \frac{a^{2x}}{\lim\limits_{n\to\infty} a^{r_n}} = \frac{a^{2x}}{a^x} \\ &= a^x. \qquad \blacksquare\end{aligned}$$

Theorem 5.6.10 *Let $a > 0$ and $x \in \mathbb{R}$. If $\{x_n\}$ is any sequence of real numbers converging to x, then* $\lim\limits_{n\to\infty} a^{x_n} = a^x$.

Proof. Suppose a, x, and $\{x_n\}$ satisfy the hypotheses.
Case 1 ($a \geq 1$): We know that $\exists$ strictly increasing sequence $\{r_n\}$ of rational numbers and $\exists$ strictly decreasing sequence $\{s_n\}$ of rational numbers such that $r_n \to x$ and $s_n \to x$. Then, by Definition 5.6.5 and Lemma 5.6.9,

$$\lim_{n\to\infty} a^{r_n} = a^x = \lim_{n\to\infty} a^{s_n}.$$

Also, in what follows, recall that the exponential function $f(x) = a^x$ is strictly increasing.

Let $\varepsilon > 0$. Then, since $r_n \to x$ and $s_n \to x$, $\exists\, n_1 \in \mathbb{N} \ni$

$$a^x - \varepsilon < a^{r_{n_1}} < a^x < a^{s_{n_1}} < a^x + \varepsilon.$$

Note that $r_{n_1} < x < s_{n_1}$. Thus, since $x_n \to x$, $\exists\, n_0 \in \mathbb{N} \ni$

$$n \geq n_0 \Rightarrow r_{n_1} < x_n < s_{n_1}.$$

Since $f(x) = a^x$ is strictly increasing, this means,

$$\begin{aligned} n \geq n_0 &\Rightarrow a^x - \varepsilon < a^{r_{n_1}} < a^{x_n} < a^{s_{n_1}} < a^x + \varepsilon \\ &\Rightarrow |a^{x_n} - a^x| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{n\to\infty} a^{x_n} = a^x$.

Case 2 ($0 < a < 1$): Then $a^{-1} > 1$ and by Case 1,

$$\lim_{n\to\infty} a^{x_n} = \lim_{n\to\infty} \frac{1}{(a^{-1})^{x_n}} = \frac{1}{\lim\limits_{n\to\infty} (a^{-1})^{x_n}} = \frac{1}{(a^{-1})^x} = a^x. \qquad \blacksquare$$

Theorem 5.6.11 (***Exponential Functions, I***) *For $a > 1$, the exponential function $f(x) = a^x$ is a continuous, strictly increasing function on $\mathbb{R}$, with range $(0, +\infty)$.*

Proof. Let $a > 1$ and define $f(x) = a^x$. All that remains to prove is that f is continuous everywhere on $\mathbb{R}$ and has range $(0, +\infty)$. Continuity follows from Theorem 5.6.10. Since f is continuous everywhere on $\mathbb{R}$, and $\mathbb{R}$ is an interval, the range of f must be an interval, by Theorem 5.3.8. By Corollary 5.6.7, this range must be $(0, +\infty)$. ∎

We get a similar, but different, result when $0 < a < 1$.

Theorem 5.6.12 (***Exponential Functions, II***) *For $0 < a < 1$, the exponential function $f(x) = a^x$ is a continuous, strictly decreasing function on $\mathbb{R}$ with range $(0, +\infty)$.*

Proof. Exercise 7.

Lemma 5.6.13 *Suppose $a > 0$ and $x \in \mathbb{R}$. Then $a^x = 1 \Leftrightarrow a = 1$ or $x = 0$.*

Proof. Part 1 ($\Leftarrow$): By Definition 5.6.5, $1^x = 1$, and by Exercise 5.5.15, $a^0 = 1$.

Part 2 ($\Rightarrow$): Suppose $a^x = 1$ and $a \neq 1$. The exponential function $f(t) = a^t$ is 1-1, by Theorems 5.6.11 and 5.6.12. Thus,

$$a^x = 1 \Rightarrow a^x = a^0 \Rightarrow x = 0.$$

Therefore, if $a^x = 1$, then either $a = 1$ or $x = 0$. ∎

CONSTANT POWER FUNCTIONS

Theorem 5.6.14 (***Positive Power Functions***) *Let $t > 0$ be fixed. The power function $f(x) = x^t$, defined with the help of 5.6.5, is positive, strictly increasing, and continuous everywhere on $(0, +\infty)$, $\lim_{x \to \infty} x^t = +\infty$, and $\lim_{x \to 0^+} x^t = 0$.*

Proof. Let $t > 0$ be fixed throughout the proof that follows.

(a) By Definition 5.6.5, the power function $f(x) = x^t$ is defined everywhere on $(0, +\infty)$, and positive there.

(b) To show that f is strictly increasing on $(0, +\infty)$, suppose $x < y$ in $(0, +\infty)$. Then, $\exists$ rational numbers q_1, q_2 such that

$$x < q_1 < q_2 < y.$$

Let $\{t_n\}$ be a sequence of positive rational numbers converging to t. By Theorem 5.6.10, $x^{t_n} \to x^t$ and $y^{t_n} \to y^t$. Since t_n is rational, the function $g_n(x) = x^{t_n}$ is strictly increasing on $(0, +\infty)$, so

$$x^{t_n} < q_1^{t_n} < q_2^{t_n} < y^{t_n}.$$

Since limits preserve inequalities,

$$x^t \le q_1^t \le q_2^t \le y^t.$$

By Exercise 5.5.15, $q_1^t < q_2^t$. Therefore

$$x^t < y^t.$$

(c) We next show that the power function $f(x) = x^t$ is continuous at every positive number x_0. Let $x_0 > 0$. Since f is strictly increasing on $(0, +\infty)$, we have by Theorem 5.2.17,

$$\lim_{x \to x_0^-} f(x) = \sup\{x^t : x < x_0\} \le f(x_0) \le \inf\{x^t : x > x_0\} = \lim_{x \to x_0^+} f(x).$$

We wish to prove that $\lim_{x \to x_0^-} f(x) = f(x_0) = \lim_{x \to x_0^+} f(x)$; that is,

$$\sup\{x^t : x < x_0\} = x_0^t = \inf\{x^t : x > x_0\}.$$

Case 1 ($x_0 \ge 1$):

(1) First, we prove that $\sup\{x^t : x < x_0\} = x_0^t$. Let $\{r_n\}$ be a monotone increasing sequence of rational numbers such that $r_n \to t$. By Definition 5.6.5,

$$x_0^{r_n} \to x_0^t.$$

Let $\varepsilon > 0$. Then, since $\{x_0^{r_n}\}$ is monotone increasing, $\exists\ n_0 \in \mathbb{N} \ni$

$$\begin{aligned} n \ge n_0 &\Rightarrow x_0^t - \varepsilon < x_0^{r_n} \le x_0^t \\ &\Rightarrow x_0^t - x_0^{r_n} < \varepsilon. \end{aligned} \tag{16}$$

Fix an integer $n \ge n_0$. By Exercise 5.5.15 (c), the power function $g_n(x) = x^{r_n}$ is continuous at x_0. Thus, $\exists\ \delta > 0 \ni$

$$|x - x_0| < \delta \Rightarrow |x^{r_n} - x_0^{r_n}| < \varepsilon - (x_0^t - x_0^{r_n}) \qquad \text{[See (16).]}$$

So,

$$\begin{aligned} x_0 - \delta < x < x_0 &\Rightarrow (x_0^t - x_0^{r_n}) - \varepsilon < x^{r_n} - x_0^{r_n} < \varepsilon - x_0^t + x_0^{r_n} \\ &\Rightarrow x_0^t - \varepsilon < x^{r_n}. \end{aligned} \tag{17}$$

Pick any $x \in (x_0 - \delta, x_0)$. Since the function $f(x) = x^t$ is strictly increasing, and $r_n < t$,

$$x^{r_n} < x^t. \tag{18}$$

Putting (17) and (18) together,

$$x_0 - \delta < x < x_0 \Rightarrow x_0^t - \varepsilon < x^t = f(x).$$

Hence $\sup\{x^t : x < x_0\} > x_0^t - \varepsilon$. By the forcing principle, this implies $\sup\{x^t : x < x_0\} \geq x_0^t$. Therefore, $\sup\{x^t : x < x_0\} = x_0^t$.

(2) Next, we must prove that $\inf\{x^t : x > x_0\} = x_0^t$. (Exercise 8)

Summarizing: in Case 1, $\lim\limits_{x \to x_0^-} f(x) = f(x_0) = \lim\limits_{x \to x_0^+} f(x)$. Therefore, the power function $f(x) = x^t$ is continuous at every $x_0 \geq 1$.

Case 2 ($0 < x_0 < 1$):

Then $\dfrac{1}{x_0} > 1$. Thus, by Case 1, $\lim\limits_{x \to \frac{1}{x_0}} x^t = \left(\dfrac{1}{x_0}\right)^t$. Let $\{x_n\}$ be any monotone decreasing sequence of real numbers converging to x_0. Then $\left\{\dfrac{1}{x_n}\right\}$ is a monotone increasing sequence converging to $\dfrac{1}{x_0}$. As proved in Case 1, the power function $f(x) = x^t$ is continuous at $\dfrac{1}{x_0}$, so $\left(\dfrac{1}{x_n}\right)^t \to \left(\dfrac{1}{x_0}\right)^t$. That is, $\dfrac{1}{x_n^t} \to \dfrac{1}{x_0^t}$. Thus, by the algebra of limits, $x_n^t \to x_0^t$. Therefore, the power function $f(x) = x^t$ is continuous at every $0 \leq x_0 < 1$.

(d) Next we prove that $\lim\limits_{x \to \infty} x^t = +\infty$. By the density of the rational numbers in $\mathbb{R}$, $\exists\ m, n \in \mathbb{N} \ni m > 1$ and $0 < \dfrac{m}{n} < t$. Let $M > 0$. Since $\lim\limits_{x \to \infty} x^m = +\infty$ (by Theorem 4.4.18) $\exists\ x_0 > 1 \ni$

$$x_0^m > M^n.$$

Then, since the power function $g(x) = x^{1/n}$ is strictly increasing (Exercise 5.5.15),

$$x_0^{m/n} > M^{n/n} = M.$$

Since the exponential function $h(u) = x_0^u$ is strictly increasing (Theorem 5.6.6),

$$x_0^t > x_0^{m/n} > M.$$

Finally, since the power function $f(x) = x^t$ is strictly increasing (proved in (c) above),

$$x > x_0 \Rightarrow x^t > x_0^t \Rightarrow x^t > M.$$

Therefore, $\lim\limits_{x \to \infty} x^t = +\infty$.

(e) Using Theorem 4.4.21 (a),

$$\lim_{x \to 0^+} x^t = \lim_{1/x \to \infty} \frac{1}{(1/x)^t} = \lim_{u \to \infty} \frac{1}{u^t} = \lim_{u \to \infty} \frac{1}{f(u)} = 0. \quad \blacksquare$$

Theorem 5.6.15 (***Negative Power Functions***) *Let $t < 0$. The power function $f(x) = x^t$, defined with the help of Definition 5.6.5, is strictly decreasing, positive, and continuous everywhere on $(0, +\infty)$, $\lim_{x \to \infty} x^t = 0$, and $\lim_{x \to 0^+} x^t = +\infty$.*

Proof. Exercise 9. ■

Theorem 5.6.16 (***A Further Algebraic Property of Exponents***) *Let $a > 0$. Then $\forall x, y \in \mathbb{R}$, $(a^x)^y = a^{xy}$.*

Proof. Part 1. We first prove the result when x is rational; call it $x = r \in \mathbb{Q}$. Let $y \in \mathbb{R}$. Let $\{s_n\}$ be a sequence of rational numbers $\ni s_n \to y$. Then, by Definition 5.6.5,

$$(a^r)^y = \lim_{n \to \infty} (a^r)^{s_n}.$$

Also $rs_n \to ry$ so by Definition 5.6.5 and Exercise 5.5.15,

$$a^{ry} = \lim_{n \to \infty} a^{rs_n} = \lim_{n \to \infty} (a^r)^{s_n}.$$

Therefore, $a^{ry} = (a^r)^y$.

Part 2. Let $y \in \mathbb{R}$ and suppose x is any real number. Then $\exists$ sequence $\{r_n\}$ of rational numbers $\ni r_n \to x$. Then, by Definition 5.6.5, $a^{r_n} \to a^x$, so by Theorem 5.6.14,

$$(a^{r_n})^y \to (a^x)^y.$$

On the other hand, $r_n y \to xy$, so by Theorem 5.6.10,

$$a^{r_n y} \to a^{xy}.$$

But by Part 1, $(a^{r_n})^y = a^{r_n y}$. Therefore, $(a^x)^y = a^{xy}$. ■

e AND e^x AS LIMITS

In Section 2.5 we proved that the sequential limit $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ exists; we called this number e. We now take a look at several function limits that also equal e.

Theorem 5.6.17 $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$.

The first thing to see is that there is something to prove. Notice that the limit described here is not the limit of a sequence, but the limit of a function. Our task will be to make the transition from the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ to the

function $f(x) = \left(1+\frac{1}{x}\right)^x$. To make this transition we will use the "greatest integer function," $\lfloor x \rfloor =$ the greatest integer $\leq x$, introduced in Example 5.2.16.

Proof of Theorem 5.6.17:

(1) Claim: $\lim\limits_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor} = e.$

Proof: Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon$. Then $x \geq n_0 \Rightarrow \lfloor x\rfloor \geq n_0 \Rightarrow \left|\left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor} - e\right| < \varepsilon.$

(2) Claim: $\lim\limits_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = e.$

Proof: $$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{-1}$$
$$= \left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}\left(\frac{\lfloor x\rfloor+1}{\lfloor x\rfloor+2}\right)$$

Thus,
$$\lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} = \lim_{x\to\infty}\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor+1}\cdot\lim_{x\to\infty}\left(\frac{\lfloor x\rfloor+1}{\lfloor x\rfloor+2}\right) = e\cdot 1 = e.$$

(3) Finally, noting that $\lfloor x\rfloor + 1 > x$, we have, for all $x > 0$,
$$\left(1+\frac{1}{\lfloor x\rfloor+1}\right)^{\lfloor x\rfloor} < \left(1+\frac{1}{x}\right)^{\lfloor x\rfloor} \leq \left(1+\frac{1}{x}\right)^{x} \leq \left(1+\frac{1}{\lfloor x\rfloor}\right)^{x} < \left(1+\frac{1}{\lfloor x\rfloor}\right)^{\lfloor x\rfloor+1}.$$

Applying (1), (2), and the squeeze principle to these inequalities, we have $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$ ■

Corollary 5.6.18 ***(Function Limit for e)*** $\lim\limits_{x\to 0}(1+x)^{1/x} = e.$

Proof. Exercise 11. ■

Lemma 5.6.19 *Let $y \neq 0$ be a fixed real number. Then $\lim\limits_{x\to 0}(1+xy)^{1/xy} = e.$*

Proof. Let $\varepsilon > 0$. By Corollary 5.6.18, $\exists \delta > 0 \ni 0 < |x| < \delta \Rightarrow \left|(1+x)^{1/x} - e\right| < \varepsilon$. Then $0 < |x| < \delta/|y| \Rightarrow 0 < |xy| < \delta \Rightarrow \left|(1+xy)^{1/xy} - e\right| < \varepsilon$. ■

Theorem 5.6.20 ***(Function Limit for e^x)*** $\forall x \in \mathbb{R}$, $\lim\limits_{t\to 0}(1+tx)^{1/t} = e^x$.

Proof. If $x = 0$ this equation is obviously true. Thus, suppose x is a (fixed) nonzero real number. Notice that $(1+tx)^{1/t} = f(g(t))$, where $g(t) = (1+tx)^{1/tx}$ and $f(u) = u^x$. The power function f is continuous on $(0, +\infty)$, by Theorems

5.6.14 and 5.6.15; moreover, by Lemma 5.6.19, $\lim_{t\to 0} g(t) = e$. By Theorem 5.1.14, $\lim_{t\to 0}(1+tx)^{1/t} = \lim_{t\to 0}\left[(1+tx)^{1/tx}\right]^x = \lim_{t\to 0} f(g(t)) = f\left(\lim_{t\to 0} g(t)\right) = f(e) = e^x$.
■

Corollary 5.6.21 (***Sequential Limit for e^x***) $\forall x \in \mathbb{R}$, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

Proof. Exercise 12. ■

LOGARITHM FUNCTIONS

Definition 5.6.22 Let $a > 0$, $a \neq 1$. We define the function $f(x) = \log_a x$ to be the inverse of the function $g(x) = a^x$, which we have shown to be a 1-1, onto, continuous function from $\mathbb{R}$ to $(0, +\infty)$. That is, $y = \log_a x \Leftrightarrow a^y = x$.

The following two theorems show that this function satisfies all the properties that we expect logarithms to have.

Theorem 5.6.23 (***Properties of Logarithms, I***) *Let $a > 0$, $a \neq 1$. The function $f(x) = \log_a x$ has the following properties:*

(a) $f : (0, +\infty) \to \mathbb{R}$;

(b) f *is 1-1, onto, and continuous on* $(0, +\infty)$;

(c) f *is strictly increasing if* $a > 1$, *and strictly decreasing if* $0 < a < 1$;

(d) $\log_a 1 = 0$;

(e) *Suppose* $a > 1$. *Then* $\log_a x > 0$ *if* $x > 1$, *and* $\log_a x < 0$ *if* $0 < x < 1$;

(f) *Suppose* $0 < a < 1$. *Then* $\log_a x < 0$ *if* $x > 1$, *and* $\log_a x > 0$ *if* $0 < x < 1$;

(g) $\forall x \in \mathbb{R}$, $\log_a (a^x) = x$;

(h) $\forall x > 0$, $a^{\log_a x} = x$.

Proof. Exercise 13. ■

Theorem 5.6.24 (***Properties of Logarithms, II***) *Let $a > 0$, $a \neq 1$. The function $f(x) = \log_a x$ satisfies the following algebraic identities, $\forall x, y > 0$, and $r \in \mathbb{R}$:*

(a) $\log_a(xy) = \log_a x + \log_a y$; (c) $\log_a\left(\frac{1}{x}\right) = -\log_a x$;
(b) $\log_a(x/y) = \log_a x - \log_a y$; (d) $\log_a(x^r) = r\log_a x$.

Proof. Exercise 14. ■

Of course, there are many possible bases to use in logarithm functions; in fact, any positive real number other than 1 may be used. Nevertheless, two special numbers are most often used as bases: 10 (in "common" logarithms) and e (in "natural" logarithms). The following theorem tells us that we can always switch from one base to another, using a conversion formula.

Theorem 5.6.25 *Suppose $a, b > 0$, $a, b \neq 1$, and $x > 0$. Then*

(a) $\log_b x = \dfrac{\log_a x}{\log_a b} = \dfrac{\ln x}{\ln a}$; (b) $\log_b a = \dfrac{1}{\log_a b}$.

Proof. Exercise 15. ■

EXERCISE SET 5.6

1. Prove Lemma 5.6.2.
2. Prove Corollary 5.6.7.
3. Prove Theorem 5.6.8 (c).
4. Prove Theorem 5.6.8 (e).
5. Prove Theorem 5.6.8 (f).
6. Suppose $a > 1$. Prove that $a^x > 1$ if $x > 0$, and $0 < a^x < 1$ if $x < 0$.
7. Prove Theorem 5.6.12.
8. Finish proving Case 1 of Theorem 5.6.14 (d), by proving (2).
9. Prove Theorem 5.6.15
10. Prove that $\lim\limits_{x \to -\infty} \left(1 + \frac{1}{x}\right)^x = e$.
11. Prove Corollary 5.6.18. [Consider one-sided limits and use Theorem 4.4.19.]
12. Prove Corollary 5.6.21.
13. Prove Theorem 5.6.23.
14. Prove Theorem 5.6.24.
15. Prove Theorem 5.6.25.
16. Suppose $a, b > 0$, $a, b \neq 1$. Prove that $\forall x \in \mathbb{R}$, $a^x = b^{x \log_b a} = e^{x \ln a}$.
17. **(Project)** Prove that if a function $f : \mathbb{R} \to (0, +\infty)$ is strictly increasing and $\forall x, y \in \mathbb{R}$, $f(x + y) = f(x)f(y)$, then $\exists\, a > 1 \ni \forall x \in \mathbb{R}$, $f(x) = a^x$. [Hint: first find $f(0)$. Then prove that $f(x) = a^x$ if $x \in \mathbb{N}$, then if $x \in \mathbb{Z}$, then if $x \in \mathbb{Q}$, and finally if $x \in \mathbb{R}$.]
18. **(Project)** Prove that if a function $f : (0, +\infty) \to \mathbb{R}$ is strictly increasing and $\forall x, y \in \mathbb{R}$, $f(xy) = f(x) + f(y)$, then $\exists\, a > 1 \ni \forall x \in \mathbb{R}$, $f(x) = \log_a x$.

5.7 *Sets of Points of Discontinuity (Project)

> This section is icing on the cake. It answers several intriguing questions but is not needed in any of the later sections of the book, except in certain advanced exercises. It is cast in the form of a project.

In this section, we shed some light on the question of what kind of set can possibly be the set of points where a given function $f : \mathcal{D}(f) \to \mathbb{R}$ is discontinuous. (We call such a set the "set of discontinuities" of f.) Exploring this question will lead to some interesting, even surprising, results. First, a routine result.

Theorem 5.7.1 *Given any finite set $A = \{a_1, a_2, \cdots, a_n\}$ of real numbers, there is a bounded function $f : \mathbb{R} \to \mathbb{R}$ having the set A as its set of discontinuities.*

Proof. Exercise. [Hint: Think of a characteristic function.] ■

As we have seen, the set of discontinuities of a function can be infinite. For example, the Dirichlet function (5.1.11) is discontinuous everywhere, and Thomae's function (5.1.12) is discontinuous on the rational numbers. The latter example leads us to ask whether the set of rational numbers is special in this regard, or whether any countable set can be the set of discontinuities of a function. Here is the answer to the second part of that question.

Theorem 5.7.2 *Given any countable set A of real numbers, there is a bounded function $f : \mathbb{R} \to \mathbb{R}$ having A as its set of discontinuities.*

Proof. Exercise. Here is a recommended approach. Let A be a countable set. The finite case is covered in Theorem 5.7.1. Thus, assume $A = \{a_1, a_2, \cdots, a_n, \cdots\}$ is infinite. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \left\{ \begin{array}{ll} \frac{1}{n} & \text{if } x = a_n \in A \\ 0 & \text{otherwise} \end{array} \right\}$. To show that f is discontinuous on A, show that A^c is dense in $\mathbb{R}$ and follow the argument given in the proof of part (b) of Theorem 5.1.12. To show that f is continuous on A^c, let $x_0 \in A^c$. Then $f(x_0) = 0$. Let $\varepsilon > 0$. Then $\exists n \in \mathbb{N} \ni \frac{1}{n} < \varepsilon$. Then there is a neighborhood N of x_0 such that $a_1 \notin N$, $a_2 \notin N, \cdots, a_n \notin N$. Then $x \in N \Rightarrow 0 \leq f(x) < \frac{1}{n} < \varepsilon$. ■

While the above result is remarkable, here is another, even more amazing result.

Theorem 5.7.3 *Given any countable set A of real numbers, there is a bounded* monotone increasing *function $f : \mathbb{R} \to \mathbb{R}$ having A as its set of discontinuities.*

Proof. The following proof is a little sketchy. You are asked to fill in the missing details. Let $A = \{a_1, a_2, \cdots, a_n, \cdots\}$ be a countable set. We'll assume A is infinite; the finite case is left to you. Define $f : \mathbb{R} \to \mathbb{R}$ as follows, using decimal notation:

$$\forall x \in \mathbb{R},\ f(x) = 0.x_1x_2x_3\cdots x_n\cdots, \text{ where } x_i = \left\{\begin{matrix} 0 \text{ if } x < a_i \\ 1 \text{ if } x \geq a_i \end{matrix}\right\}.$$

That is, $f(x)$ is the infinite decimal whose digits x_i are 0 or 1 according to whether $x < a_i$ or $x \geq a_i$.

Then (justify the following):

(a) f is bounded.

(b) $\forall x, y \in \mathbb{R},\ x \leq y \Rightarrow \forall i \in \mathbb{N},\ x_i \leq y_i$.

(c) f is monotone increasing.

(d) Suppose $x < y$. Then $f(x) = f(y) \Leftrightarrow (x, y]$ contains no points of A.

(e) Lemma: Let $x \in \mathbb{R}$. Choose any $n_0 \in \mathbb{N}$ and let $\delta = \min\{|x - a_i| : a_i \neq x,\ i = 1, 2, \cdots n_0\}$. Then $\forall y \in N_\delta(x)$, the first n_0 digits of $f(x)$ and $f(y)$ agree.

(f) CLAIM: f is continuous on A^c. To prove this, suppose $x \in A^c$.

Case 1: x is not a cluster point of A. Then $\exists\, \delta > 0 \ni N_\delta(x)$ contains no point of A. Then by (d) above, f is constant on some neighborhood of x, so f is continuous at x.

Case 2: x is a cluster point of A. Let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N} \ni 1/10^{n_0} < \varepsilon$. Let $\delta = \min\{|x - a_i| : i = 1, 2, \cdots n_0\}$. By (e) above, $|x - y| < \delta \Rightarrow$ the first n_0 digits of $f(x)$ and $f(y)$ agree so $|x - y| < \delta \Rightarrow |f(x) - f(y)| < 1/10^{n_0} < \varepsilon$. Therefore, f is continuous at x.

(g) CLAIM: f is discontinuous at every point of A. To prove this, consider a fixed $a_i \in A$.

Case 1: a_i is not a cluster point of $A \cap (-\infty, a_i)$. Then $\exists\, \delta > 0 \ni (a_i - \delta, a_i)$ contains no point of A. Let $x \in (a_i - \delta, a_i)$. Show that the decimals representing $f(x)$ and $f(a_i)$ differ only in the i^{th} place; the i^{th} digit of $f(x)$ is 0 while that of $f(a_i)$ is 1. Thus, $f(x) = f(a_i) - \dfrac{1}{10^i}$, and so $\displaystyle\lim_{x \to a_n^-} f(x) = f(a_i) - \frac{1}{10^i} \neq f(a_i)$. Therefore, f is not continuous at a_i.

Case 2: a_i is a cluster point of $A \cap (-\infty, a_i)$. Let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N} \ni 1/10^{n_0} < \varepsilon$. Let $\delta = \min\{|a_j - a_i| : j = 1, 2, \cdots n_0,\ j \neq j\}$.

Let $a_i - \delta < x < a_i$. Since a_i is a cluster point of $A \cap (-\infty, a_i)$, $\exists j \neq i \ni a_j \in (x, a_i)$. But $j > n_0$ by definition of δ. Thus, $f(x)$, $f(a_i)$, and $f(a_j)$ are decimals that agree in the first n_0 places, but the i^{th} digit of $f(a_i)$ is 1 while that of $f(a_j)$ is 0. Therefore, $f(a_i) - f(x) \geq f(a_i) - f(a_j) \geq \dfrac{1}{10^i}$, so $\forall x \in (a_i - \delta, a_i)$, $f(x) \leq f(a_i) - \dfrac{1}{10^i}$. Thus, $\displaystyle\lim_{x \to a_n^-} f(x) < f(a_i)$. Therefore, f is not continuous at a_i. ■

OSCILLATION OF A FUNCTION

Before learning more about the sets of discontinuity of functions, we make what may seem like only an interesting detour. You will eventually see the purpose of this detour. We define the **oscillation of a function at a point** as a measure of how wildly discontinuous the function is at that point. We first define the oscillation of a function on a set, and then use that definition to define the oscillation of the function at a specific point.

Definition 5.7.4 Let $f : \mathcal{D}(f) \to \mathbb{R}$, $A \subseteq \mathcal{D}(f)$. If f is bounded on A, we define the **oscillation of** f **on** A to be $W_f(A) = \sup f(A) - \inf f(A)$. If f is unbounded on A, we define $W_f(A) = +\infty$.

Exercise 5.7.5 Prove that if $A \subseteq B$, then $W_f(A) \leq W_f(B)$.

Definition 5.7.6 Let $x_0 \in \mathcal{D}(f)$. Define the function $\omega_{f,x_0} : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ by $\omega_{f,x_0}(\varepsilon) = W_f\left(N_\varepsilon(x_0) \cap \mathcal{D}(f)\right)$.

Exercise 5.7.7 Prove that $\omega_{f,x_0} : (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ is monotone increasing.

Exercise 5.7.8 Prove that $\forall x_0 \in \mathcal{D}(f)$, $\lim\limits_{\varepsilon \to 0^+} \omega_{f,x_0}(\varepsilon)$ exists in $\mathbb{R} \cup \{+\infty\}$.

Definition 5.7.9 For each $x_0 \in \mathcal{D}(f)$, define the **oscillation of** f **at** x_0 to be $\omega_f(x_0) = \lim\limits_{\varepsilon \to 0^+} \omega_{f,x_0}(\varepsilon)$.

The function ω_f is often called the **saltus function**, and $\omega_f(x_0)$ is often called the **saltus of** f **at** x_0.

Exercise 5.7.10 Prove that $\omega_f(x_0) \geq 0$, and f is continuous at x_0 iff $\omega_f(x_0) = 0$.

Theorem 5.7.11 *The set of discontinuities of a function $f : \mathcal{D}(f) \to \mathbb{R}$ is the union of a countable collection of closed sets.*

Proof. Proceed as follows. Suppose $f : \mathcal{D}(f) \to \mathbb{R}$.

(a) $\forall \varepsilon > 0$, let $S_\varepsilon(f) = \{x \in \mathcal{D}(f) : \omega_f(x) \geq \varepsilon\}$. Prove that $S_\varepsilon(f)$ is closed. [See Exercise 5.1.23.]

(b) Prove that $S_1(f) \subseteq S_{\frac{1}{2}}(f) \subseteq S_{\frac{1}{3}}(f) \subseteq \cdots \subseteq S_{\frac{1}{n}}(f) \subseteq \cdots$, and that $\bigcup\limits_{n=1}^{\infty} S_{\frac{1}{n}}(f)$ is the set of points where f is discontinuous. ■

Definition 5.7.12 A set is called an F_σ set if it is the union of countably many closed sets.[20]

Comments: An F_σ set is not necessarily closed, as shown by Example 3.2.5. Theorem 5.7.11 shows that the set of discontinuities of a function $f : \mathcal{D}(f) \to \mathbb{R}$ must be an F_σ set. We now see that our detour brought us some new insight into sets of discontinuity. To show that not every set of real numbers can be the set of discontinuities of a function we need to show that not every set of real numbers is an F_σ set. Before doing this we introduce a few technical results.

FIRST AND SECOND CATEGORY SETS

Definition 5.7.13 Let us call an interval a **proper** interval if it is of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$, where $a < b$.

Recall that a set A is **nowhere dense** if its closure $\overline{A}$ contains no proper intervals (see Definition 3.4.16). Thus, a closed set either contains a proper interval or is nowhere dense.

Lemma 5.7.14 *Suppose A is a nowhere dense set. Then, for every proper interval I, there is a proper closed interval $J \subseteq I$ such that $J \cap A = \varnothing$.*

Proof. Suppose A is a nowhere dense set. Let I be a proper interval, say $(a, b) \subseteq I \subseteq [a, b]$, where $a < b$. Since A is nowhere dense, $(a, b) \nsubseteq \overline{A}$, so $\exists x \in (a, b) - \overline{A}$. Since $x \in \overline{A}^c$, which is open, $\exists\, \delta > 0$ such that $(x - \delta, x + \delta) \subseteq \overline{A}^c$. Choosing δ' sufficiently small, we have $[x - \delta', x + \delta'] \subseteq \overline{A}^c \cap I$. Take $J = [x - \delta', x + \delta']$. ∎

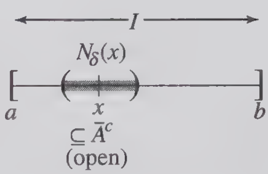

Figure 5.13

Definition 5.7.15 A set of real numbers is said to be of the **first category** if it is the union of a countable collection of nowhere dense sets; otherwise, it is said to be of the **second category**.

20. F is the first letter of the French word for "closed," and σ is the Greek equivalent of the first letter of "sum," for union.

Comments: A countable set is of the first category, since every single point set $\{x\}$ is nowhere dense. Thus, if there are any second category sets they must be uncountable. However, there are uncountable sets that are not of the second category. For example, we proved in Section 3.4 that the Cantor set is both uncountable and nowhere dense (hence, of the first category). Some authors call a first category set **meager** to indicate that it is somehow "smaller" than a second category set.

The next theorem is more powerful than it first appears. We shall put it to work to get a very interesting corollary. You may also be interested in seeing that Cantor's nested intervals theorem is useful in proving this theorem.

Theorem 5.7.16 (***Baire Category Theorem for*** $\mathbb{R}$) *Every proper interval is of the second category.*

Proof. Suppose I is a proper interval, say $(a, b) \subseteq I \subseteq [a, b]$, where $a < b$. For contradiction, suppose I is of the first category. Then $I = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense.

Since A_1 is nowhere dense, Lemma 5.7.14 guarantees that there is a proper closed interval $J_1 \subseteq I$ such that $J_1 \cap A_1 = \varnothing$. Similarly, since A_2 is nowhere dense, $\exists$ proper closed interval $J_2 \subseteq J_1$ such that $J_2 \cap A_2 = \varnothing$. Continuing in this way we get a sequence $\{J_n\}$ of proper closed intervals such that

$$J_1 \supseteq J_2 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

and such that $\forall i$, $J_i \cap A_i = \varnothing$.

By Cantor's nested intervals theorem (2.5.17) $\exists\, x_0 \in \mathbb{R}$ such that

$$x_0 \in \bigcap_{n=1}^{\infty} J_n.$$

Now, $\forall n \in \mathbb{N}$, $x_0 \in J_n$, so $x_0 \notin A_n$. Thus, $x_0 \notin \bigcup_{n=1}^{\infty} A_n = I$. But $x_0 \in I$, since $\forall n$, $x_0 \in J_n \subseteq I$. Contradiction.

Therefore, I is of the second category. ■

Lemma 5.7.17 (a) *Every subset of a first category set is of the first category.*

(b) *Every set containing a second category set is of the second category.*

(c) $\mathbb{R}$ *is of the second category.*

(d) *The union of two first category sets is a first category set. (In fact, the union of a countable collection of first category sets is a first category set.)*

(e) *The set of irrational numbers is of the second category.*

Proof. Exercise. ■

Lemma 5.7.18

(a) *A closed set either contains a proper interval or is nowhere dense.*

(b) *An F_σ set either contains a proper interval or is of the first category.*

Proof. Exercise. ■

Corollary 5.7.19 *The set of irrational numbers is not an F_σ set.*

At last, we can prove our desired result.

Corollary 5.7.20 *There does not exist a function $f : \mathcal{D}(f) \to \mathbb{R}$ having the set of irrational numbers as its set of discontinuities.*

Proof. Explain this. ■

Corollary 5.7.20 is truly remarkable! We can only marvel at the fact that there are functions continuous on the irrationals and discontinuous on the rationals, but there are NO functions continuous on the rationals and discontinuous on the irrationals!

Finally, the situation described in Theorem 5.7.11 is really an if-and-only-if condition, as expressed in the following theorem.

Theorem 5.7.21 *Let $A \subseteq \mathbb{R}$. Then there exists a function $f : \mathbb{R} \to \mathbb{R}$ having A as its set of discontinuities if and only if A is an F_σ set.*

Proof. The "⇒" direction was proved in Theorem 5.7.11. For a proof of the "⇐" direction, see Gelbaum and Olmsted [49], Section 2, # 23. ■

Example 5.7.22 Given any *closed* set A, there is a function $f : \mathbb{R} \to \mathbb{R}$ having A as its set of discontinuities.

Proof. Consider the following function:[21]

$$f(x) = \left\{ \begin{array}{c} 1 \text{ if } x \in A \cap \mathbb{Q} \\ -1 \text{ if } x \in A - \mathbb{Q} \\ 0 \text{ if } x \notin A. \end{array} \right\}. \qquad \square$$

21. The author is indebted to an anonymous reviewer of an earlier manuscript for pointing out this example.

Chapter 6

Differentiable Functions

> Sections 6.1–6.3 develop the standard introductory material on derivatives and differentiability. Section 6.4 covers important mean-value type theorems and their applications. Section 6.5 is a basic introduction to Taylor's theorem and its applications. Section 6.6 is an optional section on L'Hôpital's rule. It can be assigned as a project for independent study.

In this chapter, you will finally feel that you are studying "calculus," because our focus will be on derivatives and differentiability.

6.1 The Derivative and Differentiability

Definition 6.1.1 Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f is **differentiable at** x_0 if the limit $\displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$ exists (i.e., is finite). If this limit exists, we call it the **derivative of** f at x_0, and denote it $f'(x_0)$.

$$\text{Thus,} \quad \boxed{f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{if this limit exists.}}$$

Theorem 6.1.2 *Every "linear"[1] function $f(x) = ax + b$ is differentiable at all $x_0 \in \mathbb{R}$, and $f'(x_0) = a$.*

1. The term "linear" is time-honored but incorrect. "Affine" would be more correct. In linear algebra, "linear" has a more restrictive definition.

Proof. Let $f(x) = ax + b$. For arbitrary $x_0 \in \mathbb{R}$,

$$\begin{aligned}\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{(ax + b) - (ax_0 + b)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{ax - ax_0}{x - x_0} \\ &= \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0} = a.\end{aligned}$$

Therefore, f is differentiable at x_0 and $f'(x) = a$. ■

Corollary 6.1.3 (a) *The derivative of a constant function is* 0. *More precisely, if f is constant on a neighborhood of $x_0 \in \mathbb{R}$, then $f'(x_0) = 0$.*

(b) *The function $f(x) = x$ is differentiable everywhere, and $f'(x) = 1$.*

Example 6.1.4 The function $f(x) = |x|$ is differentiable at every x_0 except 0.

Proof. Consider the function $f(x) = |x|$. Let $x_0 \in \mathbb{R}$.
Case 1 ($x_0 > 0$): Then

$$\begin{aligned}\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} \\ &= \lim_{x \to x_0} \frac{x - x_0}{x - x_0} \quad \text{since } x > 0 \text{ as } x \to x_0 > 0 \\ &= 1.\end{aligned}$$

Thus, when $x_0 > 0$, $f'(x_0) = 1$.

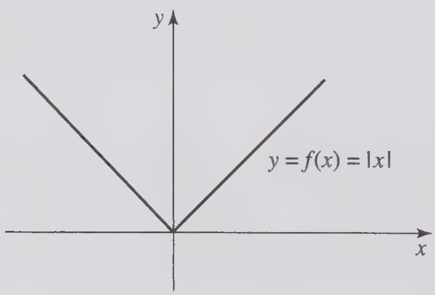

Figure 6.1

Case 2 ($x_0 < 0$): Then

$$\begin{aligned}
\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{|x|-|x_0|}{x-x_0}\\
&= \lim_{x\to x_0}\frac{-x-(-x_0)}{x-x_0} \text{ since } x<0 \text{ as } x\to x_0<0\\
&= \lim_{x\to x_0}\frac{-(x-x_0)}{x-x_0}\\
&= -1.
\end{aligned}$$

Thus, when $x_0 < 0$, $f'(x_0) = -1$.

Case 3 ($x_0 = 0$): Then

$$\begin{aligned}
\lim_{x\to 0}\frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to 0}\frac{|x|-0}{x-0}\\
&= \lim_{x\to 0}\frac{|x|}{x}, \text{ which does not exist.}\\
&\quad \text{(See Exercise 4.3.1 (a).)}
\end{aligned}$$

Thus, the function $f(x) = |x|$ is *not* differentiable at 0. □

Example 6.1.5 The function $f(x) = \sqrt{x}$ is differentiable on $(0, +\infty)$, and $\forall x_0 \in (0, +\infty)$, $f'(x_0) = \dfrac{1}{2\sqrt{x_0}}$.

Proof. Let $f(x) = \sqrt{x}$ on $(0, +\infty)$, and let $x_0 \in (0, +\infty)$. Then,

$$\begin{aligned}
\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{\sqrt{x}-\sqrt{x_0}}{x-x_0}\\
&= \lim_{x\to x_0}\frac{\sqrt{x}-\sqrt{x_0}}{(\sqrt{x}-\sqrt{x_0})(\sqrt{x}+\sqrt{x_0})}\\
&= \lim_{x\to x_0}\frac{1}{\sqrt{x}+\sqrt{x_0}} \qquad (\text{since } \sqrt{x}-\sqrt{x_0}\neq 0)\\
&= \frac{1}{\sqrt{x_0}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}} \qquad \text{since } x_0 \neq 0. \quad \square.
\end{aligned}$$

Sometimes Definition 6.1.1 is less convenient to use than another, equivalent, definition that is based on the following observation:

If g is any function, and $x_0 \in \mathbb{R}$ is any cluster point of $\mathcal{D}(g)$, then

$$\lim_{x\to x_0} g(x) = L \Leftrightarrow \lim_{h\to 0} g(x_0+h) = L.$$

(See Exercise 4.1.6.) Because of this, if we let $g(x) = \dfrac{f(x) - f(x_0)}{x - x_0}$, then we can rewrite Definition 6.1.1 in the following alternate form:

Definition 6.1.6 (Alternate Definition of Differentiability) Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f is **differentiable at** x_0 if the limit $\lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$ exists (i.e., is finite). If this limit exists, we call it the **derivative of** f **at** x_0, and denote it $f'(x_0)$.

Thus, $\boxed{f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h} \quad \text{if this limit exists.}}$

Theorem 6.1.7 (***Sequential Criterion for Differentiability***)
Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then f is ***differentiable at*** *x_0 with derivative $f'(x_0)$ iff $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f) - \{x_0\}$ such that $x_n \to x_0$,* $\dfrac{f(x_n) - f(x_0)}{x_n - x_0} \to f'(x_0)$.

Proof. This is a trivial application of Theorem 4.1.9 to Definition 6.1.6. ■

Since the definition of derivative involves the concept of limit, the concept of continuity cannot be far away. It is natural to ask whether there is a relation between these two concepts: continuity of a function f at x_0 and differentiability of f at x_0. The following theorem establishes this relationship.

Theorem 6.1.8 (***Differentiability Implies Continuity***) *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Suppose f is differentiable at x_0. Then, for all $x \in \mathcal{D}(f) - \{x_0\}$,

$$f(x) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) + f(x_0).$$

Thus, by the algebra of limits,

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x \to x_0} (x - x_0) + \lim_{x \to x_0} f(x_0) \\
&= f'(x_0)(x_0 - x_0) + f(x_0) = 0 + f(x_0) = f(x_0).
\end{aligned}$$

Thus, $\lim\limits_{x \to x_0} f(x) = f(x_0)$, which means that f is continuous at x_0. ■

Theorem 6.1.8 says that **differentiability is "stronger" than continuity**; that is, differentiability implies continuity. Differentiability of f at x_0 implies continuity of f at x_0.

Caution: Continuity does not imply differentiability. There are functions f that are continuous at x_0 but not differentiable at x_0. The absolute value function $f(x) = |x|$ at 0 is one such example. Its continuity was established in Example 5.1.9 (a), and its nondifferentiability in Example 6.1.4. The following example provides another illustration of this happening.

Example 6.1.9 The function $f(x) = \left\{ \begin{array}{ll} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{array} \right\}$ is continuous at 0, but not differentiable there. (See graph in Example 4.2.21.)

Proof. (a) Continuity at 0:

$$\begin{aligned} \lim_{x \to 0} f(x) &= \lim_{x \to 0} \left(x \sin \tfrac{1}{x} \right) \\ &= 0 \quad \text{(See Example 4.2.21.)} \\ &= f(0). \end{aligned}$$

Since $\lim\limits_{x \to 0} f(x) = f(0)$, f is continuous at 0.

(b) Nondifferentiability at 0:

$$\begin{aligned} \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{x \sin \frac{1}{x} - 0}{x - 0} \\ &= \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{x} \\ &= \lim_{x \to 0} \sin \tfrac{1}{x}, \text{ which does not exist} \qquad \text{(Example 4.1.12).} \end{aligned}$$

Therefore, f is not differentiable at 0. $\square$

The following example is an interesting variation of Example 6.1.9.

Example 6.1.10 The function $f(x) = \left\{ \begin{array}{ll} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is differentiable at 0, and $f'(0) = 0$.

Proof.

$$\begin{aligned} \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} \\ &= \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} \\ &= \lim_{x \to 0} x \sin \tfrac{1}{x} = 0. \qquad \text{(Example 4.2.21).} \end{aligned}$$

$-1 \leq \sin\frac{1}{x} \leq 1$

$-x \leq x\sin\frac{1}{x} \leq x$

$-\lim\limits_{x \to 0} x = 0 \leq$

Therefore, f is differentiable at 0, and $f'(0) = 0$. □

Another caution: Although differentiability of f implies continuity of f, it does not imply continuity of f'. In fact, the function f defined in Example 6.1.10 is differentiable at 0, but f' is not continuous there. (See Exercise 6.2.17.)

ONE-SIDED DERIVATIVES

Definition 6.1.11 Suppose $f : \mathcal{D}(f) \to \mathbb{R}$.

(a) Suppose $\mathcal{D}(f)$ includes an interval of the form $(x_0-\delta, x_0]$, for some $\delta > 0$. Then f is **differentiable from the left** at x_0 if the limit $\displaystyle\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ exists (i.e., is finite). If this limit exists, we call it the **derivative from the left of** f at x_0, and denote it $f'_-(x_0)$.

Thus, $$\boxed{f'_-(x_0) = \lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \quad \text{if this limit exists.}}$$

(b) Suppose $\mathcal{D}(f)$ includes an interval of the form $[x_0, x_0+\delta)$, for some $\delta > 0$. Then f is **differentiable from the right at** x_0 if the limit $\displaystyle\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$ exists (i.e., is finite). If this limit exists, we call it the **derivative from the right of** f at x_0, and denote it $f'_+(x_0)$.

Thus, $$\boxed{f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \quad \text{if this limit exists.}}$$

Example 6.1.12 In Example 6.1.4 we showed that for the function $f(x) = |x|$, $f'_-(0) = -1$, while $f'_+(0) = 1$.

Theorem 6.1.13 *Suppose $f : \mathcal{D}(f) \to \mathbb{R}$ and x_0 is an interior point of $\mathcal{D}(f)$. Then $f'(x_0)$ exists $\Leftrightarrow$ both $f'_-(x_0)$ and $f'_+(x_0)$ exist and are equal.*

Proof. Exercise 11. ■

***Theorem 6.1.14** (a) *If $\exists\,\delta > 0 \ni f$ is differentiable on $(x_0-\delta, x_0)$ and continuous from the left at x_0, and $\displaystyle\lim_{x\to x_0^-} f'(x)$ exists, then $f'_-(x_0)$ exists and equals $\displaystyle\lim_{x\to x_0^-} f'(x)$.*

---

*An asterisk with a theorem, proof, or other material in this chapter indicates that the item is challenging and can be omitted, especially in a one-semester course.

(b) *If* $\exists\,\delta > 0 \ni f$ *is differentiable on* $(x_0, x_0+\delta)$ *and continuous from the right at* x_0, *and* $\lim\limits_{x\to x_0^+} f'(x)$ *exists, then* $f'_+(x_0)$ *exists and equals* $\lim\limits_{x\to x_0^+} f'(x)$.

Proof. Postponed to Section 6.4, Exercise 29. ■

EXERCISE SET 6.1

1. Use Definition 6.1.1 to calculate the derivative of the given function f at $x = x_0$.

   (a) $f(x) = 3x^2 - 2x$

   (b) $f(x) = x^3$

   (c) $f(x) = \dfrac{1}{x} \quad (x \neq 0)$

   (d) $f(x) = \dfrac{x+1}{x-1} \quad (x \neq 1)$

   (e) $f(x) = \sqrt{2x+3} \quad (x > -\frac{3}{2})$

   (f) $f(x) = \dfrac{1}{\sqrt{x}} \quad (x > 0)$

2. Use (alternate) Definition 6.1.6 to calculate the derivative of the given function f at $x = x_0$.

   (a) $f(x) = 4x^2 + 3x - 5$

   (b) $f(x) = x^3$

   (c) $f(x) = \dfrac{5}{x} \quad (x \neq 0)$

   (d) $f(x) = \dfrac{1}{5x+4} \quad (x \neq -\frac{4}{5})$

   (e) $f(x) = \sqrt{4x-1} \quad (x > \frac{1}{4})$

   (f) $f(x) = \dfrac{1}{\sqrt[3]{x}} \quad (x \neq 0)$

3. Let $f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$

   (a) Sketch the graph of f.

   (b) Prove that f is differentiable at 0.

   (c) Sketch the graph of f'. Is f' continuous at 0? (Prove or disprove.)

   (d) Is f' differentiable at 0? (Prove or disprove.)

4. Let $f(x) = \left\{ \begin{matrix} x^2 & \text{if } x \geq 0, \\ x & \text{if } x < 0 \end{matrix} \right\}$. Prove that f is *not* differentiable at 0.

5. Let $f(x) = \left\{ \begin{matrix} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational} \end{matrix} \right\}$. Prove that f is differentiable at 0. Is f differentiable anywhere else? Explain.

Try using $\varepsilon - \delta$

6. Find the values of the constants a and b for which the function

$$f(x) = \left\{ \begin{array}{ll} x^2 & \text{if } x \leq 3, \\ ax+b & \text{if } x > 3 \end{array} \right\} \text{ is differentiable at 3.}$$

7. For each of the following functions, answer these questions: Where is f continuous? Where is f differentiable? What is the formula for $f'(x)$? Where is f' continuous? (Explain using graphs; omit proofs.)

   (a) $f(x) = x + |x|$ (b) $f(x) = x|x|$ (c) $f(x) = |\sin x|$

   (d) $f(x) = |x-1| + |x+1|$ (e) $f(x) = x\lfloor x \rfloor$ (f) $f(x) = x - \lfloor x \rfloor$

   where $\lfloor x \rfloor =$ the greatest integer[2] $\leq x$.

8. (a) Prove that the function $f(x) = |x^3|$ is differentiable everywhere. What is $f'(0)$?

   (b) Prove that the function $f(x) = \sqrt[3]{x}$ is not differentiable at 0, even though it is continuous there.

9. Prove that the function $f(x) = \left\{ \begin{array}{ll} x^r \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is

   (a) continuous from the right at $0 \Leftrightarrow r > 0$;

   (b) differentiable from the right at $0 \Leftrightarrow r > 1$.

10. Give alternate definitions of $f'_-(x_0)$ and $f'_+(x_0)$ along the lines of Definition 6.1.6.

11. Prove Theorem 6.1.13.

12. Prove that if f is differentiable from the left (or right) at x_0, then f is continuous from the left (or right) at x_0.

13. Find an example of a function f for which $f(x_0)$ exists, $\lim\limits_{x \to x_0^-} f'(x)$ and $\lim\limits_{x \to x_0^+} f'(x)$ exist and are equal, but $f'_-(x_0)$ and $f'_+(x_0)$ do not exist.

14. Show by example that it is possible for both $f'_-(x_0)$ and $f'_+(x_0)$ to exist (and be equal), even when $\lim\limits_{x \to x_0^-} f'(x)$ and $\lim\limits_{x \to x_0^+} f'(x)$ do not exist.

15. Use Theorem 6.1.14 to prove that if f is differentiable on some deleted neighborhood of x_0 and continuous at x_0, and $\lim\limits_{x \to x_0^-} f'(x) = \lim\limits_{x \to x_0^+} f'(x)$, then f is differentiable at x_0 and $f'(x_0) = \lim\limits_{x \to x_0^-} f'(x)$.

---

2. The "greatest integer function" $\lfloor x \rfloor$ is defined in Example 5.2.16.

16. (a) Prove that if f is differentiable at x_0, then $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0-h)}{2h}$ exists and equals $f'(x_0)$.

    (b) Find an example of a function f and a point x_0 such that $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0-h)}{2h}$ exists but f is not differentiable at x_0.

17. A function f defined on an open interval I is said to "satisfy a **Lipschitz condition**[3] **of order** α" on I if $\exists M > 0 \ni \forall x, y \in I$, $|f(x) - f(y)| \leq M|x-y|^{\alpha}$.

    (a) Prove that if f satisfies a Lipschitz condition of order α on I, for some real number $\alpha > 1$, then f is differentiable on I, and $f'(x) = 0$ on I.

    (b) Find an example of a function f that satisfies a Lipschitz condition of order $\alpha = 1$ on an interval I but f is not differentiable on I.

6.2 Rules for Differentiation

We now get down to the business of proving the familiar differentiation formulas.

Theorem 6.2.1 (***Power Rule***) *For a given natural number n, the function $f(x) = x^n$ is differentiable everywhere, and $\forall x_0 \in \mathbb{R}$, $f'(x_0) = nx_0^{n-1}$.*

Proof. The case $n = 1$ is covered by Theorem 6.1.2. Hence, assume $n \geq 2$. Let $f(x) = x^n$. Then, $\forall x_0 \in \mathbb{R}$,

$$\begin{aligned}
\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{x^n - x_0^n}{x-x_0} \\
&= \lim_{x\to x_0} \frac{(x-x_0)(x^{n-1}+x^{n-2}x_0+\cdots+xx_0^{n-2}+x_0^{n-1})}{x-x_0} \\
&= \lim_{x\to x_0} (x^{n-1}+x^{n-2}x_0+\cdots+xx_0^{n-2}+x_0^{n-1}) \\
&= x_0^{n-1}+x_0^{n-2}x_0+\cdots+x_0x_0^{n-2}+x_0^{n-1} \\
&= x_0^{n-1}+x_0^{n-1}+\cdots+x_0^{n-1}+x_0^{n-1} \quad (n \text{ terms}) \\
&= nx_0^{n-1}. \quad (\text{since there are } n \text{ terms})
\end{aligned}$$

Therefore, $f(x) = x^n$ is differentiable everywhere, and $f'(x_0) = nx_0^{n-1}$. ∎

---

3. In Exercise 5.4.10 we proved that if f satisfies a Lipschitz condition of order 1 on I then f is uniformly continuous on I.

NOTATION FOR DERIVATIVES

It is somewhat cumbersome to continue using the subscript in x_0. We shall usually write $f'(x)$ instead of $f'(x_0)$ whenever it is possible to do so without ambiguity. Then the "derivative" becomes a function, f'. It is important to understand that, in Definition 6.1.1, the derivative of a function f at a point x_0 is a *number*, whereas we are now suggesting that we may also consider the *function* f'. Of course, the two functions, f and f' may have different domains; $f(x)$ may exist where $f'(x)$ does not.

Some of the common notational devices for derivatives are:

$$f'(x) = D_x\left(f(x)\right) = \frac{d}{dx}\left(f(x)\right) = \frac{df(x)}{dx}.$$

If $y = f(x)$, then we can write

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}y.$$

You are probably aware that calculus as a formal subject was developed in the seventeenth and eighteenth centuries. One of the two principal inventors of the subject, Sir Isaac Newton (English, 1642–1727) used a notation similar to our y' for the derivative. The other principal inventor, Gottfried Wilhelm Leibniz (German, 1646–1717) developed the dy/dx notation. Both notations have advantages. The simplicity of y' is frequently offset by the suggestive power of the "differentials" in dy/dx.

ALGEBRA OF DERIVATIVES

Theorem 6.2.2 (***Algebra of Derivatives***) *Suppose f and g are differentiable at x and $c \in \mathbb{R}$. Then*

(a) *cf is differentiable at x, and $(cf)'(x) = c\left[f'(x)\right]$;*

(b) *$f + g$ is differentiable at x, and $(f+g)'(x) = f'(x) + g'(x)$;*

(c) *$f - g$ is differentiable at x, and $(f-g)'(x) = f'(x) - g'(x)$;*

(d) *fg is differentiable at x, and $(fg)'(x) = f(x)g'(x) + g(x)f'(x)$;*

(e) *if $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x, and $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$.*

[Rules (a)–(e) are called the **constant multiple rule**, **sum rule**, **difference rule**, **product rule**, and **quotient rule**, respectively.]

Proof. Suppose f and g are differentiable at x_0 and $c \in \mathbb{R}$. Then

(a)
$$\begin{aligned}\lim_{x\to x_0} \frac{cf(x)-cf(x_0)}{x-x_0} &= \lim_{x\to x_0} c\left[\frac{f(x)-f(x_0)}{x-x_0}\right]\\ &= c\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\ &= cf'(x_0).\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)+g(x)-[f(x_0)+g(x_0)]}{x-x_0}\\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)+g(x)-g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right]\\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}+\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\\ &= f'(x_0)+g'(x_0).\end{aligned}$$

(c) Exercise 1.

(d)
$$\lim_{x\to x_0}\frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$
$$\begin{aligned}&= \lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0}\left[\frac{f(x)g(x)-f(x)g(x_0)}{x-x_0}+\frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\right]\\ &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)}{x-x_0}+\lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0} f(x)\frac{g(x)-g(x_0)}{x-x_0}+\lim_{x\to x_0} g(x_0)\frac{f(x)-f(x_0)}{x-x_0}\\ &= \lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}+g(x_0)\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}.\end{aligned}$$

Now, f is differentiable at x_0. Hence, by Theorem 6.1.8, f is continuous at x_0. That is, $\lim_{x \to x_0} f(x) = f(x_0)$. Substituting this into the last line above, we have

$$(fg)'(x_0) = \lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = f(x_0)g'(x_0) + g(x_0)f'(x_0).$$

(e) Since $g(x_0) \neq 0$ and g is differentiable (hence continuous) at x_0, $g(x) \neq 0$ on some neighborhood of x_0, so

$$\lim_{x \to x_0} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{1}{x - x_0} \frac{f(x)g(x_0) - g(x)f(x_0)}{g(x)g(x_0)}\right]$$

$$= \lim_{x \to x_0} \left[\frac{1}{g(x)g(x_0)}\right] \lim_{x \to x_0} \left[\frac{f(x)g(x_0) - g(x)f(x_0)}{x - x_0}\right]$$

$$= \frac{1}{g(x_0)g(x_0)} \lim_{x \to x_0} \left[\frac{f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0} + \frac{f(x_0)g(x_0) - f(x_0)g(x)}{x - x_0}\right]$$

(Remember, differentiability $\Rightarrow$ continuity, so $\lim_{x \to x_0} g(x) = g(x_0)$.)

$$= \frac{1}{g^2(x_0)} \lim_{x \to x_0} \left[g(x_0)\frac{f(x) - f(x_0)}{x - x_0} - f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\right]$$

$$= \frac{1}{g^2(x_0)} \left[g(x_0) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}\right]$$

$$= \frac{1}{g^2(x_0)} [g(x_0)f'(x_0) - f(x_0)g'(x_0)]$$

$$= \frac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}. \quad ■$$

The next theorem, the chain rule, is easy to state and almost as easy to believe on an intuitive basis. Its proof, however, requires some finesse (to avoid dividing by zero at a crucial point in the proof). The chain rule is known to elementary calculus students in its familiar form: *if y is a differentiable function of u, and u is a differentiable function of x, then*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

For our purposes, however, we must restate this as a theorem with more precise hypotheses and a more precise conclusion.

Theorem 6.2.3 (*The Chain Rule*) *Suppose f is differentiable at an interior point x_0 of its domain, and g is differentiable at $f(x_0)$, an interior point of its domain. Then the composite function $g \circ f$ is differentiable at x_0, and $(g \circ f)'(x_0) = (g' \circ f)(x_0) \cdot f'(x_0) = g'(f(x_0)) \cdot f'(x_0)$.*

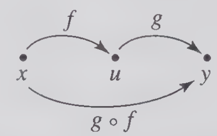

Figure 6.2

Proof. [4] Suppose f is differentiable at an interior point x_0 of its domain, and g is differentiable at $f(x_0)$, an interior point of its domain. Define the function $h : \mathcal{D}(g) \to \mathbb{R}$ by

$$h(u) = \begin{cases} \dfrac{g(u) - g\left(f(x_0)\right)}{u - f(x_0)} & \text{if } u \neq f(x_0); \\ g'\left(f(x_0)\right) & \text{if } u = f(x_0). \end{cases}$$

Then h is continuous at $f(x_0)$, since

$$\begin{aligned} \lim_{u \to f(x_0)} h(u) &= \lim_{u \to f(x_0)} \frac{g(u) - g\left(f(x_0)\right)}{u - f(x_0)} \qquad (u \neq f(x_0) \text{ as } u \to f(x_0)) \\ &= g'\left(f(x_0)\right) \quad \text{by definition of derivative} \\ &= h\left(f(x_0)\right) \quad \text{by definition of } h. \end{aligned} \tag{1}$$

Now $\forall u \in \mathcal{D}(g)$, even if $u = f(x_0)$, the definition of $h(u)$ yields

$$g(u) - g\left(f(x_0)\right) = h(u)\left(u - f(x_0)\right). \tag{2}$$

Thus, for all x in some deleted neighborhood of x_0,

$$\begin{aligned} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \frac{g\left(f(x)\right) - g\left(f(x_0)\right)}{x - x_0} \\ &= \frac{h\left(f(x)\right)\left(f(x) - f(x_0)\right)}{x - x_0} \\ &= h\left(f(x)\right) \frac{f(x) - f(x_0)}{x - x_0}. \end{aligned}$$

4. This proof may seem unnecessarily complicated. To see the fallacy of a simpler approach, see Exercise 8.

Thus,

$$
\begin{aligned}
\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \lim_{x \to x_0} h(f(x)) \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= h(f(x_0))\, f'(x_0) \\
&\quad \left(\lim_{x \to x_0} h(f(x)) = h(f(x_0)) \text{. Why?} \right) \\
&= g'(f(x_0)) f'(x_0) \quad \text{by definition of } h(f(x_0)). \ \blacksquare
\end{aligned}
$$

DERIVATIVES OF INVERSE FUNCTIONS

Suppose f is 1-1 and continuous on an open interval I. By Theorem 5.3.8, $f(I)$ is an interval J; by Theorem 5.5.4,[5] f is strictly monotone on I; and by Corollary 5.5.3, $f^{-1} : J \to I$ is continuous and strictly monotone. We now investigate the differentiability of f^{-1}.

Theorem 6.2.4 (*Inverse Function Theorem for Differentiable Functions*) *Suppose f is 1-1 and continuous on an open interval I. If f is differentiable at a point $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $f(x_0)$, and $\left(f^{-1}\right)'(f(x_0)) = \dfrac{1}{f'(x_0)}$.*

Proof. Suppose f is 1-1 and continuous on an open interval I, differentiable at a point $x_0 \in I$, and $f'(x_0) \neq 0$. Let $y_0 = f(x_0)$. We shall use the sequential criterion for differentiability (6.1.7). Let $\{y_n\}$ be a sequence in $f(I) - \{y_0\}$ converging to y_0. Then $\forall n \in \mathbb{N}$, let $x_n = f^{-1}(y_n)$; i.e., $y_n = f(x_n)$. Note that $\forall n \in \mathbb{N}$, $y_n \neq y_0$ and $x_n \neq x_0$ (Why?). Then,

$$
\begin{aligned}
\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} &= \lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} \\
&= \lim_{n \to \infty} \left(\frac{1}{\dfrac{f(x_n) - f(x_0)}{x_n - x_0}} \right) \\
&= \frac{1}{\displaystyle\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}} \\
&\quad \text{(by the algebra of limits for sequences)} \\
&= \frac{1}{f'(x_0)} \quad \text{since } f \text{ is differentiable at } x_0 \text{, and } f'(x_0) \neq 0.
\end{aligned}
$$

5. Corollary 5.5.3 and Theorem 5.5.4 were proved in (optional) Section 5.5. These results are not needed for the proofs that follow.

By the sequential criterion for derivatives, $\left(f^{-1}\right)'(f(x_0)) = \dfrac{1}{f'(x_0)}$. ■

Example 6.2.5 Consider the function $f(x) = 3x + 5$.

The inverse function is $f^{-1}(x) = \dfrac{x-5}{3}$. The derivatives of f and f^{-1} are $f'(x) = 3$ and $\left(f^{-1}\right)'(x) = \frac{1}{3}$. □

Example 6.2.6 Consider the function $f(x) = x^2$ on the interval $[0, +\infty)$.

This function is 1-1 on $[0, +\infty)$ and the inverse function for this interval is $f^{-1}(x) = \sqrt{x}$. Theorem 6.2.4 says that $\left(f^{-1}\right)'(f(x)) = \dfrac{1}{f'(x)}$; that is, $\left(f^{-1}\right)'(x^2) = \dfrac{1}{2x}$ or $\left(f^{-1}\right)'(y) = \dfrac{1}{2\sqrt{y}}$. This is consistent with the formula you remember from elementary calculus: $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$. (See Example 6.1.5.) □

DERIVATIVES OF RATIONAL POWER FUNCTIONS

Having proved Theorem 6.2.4, we are now able to extend the power rule $\frac{d}{dx}x^n = nx^{n-1}$ to rational exponents. Up to this point we have proved this rule only for integers n. In fact, until Section 5.5 (Exercise 5.5.15) we did not even have a *definition* of x^r for general $r \in \mathbb{Q}$. Our proof will be in two steps.

Theorem 6.2.7 *Let $n \in \mathbb{N}$, $n \neq 0$. The function $x^{\frac{1}{n}}$ is differentiable everywhere on its domain, except at* 0, *and $\dfrac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}\,x^{\frac{1}{n}-1}$ if $x \neq 0$.*

Proof. Let $n \in \mathbb{N}$, $n \neq 0$. Everywhere on its domain, the function $x^{\frac{1}{n}}$ is 1-1 and continuous, and is the inverse of the function x^n (see Exercise 5.5.15). Thus, letting $y = f(x) = x^n$ (restricting the domain to $[0, \infty)$ if n is even) we have $x = f^{-1}(y) = y^{\frac{1}{n}}$ and by Theorem 6.2.4,

$$\left(f^{-1}\right)'(x^n) = \frac{1}{f'(x)}.$$

By substituting $y = x^n$ $(x = y^{\frac{1}{n}})$ into this equation, we have

$$\begin{aligned}\left(f^{-1}\right)'(y) &= \frac{1}{nx^{n-1}} = \frac{1}{n\,y^{\frac{n-1}{n}}}\\ &= \tfrac{1}{n}\,y^{\frac{1-n}{n}} = \tfrac{1}{n}\,y^{\frac{1}{n}-1}.\end{aligned}$$

In summary, if $g(y) = y^{\frac{1}{n}}$, then $g'(y) = \frac{1}{n}\,y^{\frac{1}{n}-1}$. ■

Corollary 6.2.8 *$\forall r \in \mathbb{Q}$, the power function $f(x) = x^r$ is differentiable everywhere on its domain, and $\frac{d}{dx}x^r = rx^{r-1}$.*

Proof. Exercise 10. ■

LOGARITHM, EXPONENTIAL, AND POWER FUNCTIONS

Students who skipped Section 5.6 should skip the *proofs* of the following theorem and its corollaries. These results will be derived in the more customary way in Chapter 7.

We adopt the conventional notation for the natural logarithm, $\ln x = \log_e x$. This function and e^x were treated in detail in Section 5.6.

Theorem 6.2.9 *The functions $f(x) = \ln x$ and $g(x) = e^x$ are differentiable everywhere on their domains, and*

(a) $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$; (b) $\dfrac{d}{dx}e^x = e^x$.

***Proof.** (a) Using the properties of logarithms,

$$\begin{aligned}
\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} &= \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h\to 0}\ln\left(\frac{x+h}{x}\right)^{\frac{1}{h}} \\
&= \ln\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right] \quad \text{by Theorem 5.1.14 (b)} \\
&= \ln e^{1/x} \quad \text{by Corollary 5.6.20} \\
&= \frac{1}{x}.
\end{aligned}$$

(b) The exponential function e^x is the inverse of the logarithm function $\ln x$. That is, for $f(x) = \ln x$ we have $f^{-1}(x) = e^x$. Thus, by Theorem 6.2.4 with $y = \ln x$,

$$\frac{d}{dy}e^y = \frac{1}{\frac{d}{dx}\ln x} = \frac{1}{\frac{1}{x}} = x = e^y. \quad ■$$

Corollary 6.2.10 *Suppose $a > 0$, and $a \neq 1$. Then*

(a) $\forall x \in \mathbb{R}, \dfrac{d}{dx}\, a^x = a^x \ln a;$ (b) $\forall x > 0, \dfrac{d}{dx}\, \log_a x = \dfrac{1}{x \ln a}.$

Proof. Exercise 11. ■

Corollary 6.2.11 *If $c \in \mathbb{R}$, then* $\dfrac{d}{dx}\, x^c = cx^{c-1}.$

Proof. Exercise 12. ■

TRIGONOMETRIC FUNCTIONS

The trigonometric functions will be defined rigorously in Chapter 7, where the sine and cosine functions will be defined using the integral as a foundation. They will also be defined in Chapter 8 using power series. In either context their derivatives are also readily obtained. In the meantime, we shall need to use these functions as examples. We thus give the following formulas for reference.

Table 6.1

For all real numbers in the domain of the indicated functions,	
(a) $\dfrac{d}{dx}(\sin x) = \cos x$	(b) $\dfrac{d}{dx}(\cos x) = -\sin x$
(c) $\dfrac{d}{dx}(\tan x) = \sec^2 x$	(d) $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
(e) $\dfrac{d}{dx}(\cot) = -\csc^2 x$	(f) $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

The "proof" of formula (a) usually given in calculus courses is not acceptable here since at a crucial point it relies on geometric, rather than analytic, reasoning. Of course, once formula (a) has been proved, the remaining five are easily derived from it using trigonometric identities and the "Algebra of Derivatives." We shall *assume* that these functions have their familiar properties, until we actually prove them in Section 7.7. (See Exercise 16 on page 315.)

EXERCISE SET 6.2

1. Prove Theorem 6.2.2 (c).
2. Without using the chain rule, prove the general power rule for natural numbers: if f is differentiable at x_0, then $\forall n \in \mathbb{N}$, f^n is differentiable at x_0, and $\frac{d}{dx}\,[f(x)]^n = n\,[f(x)]^{n-1}\, f'(x)$. [Use mathematical induction.]

3. Prove that if f is differentiable at x_0 and $f(x_0) \neq 0$, then
$$\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \frac{-f'(x)}{f^2(x)}.$$

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0, and define $g : \mathbb{R} \to \mathbb{R}$ by the given formula. Use the theorems of this section to prove that g is differentiable at x_0 and find $g'(x_0)$ in terms of x_0, $f(\_)$, and $f'(\_)$.

   (a) $g(x) = x^3 f(x^2)$ (b) $g(x) = x^4 f^3(x)$

   (c) $g(x) = \left[f(x^3)\right]^5$ (d) $g(x) = \dfrac{f\left(\frac{1}{x}\right)}{x^4}$, if $x \neq 0$

5. We define a function $f : \mathbb{R} \to \mathbb{R}$ to be an **odd function** if $\forall x \in \mathbb{R}$, $f(-x) = -f(x)$, and an **even function** if $\forall x \in \mathbb{R}$, $f(-x) = f(x)$. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere. Prove:

   (a) If f is an odd function, then f' is an even function;

   (b) If f is an even function, then f' is an odd function.

6. State and prove a product rule for hgf. $h \cdot g \cdot f$

7. State and prove a chain rule for $h \circ g \circ f$.

8. Suppose f is differentiable at an interior point x_0 of its domain, and g is differentiable at $f(x_0)$, an interior point of its domain. Find the flaw in the following "proof" of the chain rule:
$$\begin{aligned}
&\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} \\
&\qquad = \lim_{x \to x_0} \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \frac{f(x) - f(x_0)}{x - x_0}\right) \\
&\qquad = \lim_{f(x) \to f(x_0)} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&\qquad\qquad \text{(using Theorem 4.2.23).} \\
&\qquad = g'(f(x_0) \cdot f'(x_0).
\end{aligned}$$
Then show that the proof is valid if f is strictly monotone in a neighborhood of x_0.

9. Determine where the function $\sqrt{x + \sqrt{x + \sqrt{x}}}$ is differentiable, and find its derivative.

10. Prove Corollary 6.2.8. $\left[\text{Hint: } x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m.\right]$

11. Prove Corollary 6.2.10. [Hint: Use Exercise 5.6.16 to convert to base e.]

12. Prove Corollary 6.2.11. [See hint for Exercise 11.]

13. Prove the general power rule for *real number exponents:* If f is differentiable then $\forall\, c \in \mathbb{R}$, $\frac{d}{dx}\left[f(x)\right]^c = c\left[f(x)\right]^{c-1} f'(x)$.

14. Prove that $\forall x \neq 0$, $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

15. Use the "logarithmic differentiation" technique learned in elementary calculus to prove that if f and g are differentiable and f is positive, then $\frac{d}{dx}\left[f(x)\right]^{g(x)} = \left[f(x)\right]^{g(x)} \left(g'(x) \ln f(x) + g(x)\frac{f'(x)}{f(x)}\right)$.

16. Assume formula (a) of Table 6.1. Use this, trigonometric identities, the algebra of derivatives, and the chain rule if necessary, to derive formulas (b)–(f).

17. In Example 6.1.10 we proved that the function $f(x) = \left\{ \begin{array}{ll} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is differentiable at 0.

    (a) Prove that f is differentiable everywhere on $\mathbb{R}$.

    (b) Prove that f' is continuous everywhere except at 0. [To prove that f' is not continuous at 0, you may find the sequential criterion helpful.]

18. Use mathematical induction to prove **Leibniz's rule**: $\forall n \in \mathbb{N}$, if f and g are n times differentiable at x, then $(fg)^{(n)}(x) = \sum\limits_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)$, where $f^{(k)}$ and $g^{(k)}$ denote the kth derivatives of f and g, $f^{(0)} = f$, $g^{(0)} = g$, and $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. [See Exercise 1.3.23.]

19. A function f is said to be **periodic with period** $p > 0$ if $\forall x \in \mathbb{R}$, $f(x+p) = f(x)$. [See Exercise 5.4.22.] Prove that if $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p and differentiable on some interval $[a, a+p)$, then f is differentiable everywhere on $\mathbb{R}$ and f' is periodic with period p.

*20. **Periodic Extensions:** Suppose $f : [a, b) \to \mathbb{R}$. Define $\widehat{f} : \mathbb{R} \to \mathbb{R}$ by $\widehat{f}(x) = f\left(x - (b-a)\left\lfloor \dfrac{x-a}{b-a} \right\rfloor\right)$, where $\lfloor x \rfloor$ denotes the "greatest integer function."[6] Prove that

    (a) $\forall x \in \mathbb{R}$, $x - (b-a)\left\lfloor \dfrac{x-a}{b-a} \right\rfloor \in [a, b)$.

    (b) $\widehat{f}$ is periodic, with period $b-a$, and $\widehat{f}|_{[a,b)} = f$. (We call $\widehat{f}$ the **periodic extension of** f **to** $\mathbb{R}$.)

6. The greatest integer function $\lfloor x \rfloor$ is defined in Example 5.2.16.

(c) if f is continuous on $[a, b)$ and $\lim_{x \to b^-} f(x) = f(a)$, then $\widehat{f}$ is continuous on $\mathbb{R}$.

(d) if f is differentiable on (a, b), differentiable from the right at a, and $\lim_{x \to b^-} f'(x) = \lim_{x \to a^+} f'(x)$, then $\widehat{f}$ is differentiable on $\mathbb{R}$.

6.3 Local Extrema and Monotone Functions

In this section, we state and prove theorems that justify the procedures used in elementary calculus courses to find local maximum and minimum values of functions using their derivatives.

Definition 6.3.1 Suppose f is defined in a neighborhood of x_0. Then
(a) f has a **local maximum** at x_0 if, for some neighborhood of x_0, f takes on its maximum value at x_0. That is,

$$\exists\, \delta > 0 \ni f(x_0) = \max f\,(N_\delta(x_0)); \text{ equivalently,}$$
$$\exists\, \delta > 0 \ni \forall x \in N_\delta(x_0) \cap \mathcal{D}(f),\ f(x) \le f(x_0).$$

(b) f has a **local minimum** at x_0 if, for some neighborhood of x_0, f takes on its minimum value at x_0. That is,

$$\exists\, \delta > 0 \ni f(x_0) = \min f\,(N_\delta(x_0)); \text{ equivalently,}$$
$$\exists\, \delta > 0 \ni \forall x \in N_\delta(x_0) \cap \mathcal{D}(f),\ f(x) \ge f(x_0).$$

(c) a function f has a **local extreme value** at x_0 if it has either a local maximum or a local minimum at x_0.

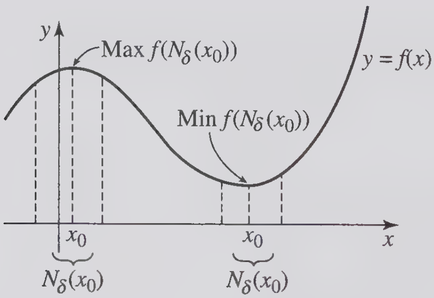

Figure 6.3

The next theorem is the basic tool needed to prove that if a differentiable function f has a local extreme point at an interior point $x_0 \in \mathcal{D}(f)$, then $f'(x_0) = 0$.

Theorem 6.3.2 *Suppose $f'(x_0) > 0$ at an interior point x_0 of $\mathcal{D}(f)$. Then* $\exists\, \delta > 0 \ni$

(a) $\forall x \in (x_0 - \delta,\, x_0), \quad f(x) < f(x_0)$, *and*

(b) $\forall x \in (x_0,\, x_0 + \delta), \quad f(x) > f(x_0)$. *(See Figure 6.4.)*

Proof. Suppose $f'(x_0) > 0$ at an interior point $x_0 \in \mathcal{D}(f)$. Then

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0.$$

By the "bounded away from zero" theorem[7] for limits of functions, $\exists\, \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow \dfrac{f(x) - f(x_0)}{x - x_0} > 0$. Then,

$$\begin{aligned}
&\text{(a)}\quad x \in (x_0 - \delta, x_0) \Rightarrow x - x_0 < 0 \text{ and } \frac{f(x) - f(x_0)}{x - x_0} > 0\\
&\qquad\qquad\qquad\qquad\ \Rightarrow f(x) - f(x_0) < 0\\
&\qquad\qquad\qquad\qquad\ \Rightarrow f(x) < f(x_0).\\
&\text{(b)}\quad x \in (x_0,\, x_0 + \delta) \Rightarrow x - x_0 > 0 \text{ and } \frac{f(x) - f(x_0)}{x - x_0} > 0\\
&\qquad\qquad\qquad\qquad\ \Rightarrow f(x) - f(x_0) > 0\\
&\qquad\qquad\qquad\qquad\ \Rightarrow f(x) > f(x_0). \qquad \blacksquare
\end{aligned}$$

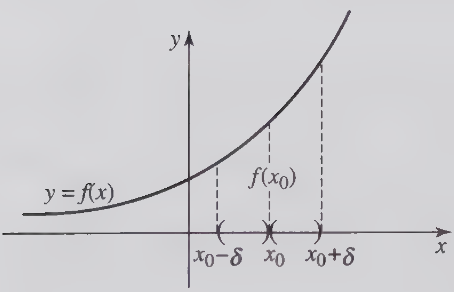

Figure 6.4

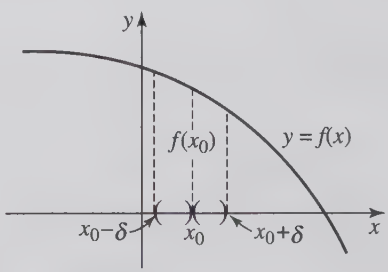

Figure 6.5

Theorem 6.3.3 *Suppose $f'(x_0) < 0$ at an interior point x_0 of $\mathcal{D}(f)$. Then* $\exists\, \delta > 0 \ni$

(a) $\forall x \in (x_0 - \delta,\, x_0), \quad f(x) > f(x_0)$, *and*

(b) $\forall x \in (x_0,\, x_0 + \delta), \quad f(x) < f(x_0)$. *(See Figure 6.5.)*

7. See Definition 4.2.8 and Theorem 4.2.9.

Proof. Exercise 1. ■

Theorem 6.3.4 (*Local Extreme Value Theorem*) *If a function f has a local extreme value at an interior point x_0 of its domain, then either $f'(x_0) = 0$ or $f'(x_0)$ does not exist.*

Proof. We shall prove the contrapositive. Suppose it is *not* true that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. That is, $f'(x_0)$ exists *and* $f'(x_0) \neq 0$. Then either $f'(x_0) > 0$ or $f'(x_0) < 0$.

Case 1 ($f'(x_0) > 0$): By Theorem 6.3.2, $\exists\, \delta > 0 \ni$

(a) $\forall x \in (x_0 - \delta, x_0),\quad f(x) < f(x_0)$, and

(b) $\forall x \in (x_0, x_0 + \delta),\quad f(x) > f(x_0)$.

By (a), f cannot have a local minimum at x_0. By (b), f cannot have a local maximum at x_0. Since f has neither a local maximum nor a local minimum at x_0, it does not have a local extreme value at x_0.

Case 2 ($f'(x_0) < 0$): Exercise 2. ■

MONOTONE FUNCTIONS

Monotone functions were defined in Section 5.2 (see Definition 5.2.15). As you recall from elementary calculus, there is a natural relationship between the monotonicity of a function and the sign of its derivative. We shall now explore that relationship.

Theorem 6.3.5 (a) *If f is differentiable at x_0 and monotone (or strictly) increasing on an open interval I containing x_0, then, $f'(x_0) \geq 0$.*

(b) *If f is differentiable and monotone (or strictly) decreasing on an open interval I containing x_0, then, $f'(x_0) \leq 0$.*

Proof. (a) Suppose f is differentiable at x_0 and monotone increasing on an open interval I containing x_0. Then $\forall x < x_0$ in I,

$$f(x) \leq f(x_0), \text{ so}$$
$$f(x) - f(x_0) \leq 0 \text{ and } x - x_0 < 0, \text{ so}$$
$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Thus, since limits from the left preserve inequalities, $\displaystyle\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$. Since f is differentiable at x_0, this limit (from the left) exists and equals $f'(x_0)$. Therefore, $f'(x_0) \geq 0$.

(b) Exercise 3. ■

Note: In the next section, after we have proved the mean value theorem, we shall prove a partial converse of this theorem.

Caution: Taken together, Theorems 6.3.2–6.3.5 might lead you to conclude erroneously that if $f'(x_0) > 0$, then f must be monotone increasing on some neighborhood of x_0, and if $f'(x_0) < 0$, then f must be monotone decreasing on some neighborhood of x_0. Indeed, your intuition may strongly suggest this. The following example will show that such a conclusion is not necessarily true.

Example 6.3.6 The function[8] $f(x) = \left\{ \begin{matrix} x + 2x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{matrix} \right\}$ is differentiable everywhere, $f'(0) > 0$, but f is not monotone on any neighborhood of 0.

Proof. See Exercise 7. $\square$

*INTERMEDIATE VALUE PROPERTY OF DERIVATIVES

Suppose f is differentiable on $I = [a, b]$, where $a < b$. Then f is continuous on $[a, b]$. The derived function $f'(x)$ exists on $[a, b]$ but is not necessarily continuous there. However, f' does have a very remarkable property in common with continuous functions: the *intermediate value property.*

Theorem 6.3.7 (***Intermediate Value Property of Derivatives***) *Suppose f is differentiable on an open interval containing a and b, where $a < b$. If k is any number between $f'(a)$ and $f'(b)$, then $\exists\, c \in (a, b) \ni f'(c) = k$.*

***Proof.** Suppose f is differentiable on an open interval I containing a and b, where $a < b$, and k is any number between $f'(a)$ and $f'(b)$. Then either $f'(a) < k < f'(b)$ or $f'(a) > k > f'(b)$.

Case 1 ($f'(a) < k < f'(b)$): Define the function g on I by

$$g(x) = kx - f(x).$$

Then g is continuous on $[a, b]$, so by the extreme value theorem (5.3.7), it has a maximum value on $[a, b]$. Observe that

(a) $g'(a) = k - f'(a) > 0$, and by hypothesis, a is an interior point of $\mathcal{D}(g)$. Thus, by Theorem 6.3.2, g cannot have its maximum value for $[a, b]$ at a.

(b) $g'(b) = k - f'(b) < 0$, and by hypothesis, b is an interior point of $\mathcal{D}(g)$. Thus, by Theorem 6.3.3, g cannot have its maximum value for $[a, b]$ at b.

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8. This example comes from [49] Gelbaum and Olmsted, *Counterexamples in Analysis*—a wonderful source of examples.

Thus, by (a) and (b), g must have its maximum value for $[a, b]$ at some point $c \in (a, b)$. Then g has a *local* maximum at c. Hence, by Theorem 6.3.4, $g'(c) = 0$.

That is,

$$k - f'(c) = 0;$$
$$f'(c) = k.$$

Case 2 ($f'(a) > k > f'(b)$): Exercise 9. ■

EXERCISE SET 6.3

1. Prove Theorem 6.3.3.
2. Prove Case 2 of Theorem 6.3.4.
3. Prove Theorem 6.3.5 (b).
4. Suppose both functions $f, g : \mathbb{R} \to \mathbb{R}$ have local maxima at x_0. Which of the following must be true? [Give a proof or a counterexample.]
   (a) $f + g$ has a local maximum at x_0.
   (b) fg has a local maximum at x_0.
5. Use Theorem 6.3.5 and its converse (not yet proved) to find the interval(s) over which the given function is increasing, and the interval(s) over which it is decreasing. Also, find the local extreme values of each function:
   (a) $f(x) = |x^2 - x - 6|$ (b) $f(x) = x + \dfrac{1}{x}$
   (c) $f(x) = \dfrac{1}{x^2 - 1}$ (d) $f(x) = \dfrac{x}{x^2 - 1}$
   (e) $f(x) = \dfrac{1}{x^2 + 1}$ (f) $f(x) = \dfrac{x}{x^2 + 1}$
6. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$, which is strictly increasing and yet $f'(x)$ is not everywhere > 0.
7. Let f be the function defined in Example 6.3.6. Prove:
   (a) f is differentiable everywhere, and $f'(0) > 0$.
   (b) f is not monotone in any neighborhood of 0. [Show that every neighborhood of 0 must contain a tail of the sequence $\left\{\dfrac{1}{n\pi}\right\}$ and apply Theorem 6.3.5.]
8. Prove that for the function f of Exercise 7, f' is not continuous at 0.
9. Complete the proof of Theorem 6.3.7 by proving Case 2.

10. Prove that there is no differentiable function f, defined on any open interval I containing 0, such that $\forall x \in I$, $f'(x) = \left\{ \begin{array}{l} 1 \text{ if } x \geq 0, \\ 2 \text{ if } x < 0 \end{array} \right\}$.

11. Prove that if f is differentiable on an open interval I and $\forall x \in I$, $f'(x) \neq 0$, then $f'(x)$ has the same sign throughout the interval I.

12. Suppose f is differentiable on some neighborhood of x_0. Prove that if the derivative f' has a discontinuity at x_0, that discontinuity cannot be a "removable" or "jump " discontinuity.[9]

13. **Provocative Example:**[10] Consider the function
$$f(x) = \left\{ \begin{array}{ll} x^2 + \frac{1}{2}x^2 \sin \frac{1}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{array} \right\}.$$
    (a) Prove that f has its absolute minimum at 0.

    (b) Prove that f is differentiable everywhere, and find $f'(0)$.

    (c) Prove that $\forall \varepsilon > 0$, $f'(x)$ has both positive and negative values in the interval $(0, \varepsilon)$ and also in the interval $(-\varepsilon, 0)$.

    Thus, f has its absolute minimum value at 0 but its derivative does not make a simple change of sign at 0!

*14. **Cantor's Function:** (for those that studied the Cantor set, Section 3.4, and the Cantor function, Section 5.5). The Cantor function $\varphi(x) : [0, 1] \to [0, 1]$ is monotone increasing on $[0, 1]$, $f(0) = 0$, and $f(1) = 1$. Prove that despite this monotonicity, $\varphi'(x) = 0$ everywhere on $[0, 1]$ except possibly on a nowhere dense set of measure 0. (In fact, φ does all its "rising" on the Cantor set.)

6.4 Mean-Value Type Theorems

You will recall (although perhaps dimly) two important theorems from elementary calculus: Rolle's theorem and the mean value theorem. You might not remember why they are considered important. You will understand and appreciate these theorems more after you have studied this section.

Theorem 6.4.1 (***Rolle's Theorem***) *Let $a < b$ and suppose $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$, and $f(a) = f(b)$. Then $\exists c \in (a, b) \ni f'(c) = 0$.*

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9. See Definitions 5.2.7 and 5.2.9.

10. For another example, see page 36 of Gelbaum & Olmsted [49].

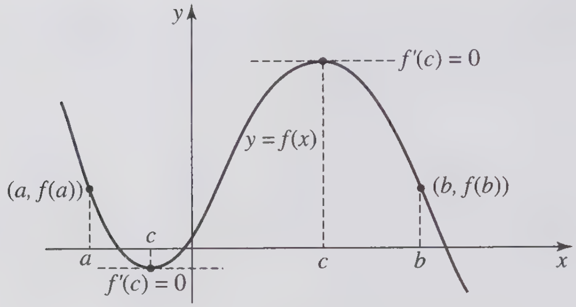


Figure 6.6

Proof. Suppose f is differentiable on (a, b) and continuous on $[a, b]$, where $a < b$, and $f(a) = f(b)$.

Case 1 (f is constant on $[a, b]$): In this case, we can take any $c \in (a, b)$; for then $f'(c) = 0$, by Corollary 6.1.3.

Case 2 (f is *not* constant on $[a, b]$): By the extreme value theorem (5.3.7), $\exists\, c, d \in [a, b] \ni$

$$f(c) = \min f[a, b] \text{ and } f(d) = \max f[a, b],$$

and since f is *not* constant on $[a, b]$, either $f(c) < f(a) = f(b)$ or $f(d) > f(a) = f(b)$.

Subcase 2a ($f(c) < f(a) = f(b)$): Then $c \neq a$ and $c \neq b$. Thus $c \in (a, b)$ and $f(c) = \min f[a, b] = \min f(a, b)$. That is, f has a *local* minimum at c. Therefore, by Theorem 6.3.4, $f'(c) = 0$.

Subcase 2b ($f(d) > f(a) = f(b)$): Exercise 1. ■

Example 6.4.2 Use Rolle's theorem to prove that the equation $5x^3 - 2x^2 + x - 56 = 0$ cannot have more than one real root.

Solution. Let $f(x) = 5x^3 - 2x^2 + x - 56$. If the equation $f(x) = 0$ has more than one real number solution, say x_1 and x_2 where $x_1 < x_2$, then by Rolle's theorem, $\exists\, c \in (x_1, x_2) \ni f'(c) = 0$. But $f'(x) = 15x^2 - 4x + 1$. The discriminant of this quadratic is $D = (-4)^2 - 4(15)(1) < 0$. Hence, there are no real numbers x for which $f'(x) = 0$. Therefore, the equation $f(x) = 0$ cannot have more than one real number solution. □

The following theorem is one of the most powerful in the entire calculus of derivatives.

Theorem 6.4.3 (***Mean Value Theorem, "MVT"***) *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b), and continuous on $[a,b]$, where $a < b$. Then $\exists c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$.*

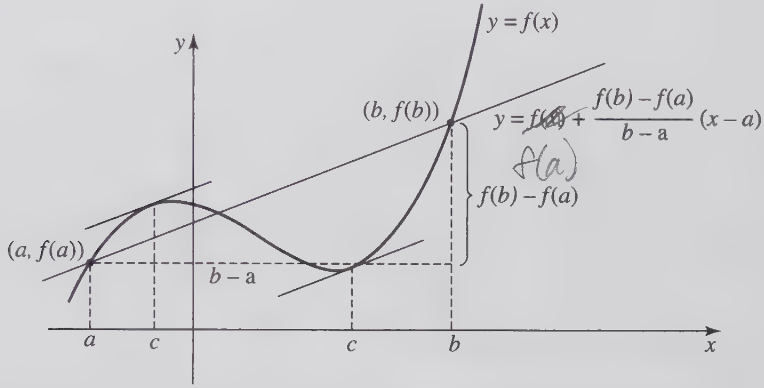


Figure 6.7

Proof. Suppose f is differentiable on (a,b), and continuous on $[a,b]$, where $a < b$. Define a new function h on $[a,b]$ by

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a). \tag{3}$$

Then h differs from f by a first degree polynomial function in x, so h is continuous wherever f is and differentiable wherever f is. Moreover,

$$h(a) = f(a) - f(a) - \frac{f(b)-f(a)}{b-a}(a-a) = 0, \text{ and}$$

$$\begin{aligned} h(b) &= f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) \\ &= f(b) - f(a) - [f(b) - f(a)] \\ &= 0. \end{aligned}$$

Thus, $h(a) = h(b)$, and h satisfies all the hypotheses of Rolle's theorem. Hence, by Rolle's theorem, $\exists\, c \in (a,b) \ni h'(c) = 0$. Now, from (3),

$$h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}.$$

Thus, $h'(c) = 0 \Rightarrow f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0$, and hence,

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \quad \blacksquare$$

APPLICATIONS OF THE MEAN VALUE THEOREM

In Corollary 6.1.3 we proved that if f is a constant, then $f'(x) = 0$. Surely you believe intuitively that the converse must also be true: if $f'(x) = 0$ on an interval, then f is constant on that interval. The proof of that converse had to wait until this point, because it is based on the mean value theorem.

Theorem 6.4.4 *Suppose f is differentiable on an interval I, and $\forall x \in I$, $f'(x) = 0$. Then f is **constant** on I.*

Proof. Suppose f is differentiable on an interval I, and $\forall x \in I$, $f'(x) = 0$. Let $x_1, x_2 \in I$. Without loss of generality, $x_1 < x_2$. By the mean value theorem applied to f on the interval $[x_1, x_2]$, $\exists\, c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But $f'(c) = 0$. That is, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$. But then $f(x_2) - f(x_1) = 0$, or $f(x_1) = f(x_2)$.

We have proved that $\forall x_1, x_2 \in I$, $f(x_1) = f(x_2)$. That is, f takes on the same value at any two points of I. But that means that f is constant on I. ▪

Corollary 6.4.5 *Suppose f and g are differentiable on an interval I, and $\forall x \in I$, $f'(x) = g'(x)$. Then $\exists$ constant $C \in \mathbb{R} \ni \forall x \in I$, $f(x) = g(x) + C$.*

Proof. Suppose f and g are differentiable on an interval I, and $\forall x \in I$, $f'(x) = g'(x)$. Define the function h on I by $h(x) = f(x) - g(x)$. By the algebra of derivatives, h is differentiable on I, and $\forall x \in I$,

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, by Theorem 6.4.4, h is constant on I. That is, $\exists C \in \mathbb{R} \ni \forall x \in I$, $h(x) = C$. Thus, $\forall x \in I$, $f(x) - g(x) = C$, or

$$f(x) = g(x) + C. \quad \blacksquare$$

The result we have just proved should be quite familiar to you. Where have you seen it before? Do you remember indefinite integration in Calculus I? Corollary 6.4.5 tells us that any two antiderivatives of the same function must have a constant difference. One antiderivative must differ from any other by a constant. This is the reason for the "C" in indefinite integration formulas such as

$$\int x^2 dx = \frac{1}{3}x^3 + C.$$

The mean value theorem is also used as the basis for proving results about monotone functions—specifically, a converse of Theorem 6.3.5.

Theorem 6.4.6 *Suppose f is differentiable on an interval I.*

(a) *If $f'(x) \geq 0$, $\forall x \in I$, then f is monotone increasing on I.*

(b) *If $f'(x) \leq 0$, $\forall x \in I$, then f is monotone decreasing on I.*

(c) *If $f'(x) > 0$, $\forall x \in I$, then f is strictly increasing on I.*

(d) *If $f'(x) < 0$, $\forall x \in I$, then f is strictly decreasing on I.*

Beware: The converses of (c) and (d) are false! (See Exercise 15.)

Proof. Suppose f is differentiable on an interval I.

(a) Suppose $f'(x) \geq 0$, $\forall x \in I$. Let $x_1 < x_2$ in I. Applying the mean value theorem to f on the interval $[x_1, x_2]$, $\exists c \in (x_1, x_2) \ni f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $f'(c) \geq 0$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0.$$

But $x_2 - x_1 > 0$ since $x_1 < x_2$. Hence, $f(x_2) - f(x_1) \geq 0$; that is,

$$f(x_2) \geq f(x_1).$$

We have proved that $\forall x_1 < x_2$ in I, $f(x_1) \leq f(x_2)$. That is, f is monotone increasing on I.

(b) Exercise 13.
(c) Exercise 14.
(d) Exercise 14. ∎

Example 6.4.7 (*Using the MVT to Prove General Inequalities*) Prove that $\forall x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$. Consequently, $\forall x \in \mathbb{R}$, $|\sin x| \leq |x|$.

Solution. Let $x, y \in \mathbb{R}$. If $x = y$, the desired inequality is true, since both sides are 0. Hence, assume $x \neq y$. Without loss of generality, assume $x < y$. The function $f(x) = \sin x$ is continuous on $[x, y]$ and differentiable on (x, y). Hence, by the mean value theorem, $\exists\, c \in (x, y) \ni$

$$\begin{aligned} f'(c) &= \frac{\sin x - \sin y}{x - y}; \text{ i.e.,} \\ \cos c &= \frac{\sin x - \sin y}{x - y}, \text{ so} \\ |\cos c| &= \left| \frac{\sin x - \sin y}{x - y} \right|. \end{aligned}$$

But $|\cos c| \leq 1$. Hence, $\dfrac{|\sin x - \sin y|}{|x - y|} \leq 1$. Therefore,

$$|\sin x - \sin y| \leq |x - y|. \quad \square$$

EXERCISE SET 6.4

1. Prove Theorem 6.4.1, Subcase 2b.

2. In each of the following, give an example of a function that fits the given conditions and for which the conclusion of Rolle's theorem does *not* hold:

   (a) f is continuous on $[a, b]$ and $f(a) = f(b)$.

   (b) f is differentiable on (a, b) and $f(a) = f(b)$.

   (c) f is continuous on $[a, b]$ and differentiable on (a, b).

3. Prove that if f is differentiable on a nonempty interval I, and $f'(x)$ is never 0 for $x \in I$, then f must be 1-1 on I.

4. Prove that the converse of Exercise 3 is not valid, by showing a counterexample: a function f that is 1-1 on a nonempty interval I but $f'(x) = 0$ for some $x \in I$.

5. Prove that the function $f(x) = 3x^5 - 2x^3 + 12x - 8$ is 1-1 on $(-\infty, +\infty)$.

6. Prove that the function $f(x) = x^3 + x^2 - 5x + 3$ is 1-1 on $[1, 5]$, but not on $(-\infty, +\infty)$.

7. Prove that the equation $7x^3 - 5x^2 + 4x - 10 = 0$ has exactly one real root. [You must prove two things: that the equation has at least one real root, and that it cannot have more than one.]

8. Prove that the equation $3x^4 - 8x^3 + 7x^2 - 45 = 0$ has exactly two real roots. [You must prove two things: that the equation has at least two real roots, and that it cannot have more than two.]

9. Suppose $f''(x) = 0$ for all $x \in \mathbb{R}$. Prove that f must be a polynomial and have degree ≤ 1.

10. Suppose $f'''(x) = 0$ for all $x \in \mathbb{R}$. Prove that f must be a polynomial and have degree ≤ 2.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$, and $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|^2$. Prove that f is a constant function.

12. Suppose f is differentiable everywhere, $f(-1) = 5$, $f(0) = 0$, and $f(1) = 10$. Prove that $\exists\, c, d \in (-1, 1) \ni f'(c) = -3$ and $f'(d) = 3$. [Hint: You will find both the MVT and Theorem 6.3.7 helpful.]

13. Prove Theorem 6.4.6 (b).

14. Prove Theorem 6.4.6 (c) and (d).

15. Prove by examples that the converses of Theorem 6.4.6 (c) and (d) are false.

16. Use the mean value theorem to prove that $\forall x, y \in \mathbb{R}$, $|\cos x - \cos y| \leq |x - y|$.

17. Use the mean value theorem to prove that $\forall x \in (0, \frac{\pi}{2})$, $\tan x > x$.

18. Use the mean value theorem to prove that $\forall\, 0 < x < y$,
$$\frac{y - x}{y} < \ln \frac{y}{x} < \frac{y - x}{x}.$$

19. Use the mean value theorem to prove that $\forall x > 1$, $\dfrac{x - 1}{x} < \ln x < x - 1$.

20. (a) Prove that the function $f(x) = \dfrac{\sin x}{x}$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right]$. [Hint: Use Exercise 17 to show that $f'(x) < 0$ on $\left(0, \frac{\pi}{2}\right]$.]

    (b) Use the result of (a) to prove that $\sin x > \dfrac{2x}{\pi}$ on $\left(0, \frac{\pi}{2}\right]$.

21. Suppose f' is continuous at some interior point x_0 of its domain. Prove that

    (a) if $f'(x_0) > 0$, f is strictly increasing on some neighborhood of x_0.

    (b) if $f'(x_0) < 0$, f is strictly decreasing on some neighborhood of x_0.

22. By Exercise 21, if f is differentiable and has a continuous derivative in some neighborhood of x_0, and $f'(x_0) \neq 0$, then f is strictly monotone on some neighborhood of x_0. Resolve the apparent contradiction between this result and Example 6.3.6.

23. Suppose f and g are differentiable on the interval $[a, +\infty)$, $f(a) \leq g(a)$, and $\forall x > a$, $f'(x) < g'(x)$. Prove that $\forall x > a$, $f(x) < g(x)$. [Hint: let $h(x) = g(x) - f(x)$.] (This remains true if "$<$" is replaced by "$\leq$" in both places.)

24. Prove that $\forall x > 1$, $e^x > ex$. [See also Exercise 28.]

25. Prove **the first derivative test**: Suppose f is continuous at x_0 and differentiable in a deleted neighborhood $N'_\delta(x_0)$, for some $\delta > 0$. Then

    (a) if $\forall x \in (x_0 - \delta, x_0)$, $f'(x) \geq 0$ and $\forall x \in (x_0, x_0 + \delta)$, $f'(x) \leq 0$, then f has a local maximum at c.

    (b) if $\forall x \in (x_0 - \delta, x_0)$, $f'(x) \leq 0$ and $\forall x \in (x_0, x_0 + \delta)$, $f'(x) \geq 0$, then f has a local minimum at c.

26. Prove **the second derivative test**: Suppose f is differentiable on a neighborhood of x_0, $f'(x_0) = 0$, and $f''(x_0)$ exists and is nonzero. Then

    (a) if $f''(x_0) < 0$, then f has a local maximum at x_0.

    (b) if $f''(x_0) > 0$, then f has a local minimum at x_0.

27. Prove the following theorem that is frequently used in elementary calculus, but rarely proved there. Suppose f is differentiable on an open interval I, f has a local maximum (minimum) at $x_0 \in I$, and x_0 is the only point of I where $f'(x) = 0$. Then f has its absolute maximum (minimum) value for I at x_0.

28. Use Exercise 27 to prove that $\forall x \neq 0$, $e^x > ex$. [See Exercise 24.]

29. Use the mean value theorem to prove Theorem 6.1.14.

30. Prove that the functions $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1 + |x|}{x}$ have the same derivative but do not differ by a constant. Does this contradict Corollary 6.4.5? Explain.

31. Derivatives (unlike continuous functions) are not necessarily bounded on compact sets. Show that there is a compact interval on which the function $f(x) = \begin{Bmatrix} x^2 \sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{Bmatrix}$ is differentiable but on which the derivative f' is unbounded.

32. (a) Prove that if f' exists and is bounded on an interval I (possibly infinite) then f satisfies a Lipschitz[11] condition of order 1 on I.

    (b) Use this result to prove that if $a > 0$, then for all $n \in \mathbb{N}$, the function $f(x) = \dfrac{1}{x^n}$ is uniformly continuous[12] on $[a, +\infty)$.

33. Prove that if f satisfies a Lipschitz[11] condition of order $\alpha > 1$ on an interval I, then f is constant on I.

6.5 Taylor's Theorem

Taylor's theorem provides a way to approximate a function f that is $(n+1)$-times differentiable in a neighborhood of a point a by a polynomial of degree $\leq n$ in powers of $(x-a)$, whose coefficients can be determined by the derivatives $f', f'', \cdots, f^{(n)}$ at a. This polynomial will be called the n^{th} **Taylor polynomial** for f about a, and will be denoted $T_n(x)$. Polynomial approximations are significant, since polynomials are the simplest kind of function to compute. They involve only three fundamental arithmetic operations: addition, subtraction, and multiplication. Taylor polynomials have many applications.

Taylor polynomials can be calculated without Taylor's theorem, using Definition 6.5.1 below. However, mere calculation of Taylor polynomials cannot justify their use in approximating functions. We need a theorem that tells us just how accurate we can expect a particular polynomial approximation to be. That is what Taylor's theorem does for us.

In this section, we will use the familiar functions e^x, $\ln x$, and the trigonometric functions. While their formal definitions are not given until Chapters 7 and 8, we need them here as examples. Thus, we shall assume that these functions are defined and differentiable everywhere in their domains, and that their derivatives obey the rules set forth in elementary calculus.

TAYLOR POLYNOMIALS

Definition 6.5.1 Suppose f and its first n derivatives $f', f'', \cdots, f^{(n)}$ exist at a. We define the n^{th} **Taylor polynomial for f about a** by the formula

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

---

11. See Exercise 6.1.17 for a definition of Lipschitz condition of order α.

12. See Exercise 5.4.10.

That is,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{4}$$

where $f^{(0)} = f$ and $f^{(k)}$ denote the k^{th} derivative of f.

It is the purpose of this section to investigate the relationship between a function f and its Taylor polynomials. We shall see that when f is "well behaved," $f(x)$ is closely approximated by $T_n(x)$ for values of x close to a. We shall understand better what this means as we progress through this section.

Example 6.5.2 Find the 4^{th} Taylor polynomial of the function $f(x) = \sin x$ about 0.

Solution. Let $f(x) = \sin x$. Then

$$\begin{aligned}
f^{(0)}(x) &= \sin x &&\Rightarrow f^{(0)}(0) = \sin 0 = 0;\\
f'(x) &= \cos x &&\Rightarrow f'(0) = \cos 0 = 1;\\
f''(x) &= -\sin x &&\Rightarrow f''(0) = -\sin 0 = 0;\\
f'''(x) &= -\cos x &&\Rightarrow f'''(0) = -\cos 0 = -1;\\
f^{(4)}(x) &= \sin x &&\Rightarrow f^{(4)}(0) = \sin 0 = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
T_4(x) &= f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3+\cdots+\frac{f^{(4)}(0)}{4!}(x-0)^4\\
&= 0+1\cdot x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\frac{0}{4!}x^4\\
&= x-\tfrac{1}{6}x^3. \quad \square
\end{aligned}$$

Example 6.5.3 Find the 4^{th} Taylor polynomial of the function $f(x) = 3 + 5x^2 - 4x^3 + x^4$ about 0.

Solution. Let $f(x) = 3 + 5x^2 - 4x^3 + x^4$. Then

$$\begin{aligned}
f^{(0)}(x) &= 3+5x^2-4x^3+x^4 &&\Rightarrow f^{(0)}(0) = 3;\\
f'(x) &= 10x-12x^2+4x^3 &&\Rightarrow f'(0) = 0;\\
f''(x) &= 10-24x+12x^2 &&\Rightarrow f''(0) = 10;\\
f'''(x) &= -24+24x &&\Rightarrow f'''(0) = -24;\\
f^{(4)}(x) &= 24 &&\Rightarrow f^{(4)}(0) = 24.
\end{aligned}$$

Thus,

$$\begin{aligned}
T_4(x) &= f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2+\frac{f'''(0)}{3!}(x-0)^3+\cdots+\frac{f^{(4)}(0)}{4!}(x-0)^4 \\
&= 3+0\cdot x+\frac{10}{2!}x^2+\frac{-24}{3!}x^3+\frac{24}{4!}x^4 \\
&= 3+\frac{10}{2}x^2+\frac{-24}{6}x^3+\frac{24}{24}x^4 \\
&= 3+5x^2-4x^3+x^4.
\end{aligned}$$

Notice that in this case, $T_4(x)$ and $f(x)$ are identical polynomials. □

Now let's take the same function f and find its 4^{th} Taylor polynomial about a different point, say 1.

Example 6.5.4 Find the 4^{th} Taylor polynomial of the function $f(x) = 3 + 5x^2 - 4x^3 + x^4$ about 1.

Solution. Let $f(x) = 3 + 5x^2 - 4x^3 + x^4$. Then

$$\begin{aligned}
f^{(0)}(x) &= 3+5x^2-4x^3+x^4 &&\Rightarrow f^{(0)}(1)=5 \\
f'(x) &= 10x-12x^2+4x^3 &&\Rightarrow f'(1)=2 \\
f''(x) &= 10-24x+12x^2 &&\Rightarrow f''(1)=-2 \\
f'''(x) &= -24+24x &&\Rightarrow f'''(1)=0 \\
f^{(4)}(x) &= 24 &&\Rightarrow f^{(4)}(1)=24
\end{aligned}$$

Then,

$$\begin{aligned}
T_4(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4 \\
&= 5+2(x-1)+\frac{-2}{2!}(x-1)^2+\frac{0}{3!}(x-1)^3+\frac{24}{4!}(x-1)^4 \\
&= 5+2(x-1)-(x-1)^2+(x-1)^4.
\end{aligned}$$

Notice that in this case, $T_4(x)$ and $f(x)$ do not appear to be identical polynomials. However, it is an easy exercise to "expand" the terms in $T_4(x)$ and show that it really is equal to $f(x)$. (Exercise 1) □

Example 6.5.5 Find the n^{th} Taylor polynomial for the function $f(x) = e^x$ about 0.

Solution. This one is easy. For all $n \in \mathbb{N}$, the n^{th} derivative of f is $f^{(n)}(x) = e^x$, and $f^{(n)}(0) = 1$. Thus,

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = \sum_{k=0}^{n} \frac{x^k}{k!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. \quad \square \end{aligned}$$

Example 6.5.6 Find the n^{th} Taylor polynomial of $f(x) = \ln(1+x)$ about 0.

Solution. We first find $f^{(n)}(0)$ for $n = 1, 2, \cdots, n$.

$$\begin{array}{llll} f(x) & = \ln(1+x) & \Rightarrow f(0) & = 0 \\ f'(x) & = \dfrac{1}{1+x} = (1+x)^{-1} & \Rightarrow f'(0) & = 1 = 0! \\ f''(x) & = -(1+x)^{-2} = \dfrac{-1!}{(1+x)^2} & \Rightarrow f''(0) & = -1! \\ f'''(x) & = 2(1+x)^{-3} = \dfrac{2!}{(1+x)^3} & \Rightarrow f'''(0) & = 2! \\ f^{(4)}(x) & = -2\cdot 3(1+x)^{-4} = \dfrac{-3!}{(1+x)^4} & \Rightarrow f^{(4)}(0) & = -3! \\ \vdots & \vdots \qquad \vdots & \vdots & \vdots \;\; \vdots \\ f^{(n)}(x) & = \dfrac{(-1)^{n+1}(n-1)!}{(1+x)^n} & \Rightarrow f^{(n)}(0) & = (-1)^{n+1}(n-1)! \end{array}$$

Thus,

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-0)^k = 0 + \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{k!}x^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}x^k \\ &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \cdots + \frac{(-1)^{n+1}}{n}x^n. \quad \square \end{aligned}$$

APPROXIMATION BY TAYLOR POLYNOMIALS

One reason to expect that $T_n(x)$ is a good approximation to $f(x)$ is that at a, $f(x)$ and $T_n(x)$ have the same value, the same derivative, the same second derivative, and so on, up to the same n^{th} derivative. The following theorem makes this explicit.

Theorem 6.5.7 *Suppose f has an n^{th} derivative at a. Then $T_n(a) = f(a)$, $T_n'(a) = f'(a)$, $T_n''(a) = f''(a)$, $\cdots$, and $T_n^{(n)}(a) = f^{(n)}(a)$.*

Proof. See Exercise 2. ■

Because f and $T_n(x)$ have the same value and first n derivatives at a we would expect the graphs of $f(x)$ and $T_n(x)$ to conform closely for values of x near a. If this is not enough to convince you that the relationship between a function and its n^{th} Taylor polynomial is a very special one, the following theorem may be enough.

Theorem 6.5.8 *The n^{th} Taylor polynomial $T_n(x)$ is the unique n^{th} degree polynomial in powers of $(x-a)$ with the properties identified in Theorem 6.5.7. That is, if $p(x) = \sum_{k=0}^{n} a_k(x-a)^k$ has the property that $f(a) = p(a)$, $f'(a) = p'(a)$, $f''(a) = p''(a)$, $\cdots$, $f^{(n)}(a) = p^{(n)}(a)$, then the coefficients in $p(x)$ are identical to the coefficients in $T_n(x)$: i.e., $\forall k = 0, 1, \cdots, n,\ \ a_k = \dfrac{f^{(k)}(a)}{k!}$.*

Proof. Suppose that

$$g(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n, \text{ and}$$
$$h(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots + b_n(x-a)^n$$

are polynomials such that

$$g(a) = h(a),\ g'(a) = h'(a),\ g''(a) = h''(a),\ \cdots,\ g^{(n)}(a) = h^{(n)}(a).$$

Then,

$$g(a) = h(a) \Rightarrow a_0 + 0 + \cdots + 0 = b_0 + 0 + \cdots + 0. \text{ Thus,}$$
$$a_0 = b_0.$$

Differentiating $g(x)$ and $h(x)$, we have

$$g'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1};$$
$$h'(x) = b_1 + 2b_2(x-a) + 3b_3(x-a)^2 + \cdots + nb_n(x-a)^{n-1}.$$

Then $g'(a) = h'(a) \Rightarrow a_1 + 0 + \cdots + 0 = b_1 + 0 + \cdots + 0$. Thus,

$$a_1 = b_1.$$

Differentiating again, we have

$$g''(x) = 2a_2 + 2 \cdot 3a_3(x-a) + 3 \cdot 4a_3(x-a)^2 + \cdots + n(n-1)a_n(x-a)^{n-2};$$
$$h''(x) = 2b_2 + 2 \cdot 3b_3(x-a) + 3 \cdot 4b_3(x-a)^2 + \cdots + n(n-1)b_n(x-a)^{n-2}.$$

Then $g''(a) = h''(a) \Rightarrow 2a_2 + 0 + \cdots + 0 = 2b_2 + 0 + \cdots + 0$. Thus,

$$a_2 = b_2.$$

Continuing in this way, we obtain $a_3 = b_3$, $a_4 = b_4$, $\cdots$, $a_n = b_n$. That is, the coefficients of g and h are identical. Hence, there can be only one n^{th} degree polynomial in powers of $(x-a)$ with the properties identified in Theorem 6.5.7. ■

To study the difference between f and its n^{th} degree Taylor polynomial about a, we introduce a notation for this difference.

Definition 6.5.9 Suppose f and its first n derivatives $f', f'', \cdots, f^{(n)}$ exist in an open interval I containing a. Then, $\forall x \in I$, we define the n^{th} Taylor **remainder** for f about a to be

$$R_n(x) = f(x) - T_n(x).$$

Thus, $\forall x \in I$,

$$f(x) = T_n(x) + R_n(x).$$

We can get some preliminary insight by looking at $T_0(x)$, the "zeroth" Taylor polynomial for f about a in light of the mean value theorem, studied in Section 6.4. Suppose f is differentiable in an interval I containing a. Let $x \in I$, $x \neq a$. By the mean value theorem applied to f on the closed interval between x and a, $\exists c$ in the open interval between x and a such that

$$\frac{f(x) - f(a)}{x - a} = f'(c).$$

Equivalently,

$$\begin{aligned} f(x) &= f(a) + f'(c)(x-a) \\ f(x) - f(a) &= f'(c)(x-a) \\ f(x) - T_0(x) &= f'(c)(x-a) \\ R_0(x) &= f'(c)(x-a). \end{aligned}$$

Thus, the mean value theorem could be rephrased as a statement about the remainder $R_0(x)$.

Lemma 6.5.10 (*Mean Value Theorem Rephrased*) *Suppose f is differentiable in an interval I containing a. Then, for all $x \neq a$ in I, $\exists c$ between x and a such that $R_0(x) = f'(c)(x-a)$.*

We are now at the point where we can state and prove Taylor's theorem. It is a statement about the remainder $R_n(x)$, the difference between $f(x)$ and $T_n(x)$. It can be considered as a generalized form of the mean value theorem; in fact, its proof will remind you of the proof of that theorem. We will use Rolle's theorem just as we used it to prove the mean value theorem (6.4.3).

Theorem 6.5.11 (***Taylor's Theorem***) *Suppose f is n times differentiable on an open interval containing a and x, where $x \neq a$, and $f^{(n+1)}(t)$ exists for all t in the open interval I between a and x. If $T_n(x)$ and $R_n(x)$ are as defined above, then $\exists c \in I \ni$*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{5}$$

[Formula (5) is called the "**Lagrange form**"[13] of the remainder.]

13. There are other formulas for $R_n(x)$ that look quite different from this one. For example, see Theorem 7.6.16.

Proof. Suppose f is n times differentiable on an open interval containing a and x, where $x \neq a$, and $f^{(n+1)}(t)$ exists for all t in the open interval I between a and x. Suppose $T_n(x)$ and $R_n(x)$ are as defined above and define the function $G(t)$ on I by the formula

$$G(t) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}. \tag{6}$$

Then G is differentiable at every $t \in I$ and continuous on the closure of I, the closed interval between a and x. Remembering that x is constant in Equation (6), the derivative of G is (fill in reasons for each step):

$$\begin{aligned}
G'(t) &= f'(t) + \sum_{k=1}^{n}\left[\frac{f^{(k)}(t)}{k!}k(x-t)^{k-1}(-1) + \frac{f^{(k+1)}(t)}{k!}(x-t)^k\right] \\
&\quad + \frac{(n+1)(x-t)^n(-1)}{(x-a)^{n+1}}R_n(x) \\
&= f'(t) - \sum_{k=1}^{n}\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \sum_{k=1}^{n}\frac{f^{(k+1)}(t)}{k!}(x-t)^k \\
&\quad - \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}R_n(x) \\
&= f'(t) - \sum_{k=0}^{n-1}\frac{f^{(k+1)}(t)}{k!}(x-t)^k + \sum_{k=1}^{n}\frac{f^{(k+1)}(t)}{k!}(x-t)^k \\
&\quad - \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}R_n(x) \\
&= f'(t) - \frac{f'(t)}{0!}(x-t)^0 + \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}R_n(x).
\end{aligned}$$

Thus, $$G'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \frac{(n+1)(x-t)^n}{(x-a)^{n+1}}R_n(x). \tag{7}$$

(Remember, in Equation (6), t is the independent variable and x is constant.)

Now, letting $t = a$ in Equation (6), we find

$$\begin{aligned}
G(a) &= f(a) + \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)\frac{(x-a)^{n+1}}{(x-a)^{n+1}} = T_n(x) + R_n(x) \\
&= f(x), \text{ by Definition 6.5.9.}
\end{aligned}$$

Also, letting $t = x$ in Equation (6), we find

$$G(x) = f(x) + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k + R_n(x)\frac{(x-x)^{n+1}}{(x-a)^{n+1}} = f(x) + 0 + 0 = f(x).$$

Thus, $G(a) = G(x)$. Hence, G satisfies all the hypotheses of Rolle's theorem on the closed interval between a and x. Therefore, by Rolle's theorem, $\exists c$ between a and x such that

$$G'(c) = 0.$$

By Equation (7) this means

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n - \frac{(n+1)(x-c)^n}{(x-a)^{n+1}} R_n(x) = 0.$$

Solving for $R_n(x)$, we have

$$\begin{aligned} R_n(x) &= \frac{(x-a)^{n+1}}{(n+1)(x-c)^n} \cdot \frac{f^{(n+1)}(c)}{n!}(x-c)^n \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad \blacksquare \end{aligned}$$

APPLICATIONS OF TAYLOR'S THEOREM

Example 6.5.12 Use Taylor's theorem to prove that $\forall x > 0$,
$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}e^x$.

Solution: Let $x > 0$. We use the Taylor polynomial $T_2(x)$ for e^x about 0. Using the notation of Definition 6.5.9, $\forall n \in \mathbb{N}$,

$$e^x = T_2(x) + R_2(x).$$

That is, $e^x = 1 + x + \frac{x^2}{2} + R_2(x)$. By Taylor's theorem, $\exists c$ between 0 and x such that $R_2(x) = \frac{e^c}{3!}x^3$. Thus, for some c between 0 and x,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{e^c}{3!}x^3.$$

Since $0 < c < x$, we have $1 < e^c < e^x$. Thus, from the above equation, we have

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}e^x. \quad \square$$

Example 6.5.13 Use Taylor's theorem to prove that

$$\forall x \in \mathbb{R}, e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}; \text{ that is, } e^x = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Solution: Let $x \in \mathbb{R}$. We use the Taylor polynomials $T_n(x)$ for e^x about 0. Using the notation of Definition 6.5.9, $\forall n \in \mathbb{N}$,

$$e^x = T_n(x) + R_n(x).$$

By Taylor's theorem, for $x \neq 0$, $\exists\, c$ between 0 and x such that

$$\begin{aligned} |R_n(x)| &= \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| = \frac{|e^c|\,|x^{n+1}|}{(n+1)!} \\ &< \frac{e^{|x|}|x^{n+1}|}{(n+1)!}. \end{aligned}$$

By Corollary 2.3.11, $\lim\limits_{n\to\infty} \dfrac{x^{n+1}}{(n+1)!} = 0$. Thus, for each x, $\lim\limits_{n\to\infty} R_n(x) = 0$.

$$\begin{aligned} \text{Therefore, } \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} &= \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} [e^x - R_n(x)] \\ &= e^x - \lim_{n\to\infty} R_n(x) = e^x. \quad \square \end{aligned}$$

Example 6.5.13 is a special case of a general theorem, which we now state.

Theorem 6.5.14 *Suppose f and all its derivatives exist on an open interval I containing a. If $\exists\, M > 0 \ni \forall x \in I$ and $\forall n \in \mathbb{N}$, $\left|f^{(n)}(x)\right| \leq M^n$, then $\lim\limits_{n\to\infty} T_n(x) = f(x)$, where $T_n(x)$ denotes the n^{th} Taylor polynomial for f about a. That is, $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$.*

Proof. Exercise 15. ■

Taylor's theorem has some unexpected applications. For example, it provides a way to prove the "Second Derivative Test," so familiar to all students who have studied maximum/minimum problems in elementary calculus. In fact, the theorem we are going to prove goes beyond this test to a more general "n^{th} Derivative Test."

Theorem 6.5.15 (*n^{th} Derivative Test for Maxima/Minima*) *Suppose that $n \geq 2$ and $f, f', f'', \cdots, f^{(n-1)}$ all exist in some neighborhood of a, $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$, and $f^{(n)}(a)$ exists but $f^{(n)}(a) \neq 0$.*

(a) *If n is even and $f^{(n)}(a) > 0$, then f has a local minimum at a.*

(b) *If n is even and $f^{(n)}(a) < 0$, then f has a local maximum at a.*

(c) *If n is odd, then f has neither local maximum nor local minimum at a.*

Proof. See Exercise 18. ■

*__Theorem 6.5.16__ (***Irrationality of e***) *e is an irrational number.*

Proof. For contradiction, suppose e is rational; say $e = \dfrac{a}{b}$, where $a, b \in \mathbb{N}$. Let n be an arbitrary integer greater than b. By Taylor's theorem [see Example 6.5.13] we have

$$e^1 = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^{c_n}}{(n+1)!}$$

for some $c_n \in (0,1)$. Multiplying both sides of this equation by $n!$, we have

$$\begin{aligned} n!\frac{a}{b} &= n!\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right] + \frac{n!e^{c_n}}{(n+1)!} \\ &= \left[n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!}\right] + \frac{e^{c_n}}{n+1}. \end{aligned}$$

Since $n > b$, $n!\dfrac{a}{b}$ is an integer. Also, the expression in square brackets in the above equation is an integer. Therefore, $\dfrac{e^{c_n}}{n+1}$ must be an integer. But, since $c_n \in (0,1)$, $1 < e^{c_n} < e$. Thus, $0 < \dfrac{e^{c_n}}{n+1} < \dfrac{e}{n+1}$, so by the squeeze principle, $\displaystyle\lim_{n\to\infty} \frac{e^{c_n}}{n+1} = 0$. Thus, we can take n sufficiently large so that

$$0 < \frac{e^{c_n}}{n+1} < 1.$$

This contradicts the assertion that $\dfrac{e^{c_n}}{n+1}$ is an integer. Therefore, e must be an irrational number. ■

SOME WORDS OF CAUTION

Taylor polynomials $T_n(x)$ about a of a function f are most reliable as approximations to $f(x)$ when f has derivatives of all orders in a neighborhood of a and $\lim_{n\to\infty} R_n(x) = 0$ for all x in this neighborhood. In the examples we have seen, this was not a severe limitation. In fact, in Example 6.5.13 we saw that for the function $f(x) = e^x$, $\lim_{n\to\infty} R_n(x) = 0$ for *all* values of x. In Exercise 13 you will prove that the same is true for the sine and cosine functions. But there are functions for which $\lim_{n\to\infty} R_n(x)$ is not zero or does not even exist.

For an extreme example, see Exercise 6.6.16. In this example, $T_n(x)$ about 0 is the constant zero function for all values of n, and thus the sequence $\{T_n(x)\}$ does not converge to the given function $f(x)$ for any $x \neq 0$.

Finally, practical concerns such as determining whether $T_n(x)$ will approximate $f(x)$ to within a prescribed degree of accuracy, how close x must be to a and how large n must be to guarantee that accuracy, and what computational procedures are most efficient, are left to specialized applied mathematics courses such as numerical analysis.

EXERCISE SET 6.5

1. Expand the terms of $T_4(x)$ obtained in Example 6.5.4, and show that $T_4(x) = f(x)$.

2. Prove Theorem 6.5.7 for $n = 4$.

3. Find the Taylor polynomials $T_{2n+1}(x)$ and $T_{2n}(x)$ about 0 for the function $f(x) = \sin x$.

4. Use Taylor's theorem and the result of Exercise 3 to prove that $\forall x \in (0, \pi)$,
$$x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$
[Hint: Calculate $T_3(x)$ about 0, and use $R_3(x)$ to obtain the first inequality; use $T_5(x)$ and $R_5(x)$ to obtain the other.]

5. Find the Taylor polynomials $T_{2n}(x)$ and $T_{2n+1}(x)$ about 0 for $f(x) = \cos x$.

6. Use the result of Exercise 5 to prove that $\forall x \in (0, \pi)$,
$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} > \cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$
[Hint: Calculate $T_5(x)$ about 0, and use $R_5(x)$ to obtain the first inequality; use $T_7(x)$ and $R_7(x)$ to obtain the other.]

7. Find the sixth Taylor polynomial $T_6(x)$ for the function $f(x) = \sqrt{x}$ about 1. Also, write the formula for the Lagrange form of $R_6(x)$.

8. Find the sixth Taylor polynomial $T_6(x)$ for the function $f(x) = \ln x$ about 1. Also, write the formula for the Lagrange form of $R_6(x)$.

9. When a Taylor polynomial $T_n(x)$ is used as an approximation to $f(x)$, $|R(x)|$ is called the "error." Use Taylor's theorem to find an upper bound on the error when $T_6(x)$ about 0 is used to approximate e^x, for $-2 \leq x \leq 2$. [See the inequality in Example 6.5.13.] Repeat for $-1 \leq x \leq 1$.

10. As in Exercise 9, use Taylor's theorem to find an upper bound on the "error" $|R(x)|$ when $T_6(x)$ about 0 is used to approximate $\sin x$, for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Repeat for $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

11. Find a Taylor polynomial about 0 that approximates e^x to within 3 decimal place accuracy for all x in $[-2, 2]$. Repeat for $[-1, 1]$.

12. Find a Taylor polynomial about 0 that approximates $\sin x$ to within 3 decimal place accuracy for all x in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Repeat for $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

13. Use Taylor's theorem and the methods of Example 6.5.13 to prove that $\forall x \in \mathbb{R}$, $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, and $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

14. Use Taylor's theorem to prove that for all x in the interval $[1, 2)$, $\ln(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$. [See Exercise 8 and Example 6.5.13.]

15. Prove Theorem 6.5.14. [Hint: See how Example 6.5.13 was done.]

16. Use the n^{th} derivative test to locate all maxima and minima of the given function. Justify your answer.

    (a) $f(x) = e^{x^3}$ (b) $f(x) = e^{x^4}$

17. Use the n^{th} derivative test to determine whether $\sin^3(x^2)$ has a local maximum, local minimum, or neither at $x = 0$.

18. **Proof of the n^{th} Derivative Test:** Prove Theorem 6.5.15, as follows:

    (a) Write the $(n-2)^{nd}$ Taylor polynomial for $f(x)$.

    (b) Show that if n is even and $f^{(n)}(a) > 0$, then for all x and c within a small enough neighborhood of a, $R_{n-1}(x) \geq 0$. Show that this implies that f has a local minimum at a.

    (c) Show that if n is even and $f^{(n)}(a) < 0$, then for all x and c within a small enough neighborhood of a, $R_{n-1}(x) \leq 0$. Show that this implies that f has a local maximum at a.

(d) Show that if n is odd, $R_{n-1}(x)$ has opposite signs for x to the left of a and to the right of a. Show that this implies that f has neither a local maximum at a nor a local minimum at a.

19. **Uniqueness of the Sine and Cosine Functions:** Suppose $f, g : \mathbb{R} \to \mathbb{R}$ and $s, c : \mathbb{R} \to \mathbb{R}$ are functions such that

(a) $f' = g$ and $g' = -f$; (b) $s' = c$ and $c' = -s$;
(c) $f(0) = s(0) = 0$; (d) $g(0) = c(0) = 1$.

Use Taylor's theorem to prove that $f = s$ and $g = c$. [Hint: To prove $f = s$, apply Taylor's theorem to $H(x) = f(x) - s(x)$ and show[14] that $\forall n \in \mathbb{N}, |H(x)| \le \dfrac{B|x^{n+1}|}{(n+1)!}$ for some constant $B > 0$.]

6.6 *L'Hôpital's Rule

> This section can be assigned as a project for independent study.

We are often interested in finding limits of the form

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} \qquad (8).$$

Throughout what follows, we allow the limits to be one-sided or even $\alpha = +\infty$ or $-\infty$. In Chapter 4, we saw how to use the "Algebra of Limits" to find the limit (8) when $\lim\limits_{x\to\alpha} f(x)$ and $\lim\limits_{x\to\alpha} g(x)$ exist but are not both 0 (or ∞).

INDETERMINATE FORMS 0/0 AND ∞/∞

In case $\lim\limits_{x\to\alpha} f(x) = \lim\limits_{x\to\alpha} g(x) = 0$ or $\lim\limits_{x\to\alpha} f(x) = \lim\limits_{x\to\alpha} g(x) = +\infty$ or $-\infty$, the limit (8) is called an **indeterminate form** because in these cases $\lim\limits_{x\to\alpha} \dfrac{f(x)}{g(x)}$ can turn out to be 0, 1, any finite number L, $+\infty$, $-\infty$, or no limit at all, as the following example shows.

Examples 6.6.1 Observe that all these possibilities can occur, as $x \to 0^+$:

(a) $\lim\limits_{x\to 0+} \dfrac{x^2}{x} = 0$ (b) $\lim\limits_{x\to 0+} \dfrac{x}{x} = 1$ (c) $\lim\limits_{x\to 0+} \dfrac{xL}{x} = L$

(d) $\lim\limits_{x\to 0+} \dfrac{x}{x^2} = +\infty$ (e) $\lim\limits_{x\to 0+} \dfrac{-x}{x^2} = -\infty$ (f) $\lim\limits_{x\to 0+} \dfrac{x \sin \frac{1}{x}}{x}$ does not exist. □

14. To see how it is done, look ahead to the proof of Theorem 7.7.34.

We can often evaluate an indeterminate form (1) by algebraically transforming $\dfrac{f(x)}{g(x)}$ into a form that is not indeterminate, and then taking the limit. We did that when finding the derivative of a function using the definition. For example,

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0}(2x - h) = 2x.$$

L'Hôpital's rule gives us a fresh approach to finding such limits. As you probably remember from your elementary calculus course, L'Hôpital's rule tells us that under certain circumstances,

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)} = \lim_{x \to \alpha} \frac{f'(x)}{g'(x)}.$$

For example, $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2.$

You will also remember that this rule is subject to certain limitations, and care must be taken not to use it when it does not apply. For example, $\displaystyle\lim_{x \to 1} \frac{x^2}{x+1} \neq \lim_{x \to 1} \frac{2x}{1}.$

L'Hôpital's rule is derived from a form of the mean value theorem, which is why the topic is located here. We begin by proving this theorem.

Theorem 6.6.2 (***Cauchy's Mean Value Theorem***): *Suppose f, g are continuous on $[a, b]$, and differentiable on (a, b), where $a < b$. Then, $\exists c \in (a, b) \ni$*

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)].$$

Proof. Suppose f, g are continuous on $[a, b]$, and differentiable on (a, b), where $a < b$. As in the proof of the mean value theorem, we define a new function h on $[a, b]$ by

$$h(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) since f and g have these properties. Moreover,

$$\begin{aligned} h(a) &= f(a)[g(b) - g(a)] - g(a)[f(b) - f(a)] \\ &= f(a)g(b) - g(a)f(b) \end{aligned}$$

and

$$\begin{aligned} h(b) &= f(b)[g(b) - g(a)] - g(b)[f(b) - f(a)] \\ &= g(b)f(a) - f(b)g(a). \end{aligned}$$

Thus $h(a) = h(b)$, so h satisfies all the hypotheses of Rolle's theorem. Hence, $\exists\, c \in (a,b) \ni h'(c) = 0$. By definition of h,

$$h'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)].$$

Since $h'(c) = 0$, we have $g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$. ∎

Geometric interpretation of the Cauchy mean value theorem: In elementary calculus, we studied curves given in parametric form: $x = f(t)$, $y = g(t)$, $a \le t \le b$. We learned there that if f and g are differentiable on (a,b), then for any $t \in (a,b)$ where $f'(t) \neq 0$, the slope of the curve at the point $(f(t), g(t))$ is $m = \dfrac{g'(t)}{f'(t)}$. Suppose that $\forall x \in (a,b)$, $f'(x) \neq 0$. Then the conclusion of the Cauchy mean value theorem can be written

$$\exists\, c \in (a,b) \ni \frac{g'(c)}{f'(c)} = \frac{g(b) - g(a)}{f(b) - f(a)}.$$

Note that when $f'(x) \neq 0$ on (a,b), Rolle's theorem guarantees that $f(a) \neq f(b)$. The Cauchy mean value theorem thus says that under the above conditions, there is some value $c \in (a,b)$ for which the slope of the line tangent to the curve at $(f(c), g(c))$ is equal to the slope of the secant line through the endpoints of the curve, $(f(a), g(a))$ and $(f(b), g(b))$. (See Figure 6.8.)

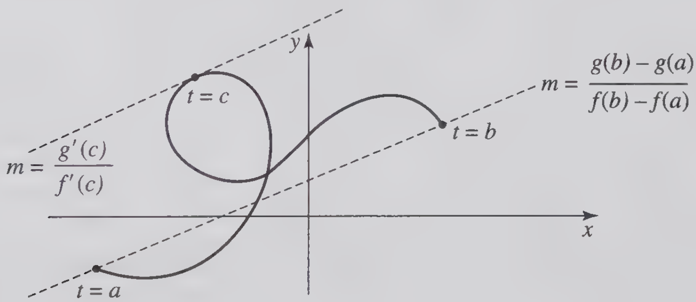


Figure 6.8

L'Hôpital's rule covers a multitude of cases. We cannot give a single proof appropriate for this course that covers all these cases; we must look at individual cases separately. Here is a comprehensive statement that incorporates all cases.

Theorem 6.6.3 (*L'Hôpital's Rule*) *Suppose $f, g : I \to \mathbb{R}$, where I is an open interval with "endpoint" α, and where*

(a) *α may be finite, $+\infty$ or $-\infty$;*

(b) *f and g are differentiable on I;*

(c) *$\forall x \in I$, $g(x)g'(x) \neq 0$ (that is, neither $g(x)$ nor $g'(x)$ can be 0 on I);*

(d) *Either* (1) $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0$ *or* (2) $\left|\lim_{x\to\alpha} g(x)\right| = \infty$;

(e) $\displaystyle\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ *(finite, $+\infty$ or $-\infty$).*

Then $\displaystyle\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L.$

Note 1: All the limits shown in the statement of this theorem are one-sided, since the domains of f and g are restricted to an open interval I with endpoint α. However, in view of the relationship between limits and one-sided limits (see Theorem 4.3.8) a proof of this theorem as stated will guarantee that the same conclusion is true for (two-sided) limits as well.

Note 2: L'Hôpital's rule involves the following cases:

- $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0$ or $\lim_{x\to\alpha} g(x) = +\infty$ or $-\infty$; (3 cases)
- $\alpha = x_0^+,\ x_0^-,\ x_0,\ +\infty$, or $-\infty$; (5 cases)
- $\displaystyle\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ (finite or $+\infty$, or $-\infty$). (3 cases)

Thus, L'Hôpital's rule will cover 45 different cases, each of which might well require its own proof! Rather than stating and proving 45 separate theorems, we shall state only two: one covering 15 cases and a second covering 30 cases. We shall prove only a small number of these cases. Simple modifications of these proofs will suffice to prove the remaining cases, and we leave them for you to prove as exercises.

Theorem 6.6.4 (*L'Hôpital's Rule I, for 0/0*) *Suppose $f, g : I \to \mathbb{R}$, where I is an open interval with "endpoint" α, and where*

(a) *α may be finite, $+\infty$ or $-\infty$;*

(b) *f and g are differentiable on I;*

(c) *$\forall x \in I$, $g(x)g'(x) \neq 0$;*

(d) $\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha} g(x) = 0;$

(e) $\lim_{x\to\alpha} \dfrac{f'(x)}{g'(x)} = L$ (*finite*, $+\infty$ *or* $-\infty$).

Then $\lim_{x\to\alpha} \dfrac{f(x)}{g(x)} = \lim_{x\to\alpha} \dfrac{f'(x)}{g'(x)} = L.$

Proof. First, note that this theorem covers 15 cases: $\alpha = x_0^+, x_0^-, x_0, +\infty$, or $-\infty$, and $L =$ a (finite) real number, $+\infty$, or $-\infty$.

Case 1: $\alpha = x_0^+$, and $L =$ a (finite) real number.

Suppose $f, g : I \to \mathbb{R}$, where f, g, and I satisfy conditions (b)–(e) specified above.

Let $\varepsilon > 0$. Then $\exists\, \delta > 0 \ni x_0 + \delta \in I$ and

$$x_0 < x < x_0 + \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon. \tag{9}$$

Suppose x is any number satisfying $x_0 < x < x_0 + \delta$, and let y be any number between x_0 and x. The Cauchy mean value theorem applies to f and g on the closed interval $[y, x]$ since f and g are differentiable on the interval I, which contains x and y. Since $g'(x) \neq 0$ on I, the Cauchy mean value theorem guarantees $\exists\, c_{x,y} \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c_{x,y})}{g'(c_{x,y})}. \tag{10}$$

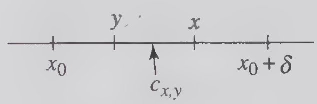


Figure 6.9

(Note that $g' \neq 0$ on I, and hence by Rolle's theorem, $g(x) - g(y) \neq 0$.) (C)

Thus, by (9) and (10),

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \varepsilon.$$

Since this is true for all $y \in (x_0, x)$, we have

$$\lim_{y\to x_0^+} \left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \varepsilon$$

$y \to 0$ case

$$\text{i.e.,} \quad \left| \frac{f(x) - 0}{g(x) - 0} - L \right| < \varepsilon.$$ why? see (d

Thus, $x_0 < x < x_0 + \delta \Rightarrow \left| \dfrac{f(x)}{g(x)} - L \right| < \varepsilon$. Therefore, $\lim_{x\to x_0^+} \dfrac{f(x)}{g(x)} = L.$

Case 2: $\alpha = x_0^+$, and $L = +\infty$.

Again, suppose $f, g : I \to \mathbb{R}$, where f, g, and I satisfy conditions (b)–(e) specified above.

Let $M > 0$. Then $\exists\, \delta > 0$ such that $x_0 + \delta \in I$ and

$$x_0 < x < x_0 + \delta \Rightarrow \frac{f'(x)}{g'(x)} > M. \tag{11}$$

Suppose x is any number satisfying $x_0 < x < x_0 + \delta$, and let y be any number between x_0 and x. As in the proof of Case 1, the Cauchy mean value theorem applies to f and g on the closed interval $[y, x]$ so $\exists\, c_{x,y} \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c_{x,y})}{g'(c_{x,y})}. \tag{12}$$

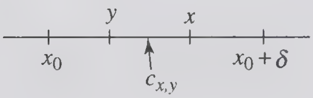


Figure 6.10

Thus, by (11) and (12),

$$x_0 < x < x_0 + \delta \Rightarrow \frac{f(x) - f(y)}{g(x) - g(y)} > M.$$

As in the proof of Case 1, we take the limit as $y \to x_0^+$, and obtain

$$x_0 < x < x_0 + \delta \Rightarrow \frac{f(x)}{g(x)} \geq M.$$

Therefore, $\displaystyle\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = +\infty$.

Case 3: $\alpha = x_0^+$, $L = -\infty$. (Exercise 1.)

Cases 4, 5, and 6: $\alpha = x_0^-$, $L =$ a (finite) real number, $+\infty$, or $-\infty$. (Exercise 3)

Cases 7, 8, and 9: $\alpha = x_0$, $L =$ a (finite) real number, $+\infty$, or $-\infty$. (Exercise 4)

Cases 10, 11, and 12: $\alpha = +\infty$, and $L =$ a real number, $+\infty$, or $-\infty$.

Our hypotheses assure us that f and g are differentiable over some interval $(a, +\infty)$, with $a > 0$. We can use Cases 1–3 to treat Cases 10–12 if we make

a change of variables and introduce two new functions. We define functions F and G on $\left(0, \frac{1}{a}\right)$ by

$$F(u) = \begin{Bmatrix} f\left(\frac{1}{u}\right) & \text{if } 0 < u < \frac{1}{a} \\ 0 & \text{if } u = 0 \end{Bmatrix} \quad \text{and} \quad G(u) = \begin{Bmatrix} g\left(\frac{1}{u}\right) & \text{if } 0 < u < \frac{1}{a} \\ 0 & \text{if } u = 0 \end{Bmatrix}.$$

By the chain rule, F and G are differentiable on $\left(0, \frac{1}{a}\right)$. By Theorem 4.4.19,

$$\lim_{u \to 0} F(u) = \lim_{u \to 0} f\left(\frac{1}{u}\right) = \lim_{x \to +\infty} f(x) = 0,$$

and similarly, $\lim\limits_{u \to 0} G(u) = 0$.

By the chain rule, $F'(u) = f'\left(\frac{1}{u}\right) \cdot \dfrac{-1}{u^2} = \dfrac{-f'\left(\frac{1}{u}\right)}{u^2}$ and $G'(u) = \dfrac{-g'\left(\frac{1}{u}\right)}{u^2}$. Thus, $G'(u) \neq 0$ on $\left(0, \frac{1}{a}\right)$. Therefore, F and G satisfy all the hypotheses of Case 1, 2, or 3 for the interval $\left(0, \frac{1}{a}\right)$. Hence,

$$\begin{aligned} \lim_{u \to 0^+} \frac{f\left(\frac{1}{u}\right)}{g\left(\frac{1}{u}\right)} = \lim_{u \to 0^+} \frac{F(u)}{G(u)} &= \lim_{u \to 0^+} \frac{F'(u)}{G'(u)} \\ &= \lim_{u \to 0^+} \frac{-f'\left(\frac{1}{u}\right)/u^2}{-g'\left(\frac{1}{u}\right)/u^2} = \lim_{u \to 0^+} \frac{f'\left(\frac{1}{u}\right)}{g'\left(\frac{1}{u}\right)}. \end{aligned}$$

That is (see Theorem 4.4.19 and Exercise 4.4-B.14),

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

Cases 13, 14, and 15: $\alpha = -\infty$, and $L =$ a (finite) real number $+\infty$, or $-\infty$. (Exercise 5) ■

Examples 6.6.5 Calculate each of the following limits. Before using L'Hôpital's rule, be sure that the hypotheses are met.

(a) $\lim\limits_{x \to 0^+} \dfrac{\sin x}{x^2}$ (b) $\lim\limits_{x \to 1} \dfrac{\ln x}{x - 1}$

(c) $\lim\limits_{x \to \pi/2} \dfrac{1 - \sin x}{\cos^2 x}$ (d) $\lim\limits_{x \to \pi/2} \dfrac{1 + \sin x}{\cos^2 x}$

Solution: (a) As $x \to 0^+$, $\sin x \to 0$ and $x^2 \to 0$. Thus, by Theorem 6.6.4,

$$\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\cos x}{2x} = +\infty$$

(As $x \to 0^+$, $\cos x \to 1$, $2x \to 0$, and $x > 0$.)

(b) As $x \to 1$, $\ln x \to 0$ and $x - 1 \to 0$. Thus, by Theorem 6.6.4,

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{1/x}{1} = 1.$$

We could have evaluated this limit without L'Hôpital's rule. Writing the numerator as $\ln x - \ln 1$, we see that this limit is just the derivative of $\ln x$ at $x = 1$.

(c) As $x \to \pi/2$, $1 - \sin x \to 0$ and $\cos^2 x \to 0$. Thus, by Theorem 6.6.4,

$$\lim_{x\to\pi/2} \frac{1-\sin x}{\cos^2 x} = \lim_{x\to\pi/2} \frac{-\cos x}{2\cos x(-\sin x)} = \lim_{x\to\pi/2} \frac{1}{2\sin x} = \frac{1}{2}.$$

As in (b), L'Hôpital's rule is not really needed to find this limit. We can evaluate it using a trigonometric identity, and the algebra of limits:

$$\lim_{x\to\pi/2} \frac{1-\sin x}{\cos^2 x} = \lim_{x\to\pi/2} \frac{1-\sin x}{1-\sin^2 x} = \lim_{x\to\pi/2} \frac{1}{1+\sin x} = \frac{1}{2}.$$

(d) As $x \to \pi/2$, $1 + \sin x \not\to 0$. Thus, L'Hôpital's rule cannot be used here. We can, however, easily evaluate this limit. As $x \to \pi/2$, $1 + \sin x \to 2$ and $\cos^2 x \to 0$, while $\cos^2 x > 0$. Thus,

$$\lim_{x\to\pi/2} \frac{1+\sin x}{\cos^2 x} = +\infty.$$

Notice that if we mistakenly apply L'Hôpital's rule to evaluate this limit, we would get the *incorrect* answer, $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{2\cos x(-\sin x)} = \lim_{x\to\pi/2} \frac{-1}{2\sin x} = \frac{-1}{2}$.

□

Theorem 6.6.6 ***(L'Hôpital's Rule II, for f/∞)*** *Suppose $f, g : I \to \mathbb{R}$, where I is an open interval with "endpoint" α, and where*

(a) *α may be finite, $+\infty$ or $-\infty$;*

(b) *f and g are differentiable on I;*

(c) *$\forall x \in I$, $g(x)g'(x) \neq 0$;*

(d) *$\lim_{x\to\alpha} g(x) = +\infty$ or $-\infty$;*

(e) *$\displaystyle\lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$ (finite, $+\infty$ or $-\infty$).*

Then $\displaystyle\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{x\to\alpha} \frac{f'(x)}{g'(x)} = L$.

Proof. First note that this theorem covers 30 cases: $\alpha = x_0^+$, x_0^-, x_0, $+\infty$, or $-\infty$; $L =$ a real number, $+\infty$, or $-\infty$; and $\lim_{x\to\alpha} g(x) = +\infty$ or $-\infty$.

Case 1: $\alpha = x_0^+$, $L =$ a finite real number, and $\lim_{x\to x_0^+} g(x) = +\infty$.

Suppose $f, g : I \to \mathbb{R}$, where f, g, and I satisfy conditions (b)–(e) specified above.

Let $\varepsilon > 0$.

By hypothesis (e), $\exists\, \delta > 0 \ni x_0 + \delta \in I$ and

$$x_0 < x < x_0 + \delta \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon. \tag{13}$$

Suppose x is any number satisfying $x_0 < x < x_0 + \delta$, and let y be any number between x_0 and x. As in the proof of Theorem 6.6.4, the Cauchy mean value theorem applies to f and g on the closed interval $[y, x]$, so $\exists\, c_{x,y} \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c_{x,y})}{g'(c_{x,y})}. \tag{14}$$

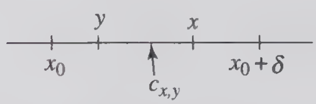


Figure 6.11

Thus, by (13) and (14), whenever $x_0 < x < x_0 + \delta$,

$$\left| \frac{f(x) - f(y)}{g(x) - g(y)} - L \right| < \varepsilon,$$

which can be transformed algebraically into

$$L - \varepsilon < \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} < L + \varepsilon. \tag{15}$$

Since this is true for all y between x_0 and x, we may consider what happens when $y \to x_0^+$. Since $g(y) \to +\infty$ as $y \to x_0^+$, we will have $1 - \dfrac{g(x)}{g(y)} > 0$ as $y \to x_0^+$. So inequality (15) is equivalent to

$$(L - \varepsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} < \frac{f(y)}{g(y)} < (L + \varepsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)}. \tag{16}$$

Now, as $y \to x_0^+$, $g(y) \to +\infty$, so the left member of this inequality approaches $(L-\varepsilon)[1-0]+0 = L-\varepsilon$, and the right member approaches $(L+\varepsilon)[1-0]+0 = L+\varepsilon$. Thus, $\exists\, \delta_1 > 0$ such that $0 < \delta_1 < \delta$ and

$$x_0 < y < x_0 + \delta_1 \Rightarrow (L-\varepsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} > (L-\varepsilon) - \varepsilon$$

$$\text{and } (L+\varepsilon)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} < (L+\varepsilon) + \varepsilon.$$

Thus, by (16) and these last two inequalities,

$$x_0 < y < x_0 + \delta_1 \Rightarrow L - 2\varepsilon < \frac{f(y)}{g(y)} < L + 2\varepsilon$$

$$\Rightarrow \left|\frac{f(y)}{g(y)} - L\right| < 2\varepsilon.$$

Therefore, $\lim\limits_{y \to x_0^+} \dfrac{f(y)}{g(y)} = L$. Equivalently, $\lim\limits_{x \to x_0^+} \dfrac{f(x)}{g(x)} = L$.

Case 2: $\alpha = x_0^+$, $L = +\infty$, and $\lim\limits_{x \to x_0^+} g(x) = +\infty$.

Again, suppose $f, g : I \to \mathbb{R}$, where f, g, and I satisfy conditions (b)–(e) specified above.

Let $M > 0$.

By hypothesis (e), $\exists\, \delta > 0 \ni x_0 + \delta \in I$ and

$$x_0 < x < x_0 + \delta \Rightarrow \frac{f'(x)}{g'(x)} > M + 1. \tag{17}$$

Suppose x is any number satisfying $x_0 < x < x_0 + \delta$, and let y be any number between x_0 and x. As in the proof of Case 1, the Cauchy mean value theorem applies to f and g on the closed interval $[y, x]$, so $\exists\, c_{x,y} \in (y, x)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c_{x,y})}{g'(c_{x,y})}. \tag{18}$$

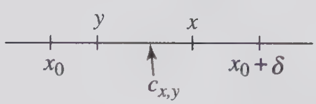


Figure 6.12

Thus, by (17) and (18), whenever $x_0 < x < x_0 + \delta$,

$$\frac{f(x) - f(y)}{g(x) - g(y)} > M + 1,$$

which can be transformed algebraically into

$$\frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} > M + 1. \tag{19}$$

Since this is true for all y between x_0 and x, we may consider what happens when $y \to x_0^+$. Since $g(y) \to +\infty$ as $y \to x_0^+$, we will have $1 - \dfrac{g(x)}{g(y)} > 0$ as $y \to x_0^+$. So inequality (19) is equivalent to

$$\frac{f(y)}{g(y)} > (M+1)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)}. \tag{20}$$

Now, as $y \to x_0^+$, $g(y) \to +\infty$, so the right member of inequality (20) approaches $(M+1)[1-0]+0 = M+1$. Thus, $\forall \varepsilon > 0$, $\exists\, \delta_1 > 0$ such that $0 < \delta_1 < \delta$ and

$$x_0 < y < x_0 + \delta_1 \Rightarrow (M+1)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} > (M+1) - \varepsilon. \tag{21}$$

Since this is true when $\forall \varepsilon > 0$, it is true when $\varepsilon = 1$. Thus, $\exists\, \delta_1 > 0$ such that $0 < \delta_1 < \delta$ and

$$x_0 < y < x_0 + \delta_1 \Rightarrow (M+1)\left[1 - \frac{g(x)}{g(y)}\right] + \frac{f(x)}{g(y)} > M. \tag{22}$$

Putting together (20) and (22),

$$x_0 < y < x_0 + \delta_1 \Rightarrow \frac{f(y)}{g(y)} > M.$$

Therefore, $\displaystyle\lim_{y \to x_0^+} \frac{f(y)}{g(y)} = +\infty = L$. That is, $\displaystyle\lim_{x \to x_0^+} \frac{f(x)}{g(x)} = \lim_{x \to \alpha} \frac{f'(x)}{g'(x)} = L$.

Case 3: $\alpha = x_0^+$, $L = -\infty$, and $\displaystyle\lim_{x \to x_0^+} g(x) = +\infty$. (Exercise 7)

Cases 4, 5, and 6: $\alpha = x_0^-$, $L =$ a (finite) real number, $+\infty$, or $-\infty$, and $\displaystyle\lim_{x \to x_0^-} g(x) = +\infty$. (Exercise 8)

Cases 7, 8, and 9: $\alpha = x_0$, $L =$ a (finite) real number, $+\infty$, or $-\infty$, and $\displaystyle\lim_{x \to x_0} g(x) = +\infty$. (Exercise 9)

Cases 10, 11, and 12: $\alpha = +\infty$, and $L =$ a (finite) real number, $+\infty$, or $-\infty$, and $\lim_{x \to +\infty} g(x) = +\infty$. (Exercise 10)

Cases 13, 14, and 15: $\alpha = -\infty$, and $L =$ a (finite) real number, $+\infty$, or $-\infty$, and $\lim_{x \to -\infty} g(x) = +\infty$. (Exercise 11)

Cases 16–30: $\lim_{x \to \alpha} g(x) = -\infty$.

Suppose $f, g : I \to \mathbb{R}$, where f, g, I, and α satisfy conditions (a)–(e) specified above, with $\lim_{x \to \alpha} g(x) = -\infty$. Define the function $h(x) = -g(x)$. Then

$$\lim_{x \to \alpha} \frac{f'(x)}{h'(x)} = \lim_{x \to \alpha} \frac{f'(x)}{-g'(x)} = -L \text{ (finite, } +\infty \text{ or } -\infty).$$

Then f and h satisfy hypotheses (a)–(e) above, with $\lim_{x \to \alpha} h(x) = +\infty$, and $\lim_{x \to \alpha} \frac{f'(x)}{h'(x)} = -L$ (finite, $+\infty$ or $-\infty$). Thus, by Cases 1–15, $\lim_{x \to \alpha} \frac{f(x)}{h(x)} = -L$; that is, $\lim_{x \to \alpha} \frac{f(x)}{g(x)} = -\lim_{x \to \alpha} \frac{f(x)}{h(x)} = -(-L) = L$. ■

Examples 6.6.7 Calculate each of the following limits. Before using L'Hôpital's rule, be sure that the hypotheses are met.

(a) $\lim_{x \to \infty} \frac{\ln x}{x}$ (b) $\lim_{x \to \infty} \frac{x^3 + 4x + 7}{5x^3 - x^2 - 3}$ (c) $\lim_{x \to 0} \frac{\ln \sin x}{\ln x}$

Solution: (a) As $x \to \infty$, the denominator $\to \infty$, so by Theorem 6.6.6,

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0.$$

(b) As $x \to \infty$, the denominator $\to \infty$, so by Theorem 6.6.6,

$$\lim_{x \to \infty} \frac{x^3 + 4x + 7}{5x^3 - x^2 - 3} = \lim_{x \to \infty} \frac{3x^2 + 4}{15x^2 - 2x} = \lim_{x \to \infty} \frac{6x}{30x - 2} = \frac{6}{30} = \frac{1}{5}.$$

Notice that this answer agrees with the answer we would obtain using the "algebra of limits" of Section 4.2.

(c) As $x \to 0$, the denominator $\to -\infty$, so by Theorem 6.6.6,

$$\lim_{x \to 0} \frac{\ln \sin x}{\ln x} = \lim_{x \to 0} \frac{\frac{1}{\sin x} \cos x}{\frac{1}{x}} = \lim_{x \to 0} \frac{x \cos x}{\sin x}.$$

We can use L'Hôpital's rule a second time, this time, because we have the indeterminate form 0/0. We get

$$\lim_{x\to 0}\frac{x(-\sin x)+\cos x}{\cos x}=\frac{0+1}{1}=1. \quad \square$$

OTHER INDETERMINATE FORMS

Sometimes other indeterminate forms such as $\infty-\infty$, $0\cdot\infty$, 0^0, etc. can be evaluated using L'Hôpital's rule, as illustrated in the following example.

Examples 6.6.8 Find each of the following limits.

(a) $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

(b) $\displaystyle\lim_{x\to 0^+} x\ln x$

(c) $\displaystyle\lim_{x\to 0^+} x^x$ [where we define $x^x=e^{x\ln x}$].

Solution: (a) This is an example of the form $\infty-\infty$. Notice that

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\sin x}{x\sin x}.$$

which is now of the form 0/0. Using L'Hôpital's rule (twice) we have

$$\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{\sin x}{-x\sin x+2\cos x}=\frac{0}{0+2}=0.$$

(b) This is an example of the form $0\cdot-\infty$.

$$\begin{aligned}\lim_{x\to 0^+} x\ln x&=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}\\&=\lim_{x\to 0^+}\frac{-x^2}{x}=\lim_{x\to 0^+}(-x)=0.\end{aligned}$$

(c) This is an example of the form 0^0. We shall use the logarithm function to convert the expression x^x to another form. Let $f(x)=x^x$. Then $\ln f(x)=x\ln x$, and $\displaystyle\lim_{x\to 0^+}\ln f(x)=\lim_{x\to 0^+}x\ln x=0$, by Part (b).

But e^x is continuous on $\mathbb{R}$; thus by Theorem 5.1.14 (b),

$$\lim_{x\to 0^+} e^{\ln f(x)}=e^{\lim\ln f(x)};\ \text{that is,}$$

$$\lim_{x\to 0^+} f(x)=e^0=1;\ \text{ i.e., }\ \lim_{x\to 0^+}x^x=1. \quad \square$$

The limits included in Example 6.6.8 were chosen for their straightforward illustrative nature. Their answers may seem predictable. In general, however, indeterminate forms are very unpredictable. Before turning to the exercise set, we give one more example in which the answer is somewhat surprising.

Example 6.6.9 Find $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x$.

Solution: Let $f(x)=\left(\frac{x+1}{x-1}\right)^x$. Then $\ln f(x)=x\ln\left(\frac{x+1}{x-1}\right)$. The function $\ln x$ is continuous on $(0,+\infty)$. Thus, by Theorem 5.1.14 (c),

$$\begin{aligned}\ln\left[\lim_{x\to\infty} f(x)\right] = \lim_{x\to\infty}\ln f(x) &= \lim_{x\to\infty} x\left[\ln(x+1)-\ln(x-1)\right]\\ &= \lim_{x\to\infty}\frac{\ln(x+1)-\ln(x-1)}{1/x}.\end{aligned}$$

By L'Hôpital's rule, this limit is equivalent to

$$\begin{aligned}\lim_{x\to\infty}\frac{\dfrac{1}{x+1}-\dfrac{1}{x-1}}{-1/x^2} &= \lim_{x\to\infty}\frac{x^2(x-1)-x^2(x+1)}{-(x+1)(x-1)}\\ &= \lim_{x\to\infty}\frac{x^3-x^2-x^3-x^2}{1-x^2}\\ &= \lim_{x\to\infty}\frac{-2x^2}{1-x^2} = \lim_{x\to\infty}\frac{-2}{\dfrac{1}{x^2}-1} = 2.\end{aligned}$$

Therefore, $\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x = e^2$. □

EXERCISE SET 6.6

1. Prove Case 3 of Theorem 6.6.4.

2. Suppose L is a (finite) real number, $+\infty$, or $-\infty$. Prove that
   (a) $\lim_{x\to x_0^-} f(x)=L \Leftrightarrow \lim_{x\to -x_0^+} f(-x)=L$.
   (b) $\lim_{x\to-\infty} f(x)=L \Leftrightarrow \lim_{x\to+\infty} f(-x)=L$.

3. Use the result of Exercise 2 (a) to prove Cases 4, 5, and 6 of Theorem 6.6.4.

4. Prove Cases 7, 8, and 9 of Theorem 6.6.4.

5. Use the result of Exercise 2 (b) to prove Cases 13, 14, and 15 of Theorem 6.6.4.

6. Calculate each of the following limits. Before using L'Hôpital's rule, be sure that the hypotheses are met.

   (a) $\displaystyle\lim_{x\to 0}\frac{1-e^x}{x}$ (b) $\displaystyle\lim_{x\to 3}\frac{\ln(x/3)}{3-x}$ (c) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{\cos x^2}$

   (d) $\displaystyle\lim_{x\to \pi}\frac{\sin x}{\sin 4x}$ (e) $\displaystyle\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}$ (f) $\displaystyle\lim_{x\to 0}\frac{x\cos x}{\sin x}$

   (g) $\displaystyle\lim_{x\to 0}\frac{2^x+1}{3^x+1}$ (h) $\displaystyle\lim_{x\to 0}\frac{2^x-1}{3^x-1}$ (i) $\displaystyle\lim_{x\to \infty}\frac{\ln(1+1/x)}{1/x}$

   (j) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x}-1}{\sqrt{x}+1}$ (k) $\displaystyle\lim_{x\to \pi}\frac{\ln(x/\pi)}{\sin x}$ (l) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

7. Prove Case 3 of Theorem 6.6.6.

8. Prove Cases 4, 5, and 6 of Theorem 6.6.6.

9. Prove Cases 7, 8, and 9 of Theorem 6.6.6.

10. Prove Cases 10, 11, and 12 of Theorem 6.6.6. 5.2 → 5.?

11. Prove Cases 13, 14, and 15 of Theorem 6.6.6.

12. Calculate each of the following limits. Before using L'Hôpital's rule, be sure that the hypotheses are met.

   (a) $\displaystyle\lim_{x\to \infty}\frac{e^x}{x}$ (b) $\displaystyle\lim_{x\to -\infty}\frac{e^x}{x}$

   (c) $\displaystyle\lim_{x\to \infty}\frac{\ln x}{\sqrt{x}}$ (d) $\displaystyle\lim_{x\to 0+}\frac{\sec x}{\ln x}$

   (e) $\displaystyle\lim_{x\to 0+}\frac{\ln x}{e^x}$ $-\infty$ (f) $\displaystyle\lim_{x\to \pi/2-}\frac{3+4\sec x}{2+\tan x}$

   (g) $\displaystyle\lim_{x\to \pi/2-}\frac{\ln\sin 2x}{\ln\cos x}$ (h) $\displaystyle\lim_{x\to 2+}\frac{1/(x-2)}{\ln(x-2)}$

13. Calculate each of the following limits. In each case, describe the indeterminate form, and transform it into a form to which L'Hôpital's rule applies. (Before using L'Hôpital's rule, be sure the hypotheses are met.)

   (a) $\displaystyle\lim_{x\to \pi/2+}(\sec x-\tan x)$ (b) $\displaystyle\lim_{x\to 0+}\left(\tfrac{1}{x}+\ln x\right)$

   (c) $\displaystyle\lim_{x\to 0+}x\ln(\sin x)$ (d) $\displaystyle\lim_{x\to 0+}x^{\sin x}$

   (e) $\displaystyle\lim_{x\to 0+}(1-x)^{1/x}$ (f) $\displaystyle\lim_{x\to \infty}\left(1-\tfrac{1}{x^2}\right)^x$ 1/2

   (g) $\displaystyle\lim_{x\to \infty}(e^x+x)^{1/x}$ (h) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

$\dfrac{\ln(e^x+x)}{x}$

14. Prove that $\forall n \in \mathbb{N}$, $\lim\limits_{t\to\infty} \dfrac{t^n}{e^t} = 0$, and use this result to prove that

(a) $\forall n \in \mathbb{N}$, $\lim\limits_{u\to 0+} \dfrac{e^{-1/u}}{u^n} = 0$ (b) $\forall n \in \mathbb{N}$, $\lim\limits_{x\to 0} \dfrac{e^{-1/x^2}}{x^n} = 0$

(c) For any polynomial $p(x)$, $\lim\limits_{x\to 0+} p\left(\frac{1}{x}\right) e^{-1/x} = 0$ and $\lim\limits_{x\to 0} p\left(\frac{1}{x}\right) e^{-1/x^2} = 0$.

15. Define the function $f(x) = \left\{ \begin{array}{ll} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{array} \right\}$.

(a) Prove that f has derivatives of every order at every $x \neq 0$. [Show that, in fact, $\forall n \in \mathbb{N}$, $\exists$ a polynomial $q(x)$ with constant term 0 such that $f^{(n)}(x) = q\left(\frac{1}{x}\right) e^{-1/x}$.]

(b) Prove that $\forall n \in \mathbb{N}$, $f^{(n)}(0) = 0$. [Thus, f has derivatives of all orders everywhere.]

(c) Find the n^{th} Taylor polynomial $T_n(x)$ of f about 0. Prove that $\forall x \in \mathbb{R}$, $\lim\limits_{n\to\infty} T_n(x)$ exists but is never equal to $f(x)$ when $x > 0$.

16. Define the function $f(x) = \left\{ \begin{array}{ll} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{array} \right\}$.

(a) Prove that f has derivatives of every order at every $x \neq 0$. [See hint for Exercise 15.]

(b) Prove that $\forall n \in \mathbb{N}$, $f^{(n)}(0) = 0$. [Thus, f has derivatives of all orders everywhere.]

(c) Find the n^{th} Taylor polynomial $T_n(x)$ of f about 0. Prove that $\forall x \in \mathbb{R}$, $\lim\limits_{n\to\infty} T_n(x)$ exists but is never equal to $f(x)$ when $x \neq 0$.

17. Use L'Hôpital's rule to find each of the following:

(a) $\lim\limits_{x\to\infty} (1+x)^{1/x}$ (b) $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$

(c) $\lim\limits_{x\to\infty} x \left[\dfrac{\left(1+\frac{1}{x}\right)^x - e}{\left(1+\frac{1}{x}\right)^x} \right]$

For (b) and (c), Exercise 6.2.15 will be helpful.

Chapter 7

The Riemann Integral

Sections 7.2–7.6 develop the essential core material on the Riemann integral. The Darboux sum approach is used because it seems the most natural at this level. Rigor is not compromised at any point, although the chapter is organized so that more esoteric matters can be skipped without sacrificing essential understanding. The elementary transcendental functions are defined rigorously in (optional) Section 7.7, and Lebesgue's criterion for Riemann integrability is proved in (optional) Section 7.9.

7.1 Refresher on Suprema, Infima, and the Forcing Principle

In defining the Riemann integral $\int_a^b f$ and establishing its properties we will make frequent use of the concepts of suprema and infima of sets of real numbers, and use them in new ways. It is a good idea to review Section 1.6 at this time. In particular, we shall need the following definitions and facts.

Definition 7.1.1 If $A \subseteq \mathbb{R}$, and $x \in \mathbb{R}$, then

$$\begin{aligned} x + A &= \{x + a : a \in A\}; \\ xA &= \{xa : a \in A\}; \\ -A &= \{-a : a \in A\}. \end{aligned}$$

Theorem 7.1.2 *Suppose $A \subseteq B \subseteq \mathbb{R}$, where B is bounded. Then*

(a) $\sup A \leq \sup B$, *and*

(b) $\inf A \geq \inf B$.

Proof. Exercise 1. ■

Theorem 7.1.3 *If $A \subseteq \mathbb{R}$ is bounded and $x \in \mathbb{R}$, then*

(a) $\sup(x + A) = x + \sup A$, *and* $\inf(x + A) = x + \inf A$

(b) *If $x > 0$, then* $\sup(xA) = x \sup A$, *and* $\inf(xA) = x \inf A$;

(c) $\sup(-A) = -\inf A$, *and* $\inf(-A) = -\sup A$;

(d) *If $x < 0$, then* $\sup(xA) = x \inf A$, *and* $\inf(xA) = x \sup A$.

Proof. Suppose $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$, and $u = \sup A \in \mathbb{R}$.

(a) Let $y \in x + A$. Then $y = x + a$ for some $a \in A$. But $x + a \leq x + u$, so $y \leq x + u$. Thus, $x + u$ is an upper bound for $x + A$.

Suppose v is another upper bound for $x + A$. Then $\forall a \in A$, $x + a \leq v$. Thus, $\forall a \in A$, $a \leq v - x$. So, $v - x$ is an upper bound for A. Hence, $u \leq v - x$. Therefore,

$$x + u \leq v.$$

Putting the above results together with Definition 1.6.3, $x+u = \sup(x+A)$. That is, $x+\sup A = \sup(x+A)$. The proof that $\inf(x+A) = x+\inf A$ is similar.

Proof of (b)–(d): Exercise 2. ■

Theorem 7.1.4 *For $A, B \subseteq \mathbb{R}$, define $A + B = \{a + b : a \in A,\ b \in B\}$.*

(a) *If A and B are bounded below, then* $\inf(A + B) = \inf A + \inf B$.

(b) *If A and B are bounded above, then* $\sup(A + B) = \sup A + \sup B$.

Proof. Exercise 3. ■

Theorem 7.1.5 *If A, B are nonempty sets of real numbers such that $\forall a \in A$, $\forall b \in B$, $a \leq b$, then $\sup A \leq \inf B$, and the following are equivalent:*

(a) $\sup A = \inf B$.

(b) $\forall \varepsilon > 0$, $\exists\, a \in A$, $b \in B \ni b - a < \varepsilon$.

(c) $\exists\, K > 0 \ni \forall \varepsilon > 0$, $\exists\, a \in A$, $b \in B \ni b - a < K\varepsilon$.

(d) $\exists$ *one and only one real number* $u \ni \forall a \in A$, $b \in B$, $a \leq u \leq b$.

(In this case, $u = \sup A = \inf B$.)

Proof. Suppose $A, B \subseteq \mathbb{R}$ such that $\forall a \in A, \forall b \in B, a \leq b$. Then $\forall a \in A$, a is a lower bound for B, so $a \leq \inf B$. That is, $\forall a \in A, a \leq \inf B$. Thus, $\inf B$ is an upper bound for A, so $\sup A \leq \inf B$.

Part 1 [$(a) \Rightarrow (b)$]: Suppose $u = \sup A = \inf B$. Let $\varepsilon > 0$. By the ε criteria for supremum and for infimum (Theorems 1.6.6 and 1.6.7) $\exists\, a \in A \ni a > u - \frac{\varepsilon}{2}$, and $\exists\, b \in B \ni b < u + \frac{\varepsilon}{2}$. Then $-a < -u + \frac{\varepsilon}{2}$. Thus,

$$b + (-a) < \left(u + \frac{\varepsilon}{2}\right) + \left(-u + \frac{\varepsilon}{2}\right) \text{ i.e.,}$$
$$b - a < \varepsilon.$$

Part 2 [$(b) \Rightarrow (c)$]: Trivial. Suppose (b) is true, and take $K = 1$.

Part 3 [$(c) \Rightarrow (d)$]: Suppose (c) is true. Let $u = \sup A$. Then, as we proved at the beginning, $u \leq \inf B$. Thus, $\forall a \in A, \forall b \in B, a \leq u \leq b$. We must now show that there is not more than one such number. For contradiction, suppose $\exists$ real number v such that $v \neq u$ and

$$\forall a \in A, \ \forall b \in B, \ a \leq v \leq b.$$

Let $\varepsilon = \dfrac{|u - v|}{K}$. Then $\varepsilon > 0$ so by (c), $\exists\, a \in A, b \in B \ni$

$$b - a < K\varepsilon$$
$$b - a < |u - v|.$$

But $u, v \in [a, b]$, so $|u - v| < b - a$. Contradiction. Therefore, (d) is true.

Part 4 [$(d) \Rightarrow (a)$]: Suppose (d) is true. We want to prove that $\sup A = \inf B$. For contradiction, suppose $\sup A \neq \inf B$. Then $\sup A < \inf B$, since we showed at the beginning that $\sup A \leq \inf B$. Then $\forall a \in A, \forall b \in B$,

$$a \leq \sup A < \inf B \leq b.$$

This contradicts (d). Therefore, $\sup A = \inf B$. ■

Finally, we shall have occasional use of the following generalized version of the forcing principle, first encountered in Theorem 1.5.9.

Theorem 7.1.6 (***Generalized Forcing Principle***)[1] *Suppose $x, a \in \mathbb{R}$.*

(a) *If $\exists\, K > 0 \ni \forall \varepsilon > 0$, $x \leq a + K\varepsilon$, then $x \leq a$.*

(b) *If $\exists\, K > 0 \ni \forall \varepsilon > 0$, $x \geq a - K\varepsilon$, then $x \geq a$.*

(c) *If $\exists\, K > 0 \ni \forall \varepsilon > 0$, $|x - a| \leq K\varepsilon$, then $x = a$.*

1. Compare with the forcing principle, Theorem 1.5.9.

Proof. Exercise 5. ■

EXERCISE SET 7.1

1. Prove Theorem 7.1.2.

2. Prove the remaining part of Theorem 7.1.3 (a), and (b)–(d).

3. Prove Theorem 7.1.4.

4. Prove that in Theorem 7.1.5 (b) and (c), "$<$" can be replaced by "$\leq$".

5. Prove Theorem 7.1.6.

7.2 The Riemann Integral Defined

The "definite" integral $\int_a^b f(x)dx$ or, more simply, $\int_a^b f$ is quite familiar to you from your elementary calculus course. In this section we give it a rigorous definition. In honor of the mathematician Bernard Riemann (1826–1866) who put this notion on a rigorous foundation, we call it the **Riemann integral**. There are other definitions of $\int_a^b f$, attributed to other mathematicians and useful for other, usually more advanced, purposes. However, Riemann's integral is by far the most universally recognized and used at this level. We emphasize that we are defining the *definite* integral, not the *indefinite* integral. The definite integral is a fundamentally significant concept, existing independently of any connection with derivatives. The "indefinite" integral $\int f(x)dx$, on the other hand, depends upon the notion of derivative for its definition. It represents the general "antiderivative" of f, and is often used as a tool in calculating the definite integral or in expressing formal solutions of differential equations.

Assumption: Throughout this section and the next, unless otherwise specified, we shall assume that any function f to be "integrated" is **defined and bounded on a compact interval** $[a, b]$, with $a < b$. Any attempt to define $\int_a^b f$ where either (a, b) is not bounded or f is not bounded on $[a, b]$ leads to what we call "improper integrals," which we do not discuss until Section 7.8.

Definition 7.2.1 A **partition** of the interval $[a, b]$ is a subset $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\} \subseteq [a, b]$, such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. (See Figure 7.1 (a).)

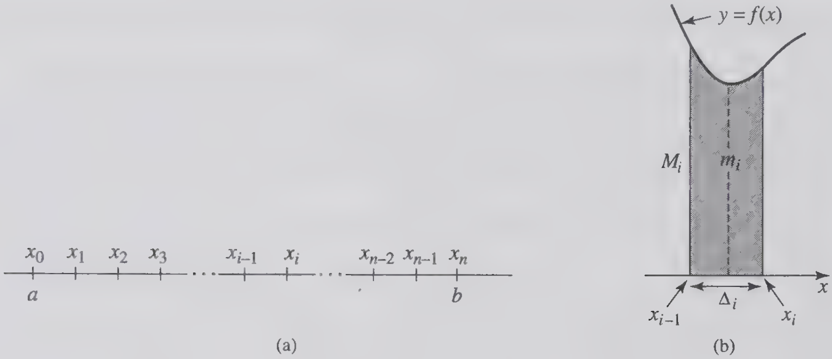


Figure 7.1

For $i = 1, 2, \cdots, n$, the i^{th} subinterval of $\mathcal{P}$ is $[x_{i-1}, x_i]$, and we define [See Figure 7.1 (b).]

$$m_i = \inf f[x_{i-1}, x_i] = \inf \{f(x) : x \in [x_{i-1}, x_i]\};$$

$$M_i = \sup f[x_{i-1}, x_i] = \sup \{f(x) : x \in [x_{i-1}, x_i]\};$$

$$\triangle_i = x_i - x_{i-1} = \text{length of the } i^{th} \text{ subinterval } [x_{i-1}, x_i].$$

For each partition $\mathcal{P}$ we define the upper and lower Darboux sums,

$$\underline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i \qquad \text{(the **lower Darboux sum** for } f \text{ over } \mathcal{P}\text{).}$$

$$\overline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i \qquad \text{(the **upper Darboux sum** for } f \text{ over } \mathcal{P}\text{).}$$

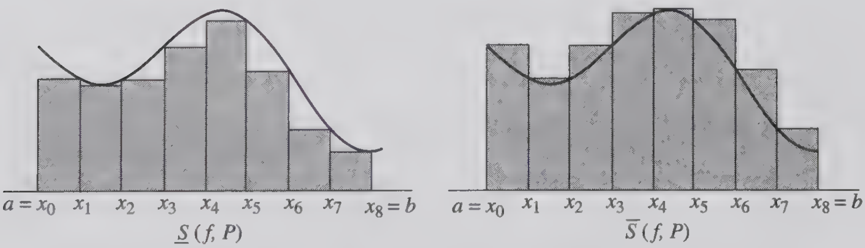


Figure 7.2

Lemma 7.2.2 *For all partitions $\mathcal{P}$ of $[a, b]$, $\underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P})$.*

Proof. Exercise 1. ■

Definition 7.2.3 A partition $\mathcal{Q}$ of $[a, b]$ is a **refinement** of a partition $\mathcal{P}$ of $[a, b]$ if $\mathcal{P} \subseteq \mathcal{Q}$.

Theorem 7.2.4 *If $\mathcal{Q}$ is a refinement of a partition $\mathcal{P}$, then*

$$\text{(a) } \underline{S}(f,\mathcal{Q}) \geq \underline{S}(f,\mathcal{P}), \qquad \textit{and} \qquad \text{(b) } \overline{S}(f,\mathcal{Q}) \leq \overline{S}(f,\mathcal{P}).$$

That is, as we refine the partitions of $[a,b]$, the lower Darboux sums "increase" and the upper Darboux sums "decrease."

Proof. Suppose $\mathcal{Q}$ is a refinement of a partition $\mathcal{P}$ of $[a,b]$. Then $\mathcal{Q}$ contains all the points of $\mathcal{P}$, and perhaps more points as well. It is sufficient to consider the case when $\mathcal{Q}$ contains exactly *one* point not contained in $\mathcal{P}$ (Exercise 2). This point will belong to exactly one of subintervals $[x_{k-1}, x_k]$ created by $\mathcal{P}$. Denote the added point by $x_k^* \in [x_{k-1}, x_k]$. Then $\mathcal{Q} = \{x_0, x_1, x_2, \cdots, x_{k-1}, x_k^*, x_k, \cdots, x_n\}$, and with the help of Theorem 7.1.2,

$\Delta i = x_i - x_{i-1}$

(a) $\underline{S}(f,\mathcal{Q})$

$$= \sum_{i=1}^{k-1} m_i \triangle_i + \inf f[x_{k-1}, x_k^*](x_k^* - x_{k-1}) + \inf f[x_k^*, x_k](x_k - x_k^*) + \sum_{i=k+1}^{n} m_i \triangle_i$$

$$\geq \sum_{i=1}^{k-1} m_i \triangle_i + m_k(x_k^* - x_{k-1}) + m_k(x_k - x_k^*) + \sum_{i=k+1}^{n} m_i \triangle_i$$

$$= \sum_{i=1}^{k-1} m_i \triangle_i + m_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} m_i \triangle_i$$

$$= \sum_{i=1}^{n} m_i \triangle_i = \underline{S}(f,\mathcal{P}). \qquad \text{Thus, } \underline{S}(f,\mathcal{Q}) \geq \underline{S}(f,\mathcal{P}). \qquad \text{Also,}$$

(b) $\overline{S}(f,\mathcal{Q})$

$$= \sum_{i=1}^{k-1} M_i \triangle_i + \sup f[x_{k-1}, x_k^*](x_k^* - x_{k-1}) + \sup f[x_k^*, x_k](x_k - x_k^*) + \sum_{i=k+1}^{n} M_i \triangle_i$$

$$\leq \sum_{i=1}^{k-1} M_i \triangle_i + M_k(x_k^* - x_{k-1}) + M_k(x_k - x_k^*) + \sum_{i=k+1}^{n} M_i \triangle_i$$

$$= \sum_{i=1}^{k-1} M_i \triangle_i + M_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} M_i \triangle_i$$

$$= \sum_{i=1}^{n} M_i \triangle_i = \overline{S}(f,\mathcal{P}). \qquad \text{Thus, } \overline{S}(f,\mathcal{Q}) \leq \overline{S}(f,\mathcal{P}). \qquad \blacksquare$$

Theorem 7.2.5 *If $\mathcal{P}$ and $\mathcal{Q}$ are any partitions of $[a, b]$, then $\underline{S}(f, \mathcal{P}) \le \overline{S}(f, \mathcal{Q})$.*[2]

Proof. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are any partitions of $[a, b]$. Then $\mathcal{P} \cup \mathcal{Q}$ is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$. Thus, by Theorem 7.2.4, $\underline{S}(f, \mathcal{P}) \le \underline{S}(f, \mathcal{P} \cup \mathcal{Q})$, and $\overline{S}(f, \mathcal{P} \cup \mathcal{Q}) \le \overline{S}(f, \mathcal{Q})$. Also, $\underline{S}(f, \mathcal{P} \cup \mathcal{Q}) \le \overline{S}(f, \mathcal{P} \cup \mathcal{Q})$ by Lemma 7.2.2. Putting these three inequalities together, we have

$$\underline{S}(f, \mathcal{P}) \le \underline{S}(f, \mathcal{P} \cup \mathcal{Q}) \le \overline{S}(f, \mathcal{P} \cup \mathcal{Q}) \le \overline{S}(f, \mathcal{Q}).$$

Therefore, $\underline{S}(f, \mathcal{P}) \le \overline{S}(f, \mathcal{Q})$. ■

Definition 7.2.6 (Upper and Lower Darboux Integrals)

Let A denote the set of lower Darboux sums for f over all possible partitions of $[a, b]$, and B denote the set of upper Darboux sums for f over all possible partitions of $[a, b]$. That is,

$$A = \{\underline{S}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}, \text{ and}$$
$$B = \{\overline{S}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$

By Theorem 7.2.5, every element of A is $\le$ every element of B. Thus, the set A is bounded above. Hence, by the completeness property of $\mathbb{R}$, A has a least upper bound. We define

$$\underline{\int_a^b} f = \sup A = \sup \{\underline{S}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$[3]

This quantity is called the **lower (Darboux) integral** of f over $[a, b]$.

Similarly, the set B is bounded below, and hence, by completeness, B has a greatest lower bound. We define

$$\overline{\int_a^b} f = \inf B = \inf \{\overline{S}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\}.$$[3]

This quantity is called the **upper (Darboux) integral** of f over $[a, b]$.

Since every element of A is $\le$ every element of B, Theorem 7.1.5 guarantees that $\sup A \le \inf B$. That is,

$$\underline{\int_a^b} f \le \overline{\int_a^b} f. \quad \square$$

We summarize these results in the following theorem.

Theorem 7.2.7 *If f is any function defined and bounded on $[a, b]$, then both $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ exist, and $\underline{\int_a^b} f \le \overline{\int_a^b} f$.*

2. See how this differs from Lemma 7.2.2.

3. Here, the supremum (or infimum) is taken over **all** partitions $\mathcal{P}$ of $[a, b]$. Don't be misled; the words "is a partition" do not mean that only one partition $\mathcal{P}$ is being considered.

Finally, we are able to define the definite integral.

Definition 7.2.8 $\left(\textbf{Darboux's Definition of } \int_a^b f\right)$ A function f defined and bounded on $[a,b]$ is **integrable** on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$. In this case, the common value of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ is called the (definite) **Riemann integral** of f over $[a,b]$, and is denoted simply $\int_a^b f$.

Note on Notation: We have not used the common notation $\int_a^b f(x)dx$, familiar to you from elementary calculus, because in defining the definite integral the symbols x and dx play no role. The notation $\int_a^b f$ correctly indicates that all we need are a function and an interval. We gain simplicity by omitting x and dx. However, in concrete examples we will often find it helpful to use the more familiar notation, $\int_a^b f(x)dx$.

Example 7.2.9 A **constant function** $f(x) = c$ is integrable over $[a,b]$, and $\int_a^b f = c(b-a)$.

Proof. For *every* partition $\mathcal{P}$ of $[a,b]$, and for every $i = 1, 2, \cdots, n$,

$$m_i = c = M_i, \text{ so}$$

$$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i = \sum_{i=1}^{n} c\triangle_i = c\sum_{i=1}^{n} \triangle_i = c(b-a),$$

and hence, $\underline{\int_a^b} f = c(b-a)$. Similarly, for every partition $\mathcal{P}$ of $[a,b]$,

$$\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i = \sum_{i=1}^{n} c\triangle_i = c\sum_{i=1}^{n} \triangle_i = c(b-a),$$

and hence $\overline{\int_a^b} f = c(b-a)$. Thus, $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a)$, from which the desired conclusion follows. $\square$

Example 7.2.10 (A Nonintegrable Function) The **Dirichlet function**[4] $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \left\{ \begin{array}{l} 1 \text{ if } x \text{ is rational,} \\ 0 \text{ if } x \text{ is irrational} \end{array} \right\}$ is *not* integrable on any closed interval $[a,b]$, where $a < b$.

4. See Example 5.1.11.

Proof. Suppose $a < b$. For *every* partition $\mathcal{P}$ of $[a,b]$, and for every $i = 1, 2, \cdots, n$, we have $m_i = 0$, and $M_i = 1$, so

$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i = \sum_{i=1}^{n} 0\triangle_i = 0$, and hence, $\underline{\int_a^b} f = 0$. Similarly,

$\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i = \sum_{i=1}^{n} 1\triangle_i = \sum_{i=1}^{n} \triangle_i = (b-a)$, and hence $\overline{\int_a^b} f = (b-a)$.

Thus, $\underline{\int_a^b} f \neq \overline{\int_a^b} f$, from which it follows that f is *not* integrable on $[a,b]$. □

Example 7.2.11 Consider the characteristic function[5] of a closed interval, say $f = \chi_{[1,3]}$, given by $f(x) = \left\{ \begin{array}{l} 1 \text{ if } 1 \leq x \leq 3, \\ 0 \text{ otherwise} \end{array} \right\}$. Prove that f is integrable on $[0,5]$ and find $\int_0^5 f$.

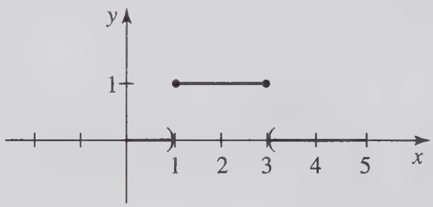


Figure 7.3

Solution. Our intuitive understanding of the integral as area (see Figure 7.3) leads us to expect that $\int_0^5 f = 2$, so we start with that expectation.

(a) Let $\mathcal{P} = \{0, 1, 3, 5\}$. Then $\mathcal{P}$ is a partition of $[0,5]$, and

$$\begin{aligned}\underline{S}(f,\mathcal{P}) &= m_1\triangle_1 + m_2\triangle_2 + m_3\triangle_3 \\ &= 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 2 \\ &= 2.\end{aligned}$$

Thus, since $\underline{\int_0^5} f$ is the supremum of all the lower sums, $\underline{\int_0^5} f \geq \underline{S}(f,\mathcal{P}) = 2$.

5. The characteristic function of a set was defined in Exercise 5.2.5.

(b) Let $0 < \varepsilon < 1$, and let $\mathcal{Q} = \{0, 1 - \frac{\varepsilon}{2}, 3 + \frac{\varepsilon}{2}, 5\}$. Then $\mathcal{Q}$ is a partition of $[0, 5]$, and

$$\begin{aligned}\overline{S}(f, \mathcal{Q}) &= M_1\triangle_1 + M_2\triangle_2 + M_3\triangle_3 \\ &= 0\left(1 - \tfrac{\varepsilon}{2}\right) + 1(2 + \varepsilon) + 0\left(2 - \tfrac{\varepsilon}{2}\right) \\ &= 2 + \varepsilon.\end{aligned}$$

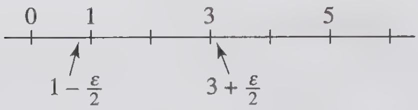


Figure 7.4

Thus, since $\overline{\int_0^5} f$ is the infimum of all the upper sums, $\overline{\int_0^5} f \leq \overline{S}(f, \mathcal{Q}) = 2 + \varepsilon$. Hence, $\forall \varepsilon > 0$, $\overline{\int_0^5} f \leq 2 + \varepsilon$. Therefore, by the forcing principle, $\overline{\int_0^5} f \leq 2$.

(c) Putting (a) and (b) together with Theorem 7.2.7,

$$2 \leq \underline{\int_0^5} f \leq \overline{\int_0^5} f \leq 2.$$

That is, $\underline{\int_0^5} f = \overline{\int_0^5} f = 2$. Therefore, f is integrable on $[0, 5]$, and $\int_0^5 f = 2$. □

We now turn our attention to the problem of determining whether a given function is integrable, and calculating the value of the integral. Sequences turn out to be quite useful in this effort. The following theorem justifies a technique often used in elementary calculus courses.

Theorem 7.2.12 (***A Sequential Criterion for Integrability and Calculating $\int_a^b f$***) *Suppose f is defined and bounded on $[a, b]$, and $L \in \mathbb{R}$.*

(a) *If there exists a sequence $\{\mathcal{P}_n\}$ of partitions of $[a, b]$ such that $\underline{S}(f, \mathcal{P}_n) \to L$, then $\underline{\int_a^b} f \geq L$.*

(b) *If there exists a sequence $\{\mathcal{Q}_n\}$ of partitions of $[a, b]$ such that $\overline{S}(f, \mathcal{Q}_n) \to M$, then $\overline{\int_a^b} f \leq M$.*

(c) *If there exist sequences $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ of partitions of $[a, b]$ such that $\underline{S}(f, \mathcal{P}_n) \to L$ and $\overline{S}(f, \mathcal{Q}_n) \to L$, then f is integrable on $[a, b]$ and $\int_a^b f = L$.*

Proof. (a) Suppose there exists a sequence $\{\mathcal{P}_n\}$ of partitions of $[a,b]$ such that $\underline{S}(f,\mathcal{P}_n) \to L$. By definition of $\underline{\int_a^b} f$, $\forall n \in \mathbb{N}$, $\underline{S}(f,\mathcal{P}_n) \leq \underline{\int_a^b} f$. Since limits preserve inequalities (Theorem 2.3.11) $\lim\limits_{n\to\infty} \underline{S}(f,\mathcal{P}_n) \leq \underline{\int_a^b} f$. That is, $L \leq \underline{\int_a^b} f$.

(b) Exercise 8.

(c) Exercise 9. ■

The next example shows how to use this sequential criterion in practice.

Example 7.2.13 Show that the function $f(x) = x^2$ is integrable on $[0,1]$ and find $\int_0^1 f$.

Solution. Consider the sequence $\{\mathcal{P}_n\}$ of partitions of $[0,1]$ given by $\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\right\}$. Then $\forall\, i = 1, 2, \cdots, n$, $\triangle_i = \frac{1}{n}$ and since f is increasing on $[0,1]$, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Then,

$$
\begin{aligned}
\text{(a) } \overline{S}(f,\mathcal{P}_n) &= \sum_{i=1}^{n} M_i \triangle_i = \sum_{i=1}^{n} f(x_i)\triangle_i \\
&= \sum_{i=1}^{n} x_i^2 \triangle_i = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \qquad \text{(See Exercise 1.3.4.)} \\
&= \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \to \frac{1}{3}.
\end{aligned}
$$

Thus, $\overline{S}(f,\mathcal{P}_n) \to \frac{1}{3}$.

(b) On the other hand,

$$
\begin{aligned}
\underline{S}(f,\mathcal{P}_n) &= \sum_{i=1}^{n} m_i \triangle_i = \sum_{i=1}^{n} f(x_{i-1})\triangle_i \\
&= \sum_{i=1}^{n} (x_{i-1})^2 \triangle_i = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \left(\frac{1}{n}\right) \\
&= \frac{1}{n^3}\sum_{i=1}^{n} (i-1)^2 = \frac{1}{n^3}\sum_{j=1}^{n-1} j^2 \\
&= \frac{1}{n^3}\frac{(n-1)n[2(n-1)+1]}{6} \qquad \text{(See Exercise 1.3.4.)} \\
&= \frac{1}{6}\left(\frac{n-1}{n}\right)\left(\frac{2n-1}{n}\right) = \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) \to \frac{1}{3}.
\end{aligned}
$$

Thus, $\underline{S}(f, \mathcal{P}_n) \to \dfrac{1}{3}$ as $n \to \infty$.

(c) Therefore, by (a), (b), and Theorem 7.2.12 (c), f is integrable over $[0,1]$ and $\int_0^1 f = \frac{1}{3}$. $\square$

In general it is quite difficult to show that a function is integrable without developing some more powerful tools to use toward that end. The next theorem is one such tool.

Theorem 7.2.14 (***Riemann's Criterion for Integrability***) *A bounded function $f\colon[a,b] \to \mathbb{R}$ is integrable on $[a,b]$ if and only if*

$$\boxed{\forall \varepsilon > 0,\ \exists \text{ partition } \mathcal{P} \text{ of } [a,b] \ni \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < \varepsilon.}$$

[Equivalently, there is some positive constant K such that $\forall \varepsilon > 0$, $\exists$ partition $\mathcal{P}$ of $[a,b] \ni \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < K\varepsilon$.]

Proof. Exercise 16. ■

The following theorem, which is equivalent to Riemann's condition, is occasionally useful.

Theorem 7.2.15 (***Equivalent Form of Riemann's Condition***) *A bounded function $f\colon[a,b] \to \mathbb{R}$ is integrable over $[a,b] \Leftrightarrow$ there is one and only one number I such that $\forall$ partitions $\mathcal{P}$ of $[a,b]$, $\underline{S}(f,\mathcal{P}) \le I \le \overline{S}(f,\mathcal{P})$. $\left(\text{In this case, } I = \int_a^b f.\right)$*

Proof. Exercise 18. ■

The next two theorems demonstrate the power of Riemann's criterion in showing that a function is integrable. These results are quite significant.

Theorem 7.2.16 ***If f is monotone on $[a,b]$, then f is integrable on $[a,b]$.***

Proof. Suppose f is monotone on $[a,b]$.

Case 1 (f is monotone *increasing* on $[a,b]$): Let $\varepsilon > 0$. Then $\forall x \in [a,b]$, $f(a) \le f(x) \le f(b)$, so f is bounded on $[a,b]$. By the Archimedean property, $\exists$ natural number $n > \dfrac{1}{\varepsilon}$; i.e., $\dfrac{1}{n} < \varepsilon$.

Consider the partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of n equally spaced points; i.e., $\triangle_i = \dfrac{b-a}{n}$. Since f is monotone increasing, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$.

Thus, $\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i = \sum_{i=1}^{n} f(x_{i-1}) \frac{b-a}{n}$,

and $\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i = \sum_{i=1}^{n} f(x_i) \frac{b-a}{n}$. Hence,

$$\begin{aligned}
\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) &= \sum_{i=1}^{n} f(x_i)\frac{b-a}{n} - \sum_{i=1}^{n} f(x_{i-1})\frac{b-a}{n} \\
&= \frac{b-a}{n}\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] \\
&= \frac{b-a}{n}\left[\cancel{f(x_1)} - f(a) + \cancel{f(x_2)} - \cancel{f(x_1)} + \cdots + f(b) - \cancel{f(x_{n-1})}\right] \\
&\quad \text{(all but two terms cancel out)} \\
&= \frac{b-a}{n}[f(b) - f(a)] \\
&< (b-a)[f(b) - f(a)]\varepsilon.
\end{aligned}$$

Therefore, by Riemann's criterion for integrability, f is integrable on $[a,b]$.

Case 2 (f is monotone *decreasing* on $[a,b]$): Exercise 19. ∎

Theorem 7.2.17 ***If f is continuous on $[a,b]$, then f is integrable on $[a,b]$.***

Proof. Suppose f is continuous on $[a,b]$. Since $[a,b]$ is compact, f is *uniformly* continuous there (see Definition 5.4.1 and Theorem 5.4.7).

Let $\varepsilon > 0$. By definition of uniform continuity, $\exists\, \delta > 0 \ni \forall x, y \in [a,b]$,

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Choose any integer $n > \dfrac{b-a}{\delta}$, and let $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ be the partition of $[a,b]$ consisting of n equally spaced points. Then each $\triangle_i = \dfrac{b-a}{n} < \delta$. By the extreme value theorem (5.3.7), f assumes minimum and maximum values on each subinterval $[x_{i-1}, x_i]$ created by $\mathcal{P}$. That is, $\exists\, x_i', x_i'' \in [x_{i-1}, x_i] \ni$

$$m_i = f(x_i') \text{ and } M_i = f(x_i'').$$

$$\begin{aligned}
\text{Then,}\quad \overline{S}(f,\mathcal{P})-\underline{S}(f,\mathcal{P}) &= \sum_{i=1}^{n}(M_i-m_i)\triangle_i \\
&= \sum_{i=1}^{n}\left(f(x_i'')-f(x_i')\right)\triangle_i \\
&< \sum_{i=1}^{n}\varepsilon\triangle_i \\
&= \varepsilon\sum_{i=1}^{n}\triangle_i=\varepsilon(b-a).
\end{aligned}$$

Hence, $\exists$ partition $\mathcal{P}$ of $[a,b] \ni \overline{S}(f,\mathcal{P})-\underline{S}(f,\mathcal{P})<(b-a)\varepsilon$. By Riemann's criterion for integrability, f is integrable over $[a,b]$. ■

EXERCISE SET 7.2

1. Prove Lemma 7.2.2. [Hint: compare the formulas for $\underline{S}(f,\mathcal{P})$ and $\overline{S}(f,\mathcal{P})$.]

2. Justify the assertion made in the proof of Theorem 7.2.4: "It is sufficient to consider the case when $\mathcal{Q}$ contains exactly *one* point not contained in $\mathcal{P}$."

3. Suppose $a<b$ and $f:[a,b]\to R$. Prove that if $\forall x\in[a,b]$, $m\le f(x)\le M$, then $m(b-a)\le \underline{\int_a^b} f\le \overline{\int_a^b} f\le M(b-a)$ and $\overline{\int_a^b} f-\underline{\int_a^b} f\le (M-m)(b-a)$.

4. Consider the characteristic function of an open interval, say $f=\chi_{(3,6)}$, defined by $f(x)=\left\{\begin{array}{l}1 \text{ if } 3<x<6,\\ 0 \text{ otherwise}\end{array}\right\}$. Prove that f is integrable on $[0,10]$ and find $\int_0^{10} f$. (See Example 7.2.11, but beware: the interval $(3,6)$ is open in this case.)

5. Consider the characteristic function of a single-point set, say $f=\chi_{\{5\}}$, defined by $f(x)=\left\{\begin{array}{l}1 \text{ if } x=5,\\ 0 \text{ otherwise}\end{array}\right\}$. Prove that f is integrable on $[2,9]$ and find $\int_2^9 f$. (See Example 7.2.11 and Exercise 4.)

6. For the function $f(x)=5-3x$, on the interval $[0,2]$, use each given partition to find $\underline{S}(f,\mathcal{P})$ and $\overline{S}(f,\mathcal{P})$: [Caution: This function is *decreasing* on the given interval.]

   (a) $\mathcal{P}=\left\{0,\frac{1}{2},1,\frac{3}{2},2\right\}$ (b) $\mathcal{P}=\left\{0,\frac{1}{2},\frac{3}{4},1,\frac{5}{4},\frac{3}{2},2\right\}$

   (c) $\mathcal{P}=\left\{0,\frac{1}{3},\frac{2}{3},1,\frac{5}{3},2\right\}$ (d) $\mathcal{P}=\left\{0,\frac{2}{n},\frac{4}{n},\frac{6}{n},\cdots,\frac{2n}{n}\right\}$

7. Repeat Exercise 6 for the function $f(x) = x^2 + 3x$.

8. Prove Theorem 7.2.12 (b).

9. Prove Theorem 7.2.12 (c).

10. Use Exercise 6 and the method of Example 7.2.13 to establish integrability and find

    (a) $\int_0^2 (5 - 3x)dx$ (b) $\int_0^2 (x^2 + 3x)dx$

11. Use the method of Example 7.2.13 to establish integrability and find

    (a) $\int_2^5 (x^2 - 2x)dx$ (b) $\int_1^6 (x^2 + x + 3)dx$

    (c) $\int_0^2 x^3 dx$ (d) $\int_{-1}^2 (1 - x^3)dx$

12. Suppose $f : [a, b] \to \mathbb{R}$ is bounded and nonnegative on $[a, b]$. Prove that

    (a) $\underline{\int_a^b} f \geq 0$.

    (b) if f is continuous at some $x_0 \in (a, b)$ and $f(x_0) > 0$, then $\underline{\int_a^b} f > 0$. [Use the neighborhood inequality property of continuous functions, Exercise 5.1.26.]

    (c) if f is continuous on $[a, b]$, then $\underline{\int_a^b} f = 0 \Leftrightarrow \forall x \in [a, b],\ f(x) = 0$. To see what can happen if f is not continuous on $[a, b]$, see Exercise 5.

13. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded $[a, b]$ and $\forall x \in [a, b],\ f(x) \leq g(x)$. Prove that $\underline{\int_a^b} f \leq \underline{\int_a^b} g$ and $\overline{\int_a^b} f \leq \overline{\int_a^b} g$.

14. Suppose f is nonnegative and integrable over $[a, b]$, and $f(r) = 0$ for all rational $r \in [a, b]$. Prove that $\int_a^b f = 0$.

15. **Another Sequential Criterion for Integrability:** Suppose $\{\mathcal{P}_n\}$ is a sequence of partitions of $[a, b]$, each of which is refined by its successor (i.e., $\forall n \in \mathbb{N},\ \mathcal{P}_n \subseteq \mathcal{P}_{n+1}$). Prove that for any bounded $f : [a, b] \to \mathbb{R}$,

    (a) both $\{\underline{S}(f, \mathcal{P}_n)\}$ and $\{\overline{S}(f, \mathcal{P}_n)\}$ converge, and $\lim\limits_{n\to\infty} \underline{S}(f, \mathcal{P}_n) \leq \lim\limits_{n\to\infty} \overline{S}(f, \mathcal{P}_n)$.

    (b) if $\lim\limits_{n\to\infty} \underline{S}(f, \mathcal{P}_n) = \lim\limits_{n\to\infty} \overline{S}(f, \mathcal{P}_n) = L$, then f is integrable on $[a, b]$, and $\int_a^b f = L$.

    (c) integrability of f on $[a, b]$ does not guarantee that $\lim\limits_{n\to\infty} \underline{S}(f, \mathcal{P}_n) = \lim\limits_{n\to\infty} \overline{S}(f, \mathcal{P}_n)$. [Show by counterexample.]

16. Prove Theorem 7.2.14. [Hint: Apply Theorems 7.1.5 and 7.2.4 to the sets A and B defined in Definition 7.2.6.]

17. Prove that Theorem 7.2.14 remains true if "$< \varepsilon$" is replaced by "$\leq \varepsilon$".

18. Prove Theorem 7.2.15. [Hint: Apply Theorem 7.1.5 to Riemann's criterion and the sets A and B of Definition 7.2.6.]

19. Prove Case 2 of Theorem 7.2.16.

20. Define the function $f : [0,1] \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = \frac{1}{n}$ if $\frac{1}{n+1} \leq x < \frac{1}{n}$; for $n \in \mathbb{N}$. Sketch the graph of f and prove that f is Riemann integrable on $[0,1]$. Notice that f has discontinuities at infinitely many points in $[0,1]$.

21. Suppose $\mathcal{Q}$ is a refinement of the partition $\mathcal{P}$ of $[a,b]$ containing just one point not in $\mathcal{P}$. Let $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$, $\mathcal{Q} = \{x_0, x_1, x_2, \cdots, x_{k-1}, x_k^*, x_k, \cdots, x_n\}$, $m_k = \inf f[x_{k-1}, x_k]$, $m_{k,1} = \inf f[x_{k-1}, x_k^*]$ and $m_{k,2} = \inf f[x_k^*, x_k]$. Prove that $\underline{S}(f,\mathcal{Q}) - \underline{S}(f,\mathcal{P}) = m_{k,1}(x_k^* - x_{k-1}) + m_{k,2}(x_k - x_k^*) - m_k(x_k - x_{k-1})$. (See the proof of (a) of Theorem 7.2.4.) State and prove a similar formula for $\overline{S}(f,\mathcal{Q}) - \overline{S}(f,\mathcal{P})$.

7.3 The Integral as a Limit of Riemann Sums

It seems that every important concept in calculus involves the notion of limit. The integral is no exception, as we shall now see. The type of limit Riemann developed for this purpose is different from any of the kinds of limits we have seen so far. Yet it will still "feel" like a limit—it will be expressed with ε and δ playing familiar roles. We begin with some technical preliminaries.

Definition 7.3.1 The **mesh** of a partition $\mathcal{P}$ of $[a,b]$ is the length of the longest subinterval $[x_{i-1}, x_i]$ between consecutive points of the partition $\mathcal{P}$; in symbols,

$$\|\mathcal{P}\| = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}.$$

Having defined the mesh of a partition, we shall immediately put this concept to work.

Theorem 7.3.2 (***Riemann/Darboux Criterion for Integrability***) *A bounded function $f\colon[a,b] \to \mathbb{R}$ is integrable over $[a,b]$* $\Longleftrightarrow$

$$\lim_{\|\mathcal{P}\|\to 0} \left(\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P})\right) = 0, \textit{ in the sense that}$$

$$\boxed{\forall \varepsilon > 0, \exists\, \delta > 0 \ni \forall \textit{ partitions } \mathcal{P} \textit{ of } [a,b],\ \|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < \varepsilon.}$$

[Equivalently, there is some $k > 0$ such that $\forall \varepsilon > 0$, $\exists\, \delta > 0 \ni \forall$ partitions $\mathcal{P}$ of $[a,b]$, $\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < k\varepsilon$.]

Proof. Suppose f is defined and bounded on $[a, b]$. Then $\exists M > 0 \ni \forall x \in [a, b]$, $|f(x)| \leq M$.

Part 1 ($\Leftarrow$): Suppose that $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \forall$ partitions $\mathcal{P}$ of $[a, b]$, $\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$. Let $\varepsilon > 0$. For the $\delta > 0$ guaranteed by our hypothesis, choose any $n \in \mathbb{N} \ni \dfrac{b-a}{n} < \delta$, and let $\mathcal{P}$ be the partition of $[a, b]$ into n subintervals of equal length. Then $\|\mathcal{P}\| = \dfrac{b-a}{n} < \delta$, so by hypothesis, $\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$. Hence, by Riemann's criterion for integrability (7.2.14), f is integrable over $[a, b]$.

Part 2 ($\Rightarrow$): Suppose f is integrable over $[a, b]$. Let $\varepsilon > 0$. By Riemann's criterion (Theorem 7.2.14), $\exists$ partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b] \ni$

$$\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \frac{\varepsilon}{2}. \tag{1}$$

Let this partition remain fixed throughout the remainder of the proof.

$$\text{Let } \delta = \frac{\varepsilon}{16nM}.$$

Now, consider any partition $\mathcal{Q}$ of $[a, b]$ such that $\|\mathcal{Q}\| < \delta$. Let $\mathcal{R}=\mathcal{Q} \cup \mathcal{P}$. If $\mathcal{R}$ has one more point than $\mathcal{Q}$, then (see Exercise 1)

$$\underline{S}(f, \mathcal{R}) - \underline{S}(f, \mathcal{Q}) \leq 2M\|\mathcal{Q}\| - (-2M)\|\mathcal{Q}\| = 4M\|\mathcal{Q}\| \tag{2}$$

Since $\mathcal{R}$ has at most n points not in $\mathcal{Q}$, we can use mathematical induction to show that

$$\begin{aligned} \underline{S}(f, \mathcal{R}) - \underline{S}(f, \mathcal{Q}) &\leq 4nM\|\mathcal{Q}\| \\ &< 4nM\delta \\ &= 4nM \cdot \frac{\varepsilon}{16nM} \\ &= \frac{\varepsilon}{4}. \end{aligned}$$

Now, $\mathcal{R} \supseteq \mathcal{Q}$ and $\mathcal{R} \supseteq \mathcal{P}$. Thus, by Theorem 7.2.4,

$$\underline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{Q}) \leq \underline{S}(f, \mathcal{R}) - \underline{S}(f, \mathcal{Q}) < \frac{\varepsilon}{4}, \text{so}$$

$$-\underline{S}(f, \mathcal{Q}) < -\underline{S}(f, \mathcal{P}) + \frac{\varepsilon}{4}. \tag{3}$$

Using a similar argument with upper sums, we can show that

$$\overline{S}(f, \mathcal{Q}) < \overline{S}(f, \mathcal{P}) + \frac{\varepsilon}{4}. \tag{4}$$

Adding together inequalities (3) and (4), we have

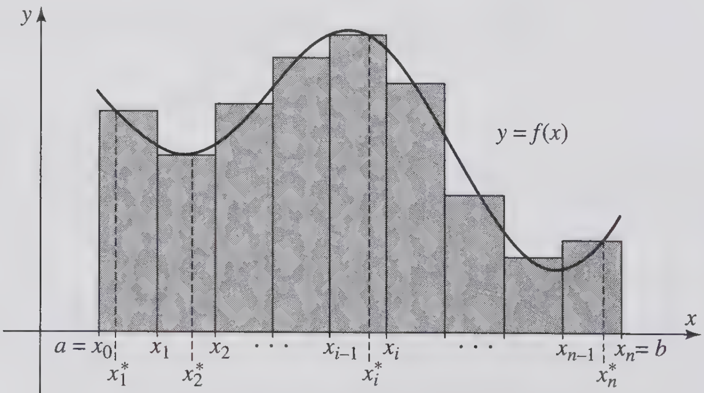
$$\begin{aligned}\overline{S}(f,\mathcal{Q}) - \underline{S}(f,\mathcal{Q}) &< \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{[from(1) above]} \\ &= \varepsilon.\end{aligned}$$

Therefore, $\forall \varepsilon > 0$, $\exists\, \delta > 0 \ni \forall$ partitions $\mathcal{Q}$ of $[a,b]$, $\|\mathcal{Q}\| < \delta \Rightarrow \overline{S}(f,\mathcal{Q}) - \underline{S}(f,\mathcal{Q}) < \varepsilon$. ■

THE INTEGRAL VIA RIEMANN SUMS

While Darboux sums lead quite effectively to a natural definition of $\int_a^b f$ there are certain advantages to be gained from a slightly different type of sum called a "Riemann sum." This type of sum will allow us to characterize the integral as a certain type of limit. It will also serve as a basis for developing numerical integration techniques in other courses (not this one).

Definition 7.3.3 A **tagged partition** $\mathcal{P}^*$ of $[a,b]$ is a partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a,b]$ along with a set of "**tags**" $x_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$. Then the sum $R(f,\mathcal{P}^*) = \sum_{i=1}^{n} f(x_i^*)\triangle_i$ is called the **Riemann sum of f over $\mathcal{P}^*$.**



The shaded area represents $R(f, P^*)$

Figure 7.5

Lemma 7.3.4 *For any partition $\mathcal{P}$ of $[a, b]$, and any selection of tags $x_1^*, x_2^*, \cdots, x_n^*$ in their respective subintervals $[x_{i-1}, x_i]$, we have*

$$\underline{S}(f, \mathcal{P}) \leq R(f, \mathcal{P}^*) \leq \overline{S}(f, \mathcal{P}).$$

That is, for a given partition $\mathcal{P}$ of $[a, b]$, all Riemann sums for f over $\mathcal{P}$ fall between the upper and lower Darboux sums for f over $\mathcal{P}$.

Proof. Exercise 2. ∎

Theorem 7.3.5 (***Limit Criterion for Integrability***) *Given any $f : [a, b] \Rightarrow \mathbb{R}$, f is integrable over $[a, b]$ and $\int_a^b f = I \iff$*

$$\boxed{\begin{array}{c}\forall \varepsilon > 0, \exists\, \delta > 0 \ni \forall \text{ tagged partitions } \mathcal{P}^* \text{ of } [a,b],\\ \|\mathcal{P}^*\| < \delta \Rightarrow |R(f, \mathcal{P}^*) - I| < \varepsilon.\end{array}}$$

That is, for all tagged partitions of sufficiently small mesh, the Riemann sum is within ε of I.

[Equivalently, $\exists\, k > 0 \ni \forall \varepsilon > 0$, $\exists\, \delta > 0 \ni \forall$ tagged partitions $\mathcal{P}^*$ of $[a, b]$, $\|\mathcal{P}^*\| < \delta \Rightarrow |R(f, \mathcal{P}^*) - I| < k\varepsilon$.]

Proof. Part 1 ($\Rightarrow$): Suppose f is integrable over $[a, b]$ and $\int_a^b f = I$.

Let $\varepsilon > 0$. By the Riemann/Darboux criterion (Theorem 7.3.2) $\exists\, \delta > 0 \ni$

$$\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon.$$

Let $\mathcal{P}^*$ be a tagged partition of $[a, b] \ni \|\mathcal{P}^*\| < \delta$. Then

$$\begin{aligned} &\underline{S}(f, \mathcal{P}) \leq R(f, \mathcal{P}^*) \leq \overline{S}(f, \mathcal{P}) \text{ by Lemma 7.3.4, so} \\ &-\overline{S}(f, \mathcal{P}) \leq -R(f, \mathcal{P}^*) \leq -\underline{S}(f, \mathcal{P}). \text{ Also,} \\ &\underline{S}(f, \mathcal{P}) \leq I \leq \overline{S}(f, \mathcal{P}) \text{ by Theorem 7.2.15.} \end{aligned}$$

Adding the last two inequalities, we have

$$-\left(\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P})\right) \leq I - R(f, \mathcal{P}^*) \leq \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}),$$

$$\begin{aligned} \text{so } |R(f, \mathcal{P}^*) - I| &\leq \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) \\ &< \varepsilon \quad \text{since } \|\mathcal{P}\| < \delta. \end{aligned}$$

Hence, $\exists\, \delta > 0 \ni \|\mathcal{P}^*\| < \delta \Rightarrow |R(f, \mathcal{P}^*) - I| < \varepsilon$.

Part 2 ($\Leftarrow$): Suppose that $\forall \varepsilon > 0, \exists \delta > 0 \ni \forall$ tagged partitions $\mathcal{P}^*$ of $[a, b]$,

$$\|\mathcal{P}^*\| < \delta \Rightarrow |R(f, \mathcal{P}^*) - I| < \varepsilon$$

$$\text{i.e., } I - \varepsilon < R(f, \mathcal{P}^*) < I + \varepsilon. \tag{5}$$

Let $\varepsilon > 0$. Choose δ as guaranteed by our hypothesis.

Let $\mathcal{P}$ be any partition of $[a, b] \ni \|\mathcal{P}\| < \delta$. By the ε-criterion for infimum (Theorem 1.6.7), for each $i = 1, 2, \cdots, n$, we can select tags $x_i^* \in [x_{i-1}, x_i] \ni$

$$f(x_i^*) < m_i + \frac{\varepsilon}{b-a}.$$

For this choice of tags x_i, the Riemann sum satisfies

$$\begin{aligned} R(f, \mathcal{P}^*) &= \sum_{i=1}^n f(x_i^*) \triangle_i \\ &< \sum_{i=1}^n (m_i + \frac{\varepsilon}{b-a}) \triangle_i \\ &= \sum_{i=1}^n m_i \triangle_i + \frac{\varepsilon}{b-a} \sum_{i=1}^n \triangle_i \\ &= \underline{S}(f, \mathcal{P}) + \tfrac{\varepsilon}{b-a}(b-a) \\ &= \underline{S}(f, \mathcal{P}) + \varepsilon. \end{aligned}$$

Thus, $\underline{S}(f, \mathcal{P}) > R(f, \mathcal{P}^*) - \varepsilon$

$\underline{S}(f, \mathcal{P}) > (I - \varepsilon) - \varepsilon$ by (5) above.

$\underline{S}(f, \mathcal{P}) > I - 2\varepsilon$. Therefore,

$$\underline{\int_a^b} f > I - 2\varepsilon.$$

Thus, $\forall \varepsilon > 0$, $I < \underline{\int_a^b} f + 2\varepsilon$. Therefore, by the forcing principle,

$$I \leq \underline{\int_a^b} f.$$

By a similar argument (Exercise 3), we can show that

$$\overline{\int_a^b} f \leq I.$$

Thus we have,

$$I \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq I,$$

from which we conclude that f is integrable over $[a, b]$, and $\int_a^b f = I$. ■

THE INTEGRAL AS A LIMIT

Note: Because of Theorem 7.3.5 it makes sense to write

$$\boxed{\begin{array}{c} \int_a^b f = \lim\limits_{\|\mathcal{P}^*\|\to 0} R(f,\mathcal{P}^*) = \lim\limits_{\|\mathcal{P}^*\|\to 0} \sum\limits_{i=1}^{n} f(x_i^*)\triangle_i. \\ \text{(if this limit exists and is independent of the tags } x_i^*\text{)} \end{array}}$$

It is important to note that this is a new kind of limit, not covered by any of our previous definitions of limits. Its definition is given in the boxed statement in Theorem 7.3.5.

Using the limit criterion for integrability, we can develop a very practical method of using sequences to calculate $\int_a^b f$ if we know that f is integrable on $[a,b]$.

Theorem 7.3.6 (***Sequential Limits for Calculating*** $\int_a^b f$)
Suppose f is integrable on $[a,b]$, and $\{\mathcal{P}_n\}$ is a sequence of partitions of $[a,b]$ such that $\|\mathcal{P}_n\| \to 0$. Then

(a) $\underline{S}(f,\mathcal{P}_n) \to \int_a^b f$ *and* $\overline{S}(f,\mathcal{P}_n) \to \int_a^b f$.

(b) *If each $\mathcal{P}_n^*$ is tagged, then $R(f,\mathcal{P}_n^*) \to \int_a^b f$, regardless of the choice of the x_i^*s.*

Proof. (a) Suppose f is integrable over $[a,b]$. Let $\varepsilon > 0$. By the Riemann/Darboux criterion for integrability, $\exists\, \delta > 0 \ni$

$$\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < \varepsilon.$$

Since $\|\mathcal{P}_n\| \to 0$, $\exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \|\mathcal{P}_n\| < \delta$. Thus,

$$n \geq n_0 \Rightarrow \overline{S}(f,\mathcal{P}_n) - \underline{S}(f,\mathcal{P}_n) < \varepsilon. \tag{6}$$

Recall from the "equivalent form" of Riemann's criterion for integrability (7.2.15) that there is a unique number $I \left(= \int_a^b f\right)$ between all upper and lower sums. Thus, we can write (6) as

$$\begin{aligned} n \geq n_0 &\Rightarrow \left[\overline{S}(f,\mathcal{P}_n) - I\right] + [I - \underline{S}(f,\mathcal{P}_n)] < \varepsilon \\ &\Rightarrow 0 \leq \overline{S}(f,\mathcal{P}_n) - I < \varepsilon \text{ and } 0 \leq I - \underline{S}(f,\mathcal{P}_n) < \varepsilon. \end{aligned}$$

Therefore, $\overline{S}(f,\mathcal{P}_n) \to I$ and $\underline{S}(f,\mathcal{P}_n) \to I$. But $I = \int_a^b f$.

(b) Exercise 6. ∎

Example 7.3.7 Use the technique of Theorem 7.3.6 to calculate $\int_1^4 (x^2 - 4x + 5)dx$.

Solution. Let $f(x) = x^2 - 4x + 5$. We know that f is integrable on $[1, 4]$ since it is continuous there. Let $\mathcal{P}_n = \{x_0, x_1, x_2, \cdots, x_n\}$ be the partition that subdivides $[1, 4]$ into n subintervals of equal length. Then $\|\mathcal{P}_n\| = \frac{4-1}{n} = \frac{3}{n}$, and for $i = 1, 2, \cdots, n$,

$$x_i = 1 + \frac{3i}{n} \quad \text{and} \quad \triangle_i = \frac{3}{n}.$$

In each subinterval $[x_{i-1}, x_i]$, choose the tag $x_i^* = x_i$. Then

$$\begin{aligned}
R(f, \mathcal{P}_n^*) &= \sum_{i=1}^{n} f(x_i^*)\triangle_i \\
&= \sum_{i=1}^{n} \left[\left(1 + \frac{3i}{n}\right)^2 - 4\left(1 + \frac{3i}{n}\right) + 5\right] \frac{3}{n} \\
&= \sum_{i=1}^{n} \left[1 + \frac{6i}{n} + \frac{9i}{n^2}^2 - 4 - \frac{12i}{n} + 5\right] \frac{3}{n} \\
&= \frac{3}{n} \sum_{i=1}^{n} \left[2 - \frac{6i}{n} + \frac{9i^2}{n^2}\right] \\
&= \frac{3}{n} \left[2 \sum_{i=1}^{n} 1 - \frac{6}{n} \sum_{i=1}^{n} i + \frac{9}{n^2} \sum_{i=1}^{n} i^2\right] \\
&= \frac{3}{n} \left[2n - \frac{6}{n} \frac{n(n+1)}{2} + \frac{9}{n^2} \frac{n(n+1)(2n+1)}{6}\right] \\
&= 6 - 9 \cdot \frac{n+1}{n} + \frac{9}{2} \cdot \frac{n+1}{n} \cdot \frac{2n+1}{n} \\
&\to 6 - 9 + 9 = 6.
\end{aligned}$$

Now, $\|\mathcal{P}_n^*\| \to 0$, so by Theorem 7.3.6, $R(f, \mathcal{P}_n^*) \to \int_1^4 f$. Therefore, $\int_1^4 f = 6$. □

*REGULAR PARTITIONS ARE SUFFICIENT

Not covering this

> This material is here only for the curious. All others should skip to Exercise Set 7.3.

*An asterisk with a theorem, proof, or other material in this chapter indicates that the material is challenging and can be omitted, especially in a one-semester course.

Definition 7.3.8 Suppose f is defined and bounded on $[a,b]$, where $a<b$. For each $n \in \mathbb{N}$, the **regular n-partition** of $[a,b]$ is the partition $\mathcal{Q}_n = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a,b]$ into n subintervals of equal length $\triangle = \|\mathcal{Q}_n\| = \dfrac{b-a}{n}$. That is, $x_i = a + i\triangle$ for $i = 0,1,2,\cdots,n$.

In this case, the upper and lower Darboux sums simplify somewhat:

$$\underline{S}(f,\mathcal{Q}_n) = \triangle \sum_{i=1}^{n} m_i \text{ and } \overline{S}(f,\mathcal{Q}_n) = \triangle \sum_{i=1}^{n} M_i$$

and thus, $\overline{S}(f,\mathcal{Q}_n) - \underline{S}(f,\mathcal{Q}_n) = \triangle \sum\limits_{i=1}^{n} (M_i - m_i)$. ∎

In elementary calculus courses, regular partitions are usually preferred over the more general partitions we have been using here, because in a first encounter they seem easier to understand and calculate. In fact, many elementary calculus courses use regular partitions exclusively. You probably noticed that we used them in finding $\int_0^1 x^2 dx$ in Example 7.2.13, in finding $\int_1^4 (x^2 - 4x + 5) dx$ in Example 7.3.7, and in proving that monotone functions and continuous functions are integrable (Theorems 7.2.16 and 7.2.17).

The ability to use general partitions is an important tool in the analyst's tool kit, both for specific calculations, as in Example 7.2.11, and for proving general theorems. Nevertheless, many students and their instructors often ask whether the exclusive use of regular partitions in the definition of Riemann integrability can be rigorously justified. Since this issue is virtually never discussed in calculus textbooks, the answer is not well known. The following theorem suggests that the answer might be "yes."

Theorem 7.3.9 *Suppose f is defined and bounded on $[a,b]$, where $a<b$. Let $\{\mathcal{Q}_n\}$ denote the sequence of regular n-partitions of $[a,b]$. Then f is integrable over $[a,b] \Leftrightarrow$ both sequences $\{\underline{S}(f,\mathcal{Q}_n)\}$ and $\{\overline{S}(f,\mathcal{Q}_n)\}$ converge, and have the same limit. In this case, $\int_a^b f = \lim\limits_{n\to\infty} \underline{S}(f,\mathcal{Q}_n) = \lim\limits_{n\to\infty} \overline{S}(f,\mathcal{Q}_n)$.*

Proof. To prove the $\Rightarrow$ direction, apply Theorem 7.3.6. To prove the $\Leftarrow$ direction, apply Theorem 7.2.12 (c). ∎

Remarks 7.3.10 It is tempting to believe that when f is integrable over $[a,b]$, the sequence $\{\underline{S}(f,\mathcal{Q}_n)\}$ of lower sums for the regular n-partitions of $[a,b]$ is monotone increasing, and the sequence $\{\overline{S}(f,\mathcal{Q}_n)\}$ of upper sums is monotone decreasing. This is not necessarily true, even though both sequences converge to $\int_a^b f$. The reason why these sequences are not necessarily monotone is that the partition $\mathcal{Q}_{n+1}$ is **not** a refinement of $\mathcal{Q}_n$ when $n>1$. A counterexample is given in Exercise 12.

The next theorem and its consequences provide conclusive evidence that regular partitions are sufficient for Riemann integrability.

Theorem 7.3.11 *Suppose f is defined and bounded on $[a,b]$, where $a < b$. Then, for every partition $\mathcal{P}$ of $[a,b]$ and $\forall \varepsilon > 0$, $\exists$ regular m-partition $\mathcal{Q}_m$ of $[a,b]$ such that*

$$\underline{S}(f,\mathcal{Q}_m) > \underline{S}(f,\mathcal{P}) - \varepsilon \text{ and } \overline{S}(f,\mathcal{Q}_m) < \overline{S}(f,\mathcal{P}) + \varepsilon.$$

Proof. Suppose f is defined and bounded on $[a,b]$, where $a < b$, and $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ is any partition of $[a,b]$. Then $\exists\, B > 0 \ni \forall x \in [a,b]$, $|f(x)| < B$. Let $\triangle_0 = \min\{\triangle_1, \triangle_2, \cdots, \triangle_n\}$.

Let $\varepsilon > 0$. Choose any natural number $m > \max\left\{\dfrac{b-a}{\triangle_0}, \dfrac{n(n+1)B(b-a)}{2\,\varepsilon}\right\}$. Let $\mathcal{Q}_m = \{y_0, y_1, y_2, \cdots, y_m\}$ be the regular m-partition of $[a,b]$. The condition $m > \dfrac{b-a}{\triangle_0}$ assures us that $\|Q_m\| = \dfrac{b-a}{m} < \triangle_0$, and thus each subinterval $[x_{i-1}, x_i]$ created by $\mathcal{P}$ contains at least one point of $\mathcal{Q}$. To see the relationship between $\underline{S}(f,\mathcal{P})$ and $\underline{S}(f,\mathcal{Q}_m)$ we introduce some notation:

$$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i, \text{ where } m_i = \inf f\left([x_{i-1}, x_i]\right) \text{ and } \triangle_i = x_i - x_{i-1},$$

$$\underline{S}(f,\mathcal{Q}_m) = \sum_{k=1}^{m} \overline{m}_k \overline{\triangle}, \text{ where } \overline{m}_k = \inf f\left([y_{k-1}, y_k]\right) \text{ and } \overline{\triangle} = \|Q_m\| = \frac{b-a}{m},$$

and for each $i = 1, 2, \cdots, n$, let $k_i = \min\{k : y_k \geq x_i\}$.

As suggested by Figure 7.6,

$$\begin{aligned} m_1\triangle_1 &\leq \overline{m}_1\overline{\triangle} + \overline{m}_2\overline{\triangle} + \cdots + \overline{m}_{k_1-1}\overline{\triangle} + B\overline{\triangle} \\ &= (\overline{m}_1 + \overline{m}_2 + \cdots + \overline{m}_{k_1})\overline{\triangle} + (B - \overline{m}_{k_1})\overline{\triangle} \\ &\leq (\overline{m}_1 + \overline{m}_2 + \cdots + \overline{m}_{k_1})\overline{\triangle} + (B - (-B))\overline{\triangle} \\ &\leq (\overline{m}_1 + \overline{m}_2 + \cdots + \overline{m}_{k_1})\overline{\triangle} + 2B\overline{\triangle}. \end{aligned} \tag{7}$$

Continuing, with the help of Figure 7.6, we see that

$$\begin{aligned} m_2\triangle_2 &\leq B\overline{\triangle} + \overline{m}_{k_1+1}\overline{\triangle} + \overline{m}_{k_1+2}\overline{\triangle} + \cdots + \overline{m}_{k_2-1}\overline{\triangle} + B\overline{\triangle}. \\ &= (\overline{m}_{k_1+1} + \overline{m}_{k_1+2} + \cdots + \overline{m}_{k_2})\overline{\triangle} + (B + B - \overline{m}_{k_2})\overline{\triangle} \\ &\leq (\overline{m}_{k_1+1} + \overline{m}_{k_1+2} + \cdots + \overline{m}_{k_2})\overline{\triangle} + (B + B - (-B))\overline{\triangle} \\ &\leq (\overline{m}_{k_1+1} + \overline{m}_{k_1+2} + \cdots + \overline{m}_{k_2})\overline{\triangle} + 3B\overline{\triangle}. \end{aligned} \tag{8}$$

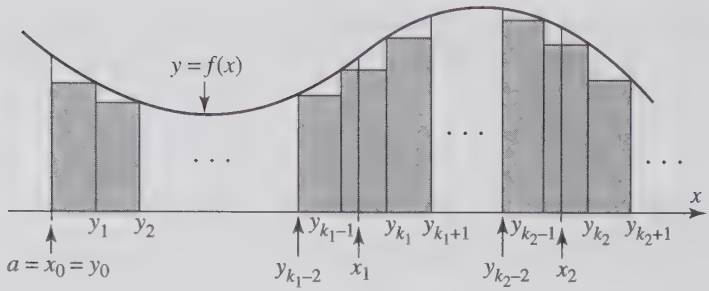


Figure 7.6

Putting (7) and (8) together, we have

$$m_1\triangle_1 + m_2\triangle_2 \le (\overline{m}_1 + \overline{m}_2 + \cdots + \overline{m}_{k_2})\,\overline{\triangle} + (2+3)B\overline{\triangle}.$$

Continuing in this way, we see that

$$\begin{aligned} m_1\triangle_1 + m_2\triangle_2 + \cdots + m_n\triangle_n &\le (\overline{m}_1 + \cdots + \overline{m}_{k_n})\,\overline{\triangle} + (2+3+\cdots+(n+1))B\overline{\triangle} \\ &= (\overline{m}_1 + \overline{m}_2 + \cdots + \overline{m}_{k_n})\,\overline{\triangle} + \frac{n(n+1)}{2}B\overline{\triangle}. \end{aligned}$$

So,
$$\underline{S}(f,\mathcal{P}) < \underline{S}(f,\mathcal{Q}_m) + \frac{n(n+1)}{2}B\overline{\triangle}.$$

Now, $\dfrac{n(n+1)}{2}B\overline{\triangle} = \dfrac{n(n+1)B(b-a)}{2m}$. But $m > \dfrac{n(n+1)B(b-a)}{2\varepsilon}$, so $\dfrac{n(n+1)B(b-a)}{2m} < \varepsilon$. Thus,

$$\underline{S}(f,\mathcal{P}) < \underline{S}(f,\mathcal{Q}_m) + \varepsilon$$

$$\text{i.e., } \underline{S}(f,\mathcal{Q}_m) > \underline{S}(f,\mathcal{P}) - \varepsilon.$$

Similarly (Exercise 15) we can show that $\overline{S}(f,\mathcal{Q}_m) < \overline{S}(f,\mathcal{P}) + \varepsilon$. ■

Corollary 7.3.12 *If f is defined and bounded on $[a,b]$, where $a<b$, then*

(a) $\underline{\int_a^b} f = \sup\{\underline{S}(f,\mathcal{Q}_m) : \mathcal{Q}_m$ *is a regular partition of* $[a,b]\}$;

(b) $\overline{\int_a^b} f = \inf\{\overline{S}(f,\mathcal{Q}_m) : \mathcal{Q}_m$ *is a regular partition of* $[a,b]\}$.

Proof. (a) By definition, $\underline{\int_a^b} f = \sup\{\underline{S}(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\}$.

Let $\varepsilon > 0$. By the ε-criterion for supremum (Theorem 1.6.6) $\exists$ partition $\mathcal{P}$ of $[a,b]$ such that

$$\underline{S}(f,\mathcal{P}) > \underline{\int_a^b} f - \frac{\varepsilon}{2}.$$

By Theorem 7.3.11, $\exists$ regular partition $\mathcal{Q}_m$ of $[a,b]$ such that

$$\begin{aligned}\underline{S}(f,\mathcal{Q}_m) &> \underline{S}(f,\mathcal{P}) - \frac{\varepsilon}{2}\\ &> \underline{\int_a^b} f - \varepsilon.\end{aligned}$$

Therefore, $\sup\{\underline{S}(f,\mathcal{Q}_m) : \mathcal{Q}_m \text{ is a regular partition of } [a,b]\} \geq \underline{\int_a^b} f$. (9)

On the other hand,

$$\begin{aligned}&\sup\{\underline{S}(f,\mathcal{Q}_m) : \mathcal{Q}_m \text{ is a regular partition of } [a,b]\}\\ &\qquad \leq \sup\{\underline{S}(f,\mathcal{P}) : \mathcal{P} \text{ is a partition of } [a,b]\} = \underline{\int_a^b} f. \end{aligned} \tag{10}$$

Putting (9) and (10) together, we have the desired result.

(b) Exercise 16. ■

With the help of these results, we can prove modified forms of Riemann's condition in terms of regular partitions. In what follows, the notation $\mathcal{Q}_n$ will represent the regular n-partition of $[a,b]$.

Theorem 7.3.13 (***Regular Partition Riemann's Criterion for Integrability***) *A bounded function $f:[a,b] \to \mathbb{R}$ is integrable on $[a,b]$ if and only if*

$$\boxed{\forall \varepsilon > 0,\ \exists n \in \mathbb{N} \ni \overline{S}(f,\mathcal{Q}_n) - \underline{S}(f,\mathcal{Q}_n) < \varepsilon.}$$

Proof. Exercise 17. ■

Theorem 7.3.14 (***Regular Partition Equivalent Form of Riemann's Criterion***) *A bounded function $f:[a,b] \to \mathbb{R}$ is integrable on $[a,b]$ $\Leftrightarrow$ there is one and only one number I such that $\forall$ regular partitions $\mathcal{Q}$ of $[a,b]$, $\underline{S}(f,\mathcal{Q}) \leq I \leq \overline{S}(f,\mathcal{Q})$. $\left(\text{In this case, } I = \int_a^b f.\right)$*

Proof. Exercise 18. ■

Similarly, we can restate the Riemann/Darboux criterion for integrability in terms of regular partitions. When we do so it becomes clear that, when expressed in terms of regular partitions, the attention is focused on the *sequences* $\{\underline{S}(f,\mathcal{Q}_n)\}$ and $\{\overline{S}(f,\mathcal{Q}_n)\}$.

Theorem 7.3.15 (*Regular Partition Riemann/Darboux Criterion for Integrability*) *A bounded function $f\colon[a,b] \to \mathbb{R}$ is integrable over $[a,b] \iff$*

$$\boxed{\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \overline{S}(f, \mathcal{Q}_n) - \underline{S}(f, \mathcal{Q}_n) < \varepsilon.}$$

Proof. Exercise 19. ∎

Continuing along this line, the limit criterion for integrability comes out in the following form using regular partitions.

Theorem 7.3.16 (*Regular Partition Limit Criterion for Integrability*) *A bounded $f\colon[a,b] \to \mathbb{R}$ is integrable over $[a,b]$ and $\int_a^b f = I \iff$*

$$\boxed{\begin{array}{c}\forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |R(f, \mathcal{Q}_n^*) - I| < \varepsilon. \\ (\textit{regardless of the choice of the tags } x_i^*)\end{array}}$$

Proof. Exercise 20. ∎

Finally, because of Theorem 7.3.16, it makes sense to write

$$\boxed{\begin{array}{c}\int_a^b f = \lim\limits_{n\to\infty} R(f, \mathcal{Q}_n^*) = \lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{i=1}^{n} f(x_i^*) \\ \text{if this limit exists and is independent of the tags } x_i^*.\end{array}}$$

It must be emphasized that this "limit" is not the ordinary limit of a sequence, as defined in Chapter 2. It is a special kind of limit, whose definition is given in the statement boxed in Theorem 7.3.16. Instead of a statement about *one* sequence, it is a statement about all possible sequences $\{R(f, \mathcal{Q}_n^*)\}$ generated by different selections of the tags x_i^*. There are infinitely many such sequences.

Encouraged by the news that regular partitions are sufficient, the reader might also naively hope that in the Riemann sum approach, the common practice of using left endpoints, right endpoints, or midpoints as the tags x_i^* is sufficient. The following example dispels that hope.

Example 7.3.17 Consider the Dirichlet function,

$$f(x) = \left\{ \begin{array}{l} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{array} \right\}$$

on the interval $[0,1]$. If $\mathcal{Q}$ is any **regular** partition of $[0,1]$, then all Riemann sums using left endpoints, right endpoints, or midpoints of the subintervals

$[x_{i-1}, x_i]$ created by $\mathcal{Q}$ are equal (to 1). Further, all trapezoidal[6] approximations and Simpson's rule[7] approximations (if n is even) using the partition $\mathcal{Q}$ are also equal (to 1). But f is **not** integrable over $[0,1]$, as was shown in Example 7.2.10. [See Exercise 21.]

EXERCISE SET 7.3

1. Prove Inequality (2) in the proof of Theorem 7.3.2. [Use Exercise 7.2.21.]

2. Prove Lemma 7.3.4.

3. Complete the proof of Part 2 of Theorem 7.3.5 by showing that $\overline{\int_a^b} f \leq I$ by the methods used there to prove that $\underline{\int_a^b} f \geq I$.

4. Use the methods of Example 7.3.7 to evaluate each of the following:

   (a) $\int_0^3 (2x+7)dx$ (b) $\int_0^4 (4-5x)dx$

   (c) $\int_2^5 (2x^2 - x)dx$ (d) $\int_{-1}^3 (x^3 + 2x)dx$

5. Use the methods of Example 7.3.7 to prove that when $0 < a < b$,

   (a) $\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$ (b) $\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3)$

6. Prove Theorem 7.3.6 (b).

7. Apply the technique of Theorem 7.3.6 to the function $f(x) = \sqrt{x}$ to find $\int_0^1 f$ using the partitions $\mathcal{P}_n = \left\{0, \frac{1}{n^2}, \frac{2^2}{n^2}, \frac{3^2}{n^2}, \cdots, 1\right\}$.

8. Let $f(x) = \sqrt[3]{x}$. Use a procedure similar to that used in Exercise 7 to find $\int_0^1 f$.

9. Prove that if f is integrable on $[0,1]$, then $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f$.

10. Use the result of Exercise 9, and the integral formulas you learned in elementary calculus, to evaluate each of the following limits:

    (a) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k}{n^2}$ (b) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^2}{n^3}$

    (c) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k^5}{n^6}$ (d) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{n+k}$

    (e) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k}{n^2+k^2}$ (f) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{n}{n^2+k^2}$

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6. See Exercise 13.

7. See Exercise 14.

11. Modify the techniques used in Exercise 10, as needed, to evaluate each of the following:

(a) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \cos\left(\frac{k\pi}{2n}\right)$ (b) $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \sin\left(\frac{k\pi}{3n}\right)$

(c) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{2k+3n}{n^2}$ (d) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \sqrt{\frac{18k}{n^3}}$

12. Consider the function $f(x) = \left\{ \begin{array}{l} 1 \text{ if } 1 \le x < 3 \\ 0 \text{ otherwise} \end{array} \right\}$.

A slight modification of the argument given in Example 7.2.11 will prove that f is integrable over $[0, 4]$. Let $\{\mathcal{Q}_n\}$ denote the sequence of regular n-partitions of $[0, 4]$. Show that the sequence of lower sums $\{\underline{S}(f, \mathcal{Q}_n)\}$ is *not* monotone increasing, and that the sequence of upper sums $\{\overline{S}(f, \mathcal{Q}_n)\}$ is *not* monotone decreasing. (See Remarks 7.3.10.)

13. Prove the **trapezoidal rule**: If f is integrable over $[a, b]$, and $\mathcal{Q}_n = \{x_0, x_1, \cdots, x_n\}$ is the regular n-partition of $[a, b]$, then

$\int_a^b f = \lim\limits_{n\to\infty} \frac{b-a}{2n} \left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$

Hint: The expression in brackets is $2\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + f(x_n)\right] - [f(b) + f(a)]$.

14. Prove **Simpson's rule**: If f is integrable over $[a, b]$, and $\mathcal{Q}_n = \{x_0, x_1, \cdots, x_n\}$ is the regular n-partition of $[a, b]$ into an *even* number of subintervals, then

$\int_a^b f = \lim\limits_{n\to\infty} \frac{b-a}{3n} \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) \cdots + 4f(x_{n-1}) + f(x_n)\right].$

Hint: Try something like the hint given in Exercise 13.

15. Finish the proof of Theorem 7.3.11 by showing that $\overline{S}(f, \mathcal{Q}) < \overline{S}(f, \mathcal{P}) + \varepsilon$.

16. Prove Corollary 7.3.12 (b).

17. Prove Theorem 7.3.13.

18. Prove Theorem 7.3.14.

19. Prove Theorem 7.3.15.

20. Prove Theorem 7.3.16.

21. Prove the claims made in Example 7.3.17.

7.4 Basic Existence and Additivity Theorems

> Learning the **results** of this section may be more important than learning their proofs. Let your instructor detemine which proofs to study.

The first result of this section is a technical lemma that paves the way for an important property of the integral known as the "additive property."

Lemma 7.4.1 (*Additivity of Upper and Lower Integrals*) *Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and $a < c < b$. Then*

$$\text{(a)}\ \overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f \qquad \text{and} \qquad \text{(b)}\ \underline{\int_a^b} f = \underline{\int_a^c} f + \underline{\int_c^b} f.$$

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and $a < c < b$. Suppose $\mathcal{P}$ is a partition of $[a, b]$. Let $\mathcal{P}' = \mathcal{P} \cup \{c\}$. Then $\mathcal{P}'$ is a partition of $[a, b]$ that refines $\mathcal{P}$; moreover, $\mathcal{P}' = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ is a partition of $[a, c]$ and $\mathcal{P}_2$ is a partition of $[c, d]$. Then

$$\begin{aligned} \overline{S}(f, \mathcal{P}) &\geq \overline{S}(f, \mathcal{P}') = \overline{S}(f, \mathcal{P}_1) + \overline{S}(f, \mathcal{P}_2) \\ &\geq \overline{\int_a^c} f + \overline{\int_c^b} f. \end{aligned}$$

Hence, $\inf\{\overline{S}(f, \mathcal{P}) : \mathcal{P} \text{ is a partition of } [a, b]\} \geq \overline{\int_a^c} f + \overline{\int_c^b} f.$

$$\text{i.e., } \overline{\int_a^b} f \geq \overline{\int_a^c} f + \overline{\int_c^b} f. \tag{11}$$

To prove the reverse inequality, let $\varepsilon > 0$. By the ε-criterion for infimum (Theorem 1.6.7) $\exists$ partitions $\mathcal{P}_3$ of $[a, c]$, and $\mathcal{P}_4$ of $[c, b]$, such that

$$\overline{S}(f, \mathcal{P}_3) < \overline{\int_a^c} f + \frac{\varepsilon}{2} \quad \text{and} \quad \overline{S}(f, \mathcal{P}_4) < \overline{\int_c^b} f + \frac{\varepsilon}{2}.$$

Then $\mathcal{P}_3 \cup \mathcal{P}_4$ is a partition of $[a, b]$, and

$$\overline{S}(f, \mathcal{P}_3 \cup \mathcal{P}_4) = \overline{S}(f, \mathcal{P}_3) + \overline{S}(f, \mathcal{P}_4) < \overline{\int_a^c} f + \overline{\int_c^b} f + \varepsilon.$$

But $\overline{\int_a^b} f \leq \overline{S}(f, \mathcal{P}_3 \cup \mathcal{P}_4)$. Therefore, $\overline{\int_a^b} f \leq \overline{\int_a^c} f + \overline{\int_c^b} f + \varepsilon.$

Since this happens $\forall \varepsilon > 0$, we conclude by the forcing principle that

$$\overline{\int_a^b} f \leq \overline{\int_a^c} f + \overline{\int_c^b} f. \tag{12}$$

By (11) and (12) together, we conclude that

$$\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f.$$

(b) Exercise 1. ■

Theorem 7.4.2 (***Additivity of the Integral, I***) *If f is integrable on $[a,b]$ then $\forall c \in (a,b)$, f is integrable on $[a,c]$ and $[c,b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. Suppose f is integrable on $[a,b]$ and $c \in (a,b)$. Since f is integrable on $[a,b]$, $\underline{\int_a^b} f = \overline{\int_a^b} f$. By Lemma 7.4.1, this implies

$$\underline{\int_a^c} f + \underline{\int_c^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f. \text{ Thus,}$$

$$\underline{\int_a^c} f - \overline{\int_a^c} f = \overline{\int_c^b} f - \underline{\int_c^b} f. \tag{13}$$

By Theorem 7.2.7, the left side of Equation (13) is ≤ 0, and the right side is ≥ 0. Thus, both sides must be 0, from which we conclude that

$$\underline{\int_a^c} f = \overline{\int_a^c} f \qquad \text{and} \qquad \overline{\int_c^b} f = \underline{\int_c^b} f.$$

Therefore, f is integrable on $[a,c]$ and $[c,b]$. Finally, using Lemma 7.4.1 and the definition of integrability, we have

$$\int_a^b f = \overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_c^b} f = \int_a^c f + \int_c^b f. \qquad \blacksquare$$

Corollary 7.4.3 *If f is integrable on $[a,b]$, and $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is any partition of $[a,b]$, then f is integrable on $[x_0,x_1]$, $[x_1,x_2]$, $\cdots$, and $[x_{n-1},x_n]$, and*

$$\int_a^b f = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f.$$

Proof. Exercise 2. $\blacksquare$

Corollary 7.4.4 *If f is integrable on $[a,b]$, then f is integrable on any closed subinterval $[c,d] \subseteq [a,b]$, where $c < d$.*

Proof. Exercise 3. $\blacksquare$

The next theorem looks like one we have already seen (Theorem 7.4.2), but is more like a converse of it. You may have to look twice to see the difference.

Theorem 7.4.5 (***Additivity of the Integral, II***) *If f is integrable on $[a,c]$ and on $[c,b]$, where $a < c < b$, then f is integrable on $[a,b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. Suppose f is integrable on $[a,c]$ and on $[c,b]$, where $a < c < b$. Then f is bounded on $[a,b]$ and

$$\begin{aligned} \underline{\int_a^b} f &= \underline{\int_a^c} f + \underline{\int_c^b} f \text{ by Lemma 7.4.1} \\ &= \overline{\int_a^c} f + \overline{\int_c^b} f \text{ since } f \text{ is integrable on } [a,c] \text{ and } [c,b] \\ &= \overline{\int_a^b} f. \qquad \text{by Lemma 7.4.1} \end{aligned}$$

Therefore, f is integrable on $[a,b]$. The desired equation follows immediately from Theorem 7.4.2. ■

Corollary 7.4.6 *If $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is any partition of $[a,b]$, and f is integrable on each subinterval $[x_{i-1}, x_i]$ created by this partition, then f is integrable on $[a,b]$ and*

$$\int_a^b f = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f.$$

Proof. Exercise 5. ■

The following theorem may seem rather innocuous, but it has amazing consequences, as we shall see.

Theorem 7.4.7 *Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and is integrable on every proper*[8] *closed subinterval of the open interval (a,b). Then*

(a) *f is integrable on $[a,b]$, and*

(b) $\int_a^b f = \lim_{h\to 0+} \int_{a+h}^b f = \lim_{h\to 0+} \int_a^{b-h} f = \lim_{h\to 0+} \int_{a+h}^{b-h} f.$

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and is integrable on every closed subinterval of (a,b).

(a) Let $0 < \varepsilon < (b-a)/2$. Then $0 < 2\varepsilon < b-a$, so $a+\varepsilon < b-\varepsilon$ and

$$\overline{\int_a^b} f = \overline{\int_a^{a+\varepsilon}} f + \overline{\int_{a+\varepsilon}^{b-\varepsilon}} f + \overline{\int_{b-\varepsilon}^{b}} f \quad \text{and} \quad \underline{\int_a^b} f = \underline{\int_a^{a+\varepsilon}} f + \underline{\int_{a+\varepsilon}^{b-\varepsilon}} f + \underline{\int_{b-\varepsilon}^{b}} f.$$

8. A proper interval is bounded and contains more than one point. See Definition 5.7.13.

Since f is integrable on $[a+\varepsilon, b-\varepsilon]$, the middle terms of the right-hand sides of these two equations are equal. Thus, when we subtract these two equations we get

$$\begin{aligned}\overline{\int_a^b} f - \underline{\int_a^b} f &= \left(\overline{\int_a^{a+\varepsilon}} f - \underline{\int_a^{a+\varepsilon}} f\right) + \left(\overline{\int_{b-\varepsilon}^b} f - \underline{\int_{b-\varepsilon}^b} f\right) \\ &\le (M-m)\varepsilon + (M-m)\varepsilon = 2(M-m)\varepsilon,\end{aligned}$$

where $M = \sup f[a,b]$ and $m = \inf f[a,b]$ (see Exercise 7.2.3).

Thus, by the generalized forcing principle (7.1.6), $\overline{\int_a^b} f - \underline{\int_a^b} f \le 0$. Therefore $\overline{\int_a^b} f \le \underline{\int_a^b} f$, from which we conclude that f is integrable on $[a,b]$.

(b) Since f is bounded on $[a,b]$, $\exists M > 0 \ni \forall x \in [a,b]$, $|f(x)| \le M$. Let $0 < h < b-a$. By Theorem 7.4.2, f is integrable on $[a, a+h]$ and $[a+h, b]$, and

$$\int_a^b f = \int_a^{a+h} f + \int_{a+h}^b f.$$

Then,

$$\left|\int_a^b - \int_{a+h}^b f\right| = \left|\int_a^{a+h} f\right| \le Mh. \quad \text{(See Exercise 6.)}$$

Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{M}$. Then

$$\begin{aligned}0 < h < \delta &\Rightarrow \left|\int_{a+h}^b f - \int_a^b f\right| \le Mh < M\delta = M\frac{\varepsilon}{M} \\ &\Rightarrow \left|\int_{a+h}^b f - \int_a^b f\right| < \varepsilon.\end{aligned}$$

Therefore, $\lim\limits_{h\to 0^+} \int_{a+h}^b f = \int_a^b f$. This proves the first equality in (b).

The remaining equalities in (b) are proved similarly. (Exercise 7) ∎

Corollary 7.4.8 (***Irrelevance of Endpoint Values***) *Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded on $[a,b]$ and $\forall x \in (a,b)$, $f(x) = g(x)$. Then if f is integrable on $[a,b]$ so is g, and $\int_a^b f = \int_a^b g$.*

Proof. Suppose $f, g : [a,b] \to \mathbb{R}$ are bounded on $[a,b]$, f is integrable on $[a,b]$, and $\forall x \in (a,b)$, $f(x) = g(x)$. Then g is integrable on every closed subinterval of (a,b), so by Theorem 7.4.7, $\int_a^b g$ exists. Further,

$$\begin{aligned}\int_a^b g &= \lim_{h\to 0^+} \int_{a+h}^{b-h} g \quad \text{by 7.4.7 (b)} \\ &= \lim_{h\to 0^+} \int_{a+h}^{b-h} f \quad \text{since } f(x) = g(x) \text{ on } [a+h, b-h] \\ &= \int_a^b f \qquad \text{by 7.4.7 (b).}\end{aligned}$$ ∎

The next theorem is really rather remarkable, and perhaps unexpected. The result shows the power of the way of thinking we have developed in this course.

Theorem 7.4.9 *Changing the values of a function at finitely many points of $[a, b]$ affects neither its integrability nor the value of its integral there. More precisely, suppose $f, g : [a, b] \to \mathbb{R}$ are bounded on $[a, b]$ and $f(x) = g(x)$ for all but finitely many points of $[a, b]$. Then f is integrable on $[a, b]$ if and only if g is integrable on $[a, b]$.*[9] *Moreover, in case of integrability, $\int_a^b f = \int_a^b g$.*

Proof. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded on $[a, b]$ and $f(x) = g(x)$ for all $x \in [a, b]$ except at the points $\{x_1, x_2, \cdots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$. Let $x_0 = a$ and $x_{n+1} = b$. Then $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n, x_{n+1}\}$ is a partition of $[a, b]$, and on each subinterval $[x_{i-1}, x_i]$ created by this partition, f and g agree except on the endpoints. Apply Corollary 7.4.8 to each $[x_{i-1}, x_i]$. Corollaries 7.4.3 and 7.4.6 yield the desired conclusion. ■

Thus, a function can have discontinuities at, for example, a million points of $[a, b]$ and still be integrable there. It is natural to ask whether it is possible for a function to have discontinuities at an *infinite* number of points of $[a, b]$ and still be integrable there. Remembering Dirichlet's function, Example 7.2.10, one might expect the answer to be no. However, the following remarkable example shows that the answer is yes.

Example 7.4.10 (A function with infinitely many points of discontinuity in an interval on which it is integrable.) Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \left\{ \begin{array}{l} 1 \text{ if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 \text{ otherwise} \end{array} \right\}.$$

Then, for each $0 < c < d < 1$, f equals the constant function $g(x) = 0$, except for at most finitely many points. So, by Theorem 7.4.9, f is integrable on $[c, d]$ and $\int_c^d f = \int_c^d g = 0$. Therefore, by Theorem 7.4.7, f is integrable on $[a, b]$ and $\int_0^1 f = \lim_{h \to 0^+} \int_h^{1-h} f = 0$. However, f is discontinuous on the *infinite* set $\left\{ \dfrac{1}{n} : n \in \mathbb{N} \right\}$. □

For examples of functions that are integrable on $[a, b]$ yet discontinuous on a *dense* subset of $[a, b]$, see Exercises 18 and 19.

9. In fact, we shall say that f is integrable on $[a, b]$ even if it is not defined at some points where it differs from an integrable function g. For example, we shall say that $|x|/x$ is integrable on $[-1, 1]$ even though it is not defined at 0.

Definition 7.4.11 A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **piecewise continuous** on $[a, b]$ if there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b]$ such that $\forall\ 1 \leq i \leq n$, f is continuous on (x_{i-1}, x_i). Notice that one-sided continuity of f at the partition points x_i is not required by this definition.

Similarly, a bounded function $f : [a, b] \to \mathbb{R}$ is said to be **piecewise monotone** on $[a, b]$ if there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b]$ such that $\forall\, 1 \leq i \leq n$, f is monotone on (x_{i-1}, x_i).

Definition 7.4.12 A function $\tau : [a, b] \to \mathbb{R}$ is said to be a **step function** if there is a partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b]$ and $\exists$ real numbers $c_1, c_2, \cdots, c_n$ such that $\forall\ 1 \leq i \leq n$,

$$\tau(x) = c_i \text{ if } \ x_{i-1} < x < x_i.$$

That is, a step function is constant on the interior of each subinterval created by consecutive points of the partition $\mathcal{P}$. We could have called a step function a "piecewise constant" function. Notice that the values of $\tau(x_i)$ for $x_i \in \mathcal{P}$ are completely unconstrained by this definition.

Theorem 7.4.13 *(Bounded) piecewise continuous functions, piecewise monotone functions, and step functions relative to a partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b]$ are all integrable on $[a, b]$. Their integrals obey the formula*

$$\int_a^b f = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f.$$

Proof. Exercise 8. ■

Using the concept of step functions we can gain greater geometric insight into the nature of Riemann integrability of functions. Step functions allow us to formulate a geometrically appealing condition equivalent to integrability.

Theorem 7.4.14 *(Step Function Squeeze Criterion for Integrability) A bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if $\forall \varepsilon > 0$, $\exists$ step functions σ, τ relative to some partition $\mathcal{P}$ of $[a, b]$ such that*

(a) $\forall x \in [a, b], \ \ \sigma(x) \leq f(x) \leq \tau(x)$;

(b) $\int_a^b (\tau - \sigma) < \varepsilon$. *(See Figure 7.7.)*

[In words, a bounded function is integrable on $[a, b]$ if and only if it can be squeezed between two step functions that enclose an arbitrarily small area.]

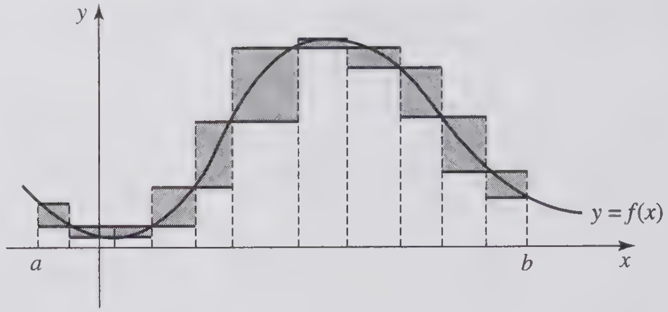


Figure 7.7

Proof. Exercise 14. ■

*REGULATED FUNCTIONS

While each of the conditions given in (7.4.13) and (7.4.14) is sufficient to guarantee that a function $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$, it would be helpful to have one straightforward condition for integrability that includes all of these as special cases. We shall see that the following condition is sufficient.

Definition 7.4.15 A function $f : [a,b] \to \mathbb{R}$ is said to be **regulated** on $[a,b]$ if $\forall x_0 \in [a,b)$, $\lim\limits_{x \to x_0^+} f(x)$ exists, and $\forall x_0 \in (a,b]$, $\lim\limits_{x \to x_0^-} f(x)$ exists.

Given a regulated function $f : [a,b] \to \mathbb{R}$, and a point $x_0 \in [a,b]$, we use the notation

$$f(x_0-) = \lim_{x \to x_0^-} f(x) \quad \text{and} \quad f(x_0+) = \lim_{x \to x_0^+} f(x),$$

and $\forall x_0 \in (a,b)$ we define the **jump** of f at x_0 to be

$$j(f, x_0) = \max\{|f(x_0+) - f(x_0)|, |f(x_0-) - f(x_0)|, |f(x_0+) - f(x_0-)|\}.$$

We define the **jump** of f at a and b to be

$$j(f,a) = |f(a+) - f(a)| \text{ and } j(f,b) = |f(b-) - f(b)|.$$

Remark 7.4.16 Given any regulated $f : [a,b] \to \mathbb{R}$ and $x_0 \in [a,b]$, $j(f, x_0) \geq 0$. Moreover, f is continuous at $x_0 \Leftrightarrow j(f, x_0) = 0$.

Examples 7.4.17

(a) The function $f(x) = \left\{\begin{array}{cl} \dfrac{|x-1|}{x-1} & \text{if } x \neq 1; \\ 0 & \text{if } x = 1 \end{array}\right\}$ is regulated on $[0,5]$, and $\forall x_0 \in [0,5],\ j(f,x_0) = \left\{\begin{array}{l} 0 \text{ if } x_0 \neq 1; \\ 2 \text{ if } x_0 = 1 \end{array}\right\}$.

(b) The Dirichlet function $f(x) = \left\{\begin{array}{l} 1 \text{ if } x \text{ is rational;} \\ 0 \text{ if } x \text{ is irrational} \end{array}\right\}$ is not regulated on $[a,b]$ if $a < b$. (See Exercise 4.1.4.)

(c) Every monotone function $f : [a,b] \to \mathbb{R}$ is regulated on $[a,b]$. (See Theorems 5.2.17 and 5.2.18.)

(d) Thomae's function T defined in Example 5.1.12 is regulated on any $[a,b]$. (See Exercise 5.1.30.) Note that $\forall x \in \mathbb{R},\ j(T,x) = T(x)$. $\square$

We shall find the following theorem useful in showing that regulated functions are integrable.

Theorem 7.4.18 *A function $f : [a,b] \to \mathbb{R}$ is regulated $\Leftrightarrow \forall \varepsilon > 0,\ \exists$ step function $\sigma : [a,b] \to \mathbb{R} \ni \forall x \in [a,b],\ |f(x) - \sigma(x)| < \varepsilon$.*

***Proof.** Part 1 ($\Rightarrow$): Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is regulated. Let $\varepsilon > 0$. Define

$$A = \{x \in [a,b] : \exists \text{ step function } \sigma \text{ on } [a,b] \ni \forall t \in [a,x],\ |f(t) - \sigma(t)| < \varepsilon\}.$$

The (constant) step function $\sigma(x) = f(a)$ shows that $a \in A$, so A is a nonempty bounded set. Hence, $\exists\, c = \sup A$. First, we show that $c > a$. Since f is regulated on $[a,b]$, $\lim_{x \to a^+} f(x) = L$ exists, so $\exists\, \delta > 0 \ni a + \delta < b$, and $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$. Then the step function $\sigma(a) = f(a)$, $\sigma(x) = L$ when $x > a$, satisfies $|f(x) - \sigma(x)| < \varepsilon$ on $\left[a, a + \frac{\delta}{2}\right]$, so $a + \frac{\delta}{2} \in A$. Thus, $c > a$.

Note that A is an interval since $\forall x \in A$, if $a \leq x' < x$, then any step function satisfying the given ε-condition on $[a,x]$ must satisfy that condition on $[a,x']$, so $x' \in A$. Thus,

$$A = [a,c) \text{ or } A = [a,c].$$

Since f is regulated on $[a,b]$, f has a limit from the left at c, say $L' = \lim_{x \to c^-} f(x)$. Then $\exists\, \delta > 0 \ni a + \delta < c$ and $c - \delta < x < c \Rightarrow |f(x) - L'| < \varepsilon$. Now $c - \delta \in A$ since $a < c - \delta < c$. So $\exists$ step function σ on $[a,b]$ such that

$$\forall t \in [a, c-\delta],\ |f(t) - \sigma(t)| < \varepsilon.$$

If we define the step function σ' on $[a,b]$ by

$$\sigma'(x) = \left\{ \begin{array}{ll} \sigma(x) & \text{on } [a, c-\delta], \\ L' & \text{on } (c-\delta, c), \\ f(c) & \text{on } [c, b], \end{array} \right\}$$

then σ' satisfies the required ε-condition on $[a,c]$. Thus, $c \in A$. Therefore, $A = [a,c]$.

We shall conclude the proof of Part 1 by showing that $c = b$. Suppose $c \neq b$. Then $a < c < b$ so f has a limit from the right at c. Let $L'' = \lim\limits_{x \to c^+} f(x)$. Then $\exists\, \delta > 0 \ni c + \delta < b$ and $c < x < c + \delta \Rightarrow |f(x) - L''| < \varepsilon$. If we define

$$\tau(x) = \left\{ \begin{array}{ll} \sigma'(x) & \text{on } [a, c], \\ f(c) & \text{at } x = c, \\ L'' & \text{on } (c, b], \end{array} \right\}$$

then τ is a step function satisfying the required ε-condition on $[a, c+\delta]$. So $c + \delta \in A$. But $c + \delta > c = \sup A$. Contradiction. Therefore, $c = b$.

Part 2 ($\Leftarrow$): Suppose $a < b$ and $f : [a,b] \to \mathbb{R}$ is such that $\forall \varepsilon > 0$, $\exists$ step function $\sigma : [a,b] \to \mathbb{R} \ni \forall x \in [a,b]$, $|f(x) - \sigma(x)| < \varepsilon$. We shall prove that f is regulated.

First, consider any $x_0 \in [a,b)$. We shall prove that $\lim\limits_{x \to x_0^+} f(x)$ exists. Let $\varepsilon > 0$. By hypothesis, $\exists$ step function $\sigma_\varepsilon : [a,b] \to \mathbb{R} \ni$

$$\forall x \in [a,b], \ |f(x) - \sigma_\varepsilon(x)| < \varepsilon/2.$$

Since σ_ε is a step function, $\exists\, \delta > 0 \ni \sigma_\varepsilon$ is constant on $(x_0, x_0 + \delta)$. Thus, $x_0 < x, y < x_0 + \delta \Rightarrow \sigma_\varepsilon(x) = \sigma_\varepsilon(y)$

$$\begin{aligned} \Rightarrow |f(x) - f(y)| &\le |f(x) - \sigma_\varepsilon(x)| + |\sigma_\varepsilon(x) - \sigma_\varepsilon(y)| + |\sigma_\varepsilon(y) - f(y)| \\ &< \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus, by the Cauchy criterion for one-sided limits[10] at x_0, $\lim\limits_{x \to x_0^+} f(x)$ exists.

Next, consider any $x_0 \in (a,b]$. An argument similar to the one just given will prove that $\lim\limits_{x \to x_0^-} f(x)$ exists (Exercise 16). Therefore, f is regulated. ■

10. See Exercise 4.2.22.

Corollary 7.4.19 *If f is a regulated function on $[a,b]$, then f is integrable on $[a,b]$.*

Proof. Suppose f is regulated on $[a,b]$ with $a < b$. Let $\varepsilon > 0$. By Theorem 7.4.18, $\exists$ step function $\sigma : [a,b] \to \mathbb{R} \ni \forall x \in [a,b]$,

$$|f(x) - \sigma(x)| < \frac{\varepsilon}{4(b-a)}$$

$$\sigma(x) - \frac{\varepsilon}{4(b-a)} < f(x) < \sigma(x) + \frac{\varepsilon}{4(b-a)}.$$

Define step functions $\tau_1, \tau_2 : [a,b] \to \mathbb{R}$ by

$$\tau_1(x) = \sigma(x) - \frac{\varepsilon}{4(b-a)} \quad \text{and} \quad \tau_2(x) = \sigma(x) + \frac{\varepsilon}{4(b-a)}.$$

Then $\forall x \in [a,b]$, $\tau_1(x) < f(x) < \tau_2(x)$ and

$$\int_a^b \tau_2 - \tau_1 = \int_a^b \frac{\varepsilon}{2(b-a)} = \frac{\varepsilon(b-a)}{2(b-a)} < \varepsilon.$$

Therefore, by Theorem 7.4.14, f is integrable on $[a,b]$. ∎

Corollary 7.4.20 *Thomae's function is integrable on every compact interval.*

Not all integrable functions are regulated. For an example of a function that is integrable on $[a,b]$ but not regulated there, see Exercise 17.

EXERCISE SET 7.4

1. Prove Lemma 7.4.1 (b).

2. Prove Corollary 7.4.3. [Use mathematical induction.]

3. Prove Corollary 7.4.4.

4. How do Theorems 7.4.2 and 7.4.5 differ?

5. Prove Corollary 7.4.6. [Use mathematical induction.]

6. Prove that if f is integrable on $[a,b]$, then $\exists\, M > 0 \ni$ for all subintervals $[c,d] \subseteq [a,b]$, $\left|\int_c^d f\right| \leq M(d-c)$. [See Exercise 7.2.3.]

7. Prove the remaining equalities in Theorem 7.4.7 (b).

8. Prove Theorem 7.4.13.

9. Determine whether each of the following functions is integrable over $[-1,1]$:

   (a) $f(x) = \left\{ \begin{array}{ll} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ (b) $g(x) = \left\{ \begin{array}{ll} \frac{1}{x}\sin x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$

10. Draw the graph of the function $f(x) = \left\{ \begin{array}{ll} \dfrac{|x^2-1|}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{array} \right\}$ on the interval $[-2, 2]$. Use theorems from this section to prove that f is integrable over $[-2, 2]$, and use your knowledge of integral as "area" to find $\int_{-2}^{2} f$.

11. Let $\lfloor x \rfloor$ = the greatest integer $\leq x$, the so-called "greatest integer function." Use geometry to determine the existence, and the value, of each of the following:

    (a) $\int_0^5 \lfloor x \rfloor \, dx$ (b) $\int_0^5 x - \lfloor x \rfloor \, dx$ (c) $\int_{-2}^{3} x \lfloor x \rfloor \, dx$

12. Suppose f is integrable and nonnegative (or nonpositive) on $[a, b]$. Prove that $\forall [c, d] \subseteq [a, b]$, $\int_c^d f \leq \int_a^b f$ $\left(\text{or } \int_c^d f \geq \int_a^b f \text{ in the nonpositive case}\right)$.

13. Suppose f is bounded on $[a, b]$ and $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ is a partition of $[a, b]$. Let m_i and M_i be as in Definition 7.2.1 and let χ_A denote the characteristic function of a set A. Define $\sigma, \tau : [0, 1] \to \mathbb{R}$ by $\sigma(x) = \sum\limits_{i=1}^{n} m_i \chi_{[x_{i-1}, x_i)}(x) + f(b)\chi_{\{b\}}(x)$ and $\tau(x) = \sum\limits_{i=1}^{n} M_i \chi_{[x_{i-1}, x_i)}(x) + f(b)\chi_{\{b\}}(x)$. Show that σ and τ are step functions and $\forall x \in [a, b]$, $\sigma(x) \leq f(x) \leq \tau(x)$. Also show that $\int_a^b \sigma = \underline{S}(f, \mathcal{P})$ and $\int_a^b \tau = \overline{S}(f, \mathcal{P})$.

14. Prove Theorem 7.4.14. [Hint: Show that this theorem is a rewording of Riemann's condition (7.2.14). Use Exercise 13 as needed.]

15. Find a function f that is integrable on every closed subinterval of $(0, 1)$ but that is not integrable on $[0, 1]$. Does this contradict Theorem 7.4.7?

16. Finish Part 2 of the proof of Theorem 7.4.18.

17. Show that the function $f(x) = \left\{ \begin{array}{ll} \sin \frac{1}{x} & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0 \end{array} \right\}$ is integrable on $[0, 1]$ but is not regulated on $[0, 1]$.

18. Find $\int_0^1 T$ of **Thomae's function**, defined in Example 5.1.12, as follows: $\forall \varepsilon > 0$, define $\sigma_\varepsilon : [0, 1] \to \mathbb{R}$ by $\sigma_\varepsilon(x) = \max\{T(x), \varepsilon\}$. Prove that σ_ε is a step function on $[0, 1]$, and observe that $0 \leq \int_0^1 T \leq \int_0^1 \sigma_\varepsilon \leq \varepsilon$. [See Exercise 7.2.13.]

    Note: this is an example of a function that is integrable on $[0, 1]$ yet discontinuous on a *dense* subset of $[0, 1]$.

19. Use Theorem 5.7.3 to prove that $\forall a < b$, if A is any countable dense subset of $[a, b]$, there is a function that is integrable on $[a, b]$ yet is discontinuous on A.

7.5 Algebraic Properties of the Integral

You may have noticed a pattern in earlier chapters. After introducing each of several big ideas of analysis (limits of sequences, limits of functions, continuous functions, and derivatives) we included results under the heading "algebra of [*big idea*]." This pattern was intentional, to show the similarity and unity of the concepts and techniques. You may have noticed that even the proofs of parallel results in the various sections exhibited some similarity. In the present section we develop similar algebraic results about the Riemann integral.

Theorem 7.5.1 (*Algebra of the Integral, I—Linearity*) *If f and g are integrable over $[a, b]$ and if $c \in \mathbb{R}$, then*

(a) *cf is integrable over $[a, b]$, and $\int_a^b cf = c\int_a^b f$;*

(b) *$f + g$ is integrable over $[a, b]$, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.*

Proof. (a) Suppose f and g are integrable over $[a, b]$ and $c \in \mathbb{R}$. Let $\mathcal{P}^*$ be any tagged partition of $[a, b]$. Then

$$R(cf, \mathcal{P}^*) = cR(f, \mathcal{P}^*).$$

Hence, in the sense of Theorem 7.3.5,

$$\lim_{\|\mathcal{P}\|\to 0} R(cf, \mathcal{P}^*) = \lim_{\|\mathcal{P}\|\to 0} cR(f, \mathcal{P}^*) = c \lim_{\|\mathcal{P}\|\to 0} R(f, \mathcal{P}^*).$$

That is, cf is integrable over $[a, b]$, and $\int_a^b cf = c\int_a^b f$.

(b) Similarly, if $\mathcal{P}^*$ is any tagged partition of $[a, b]$, then

$$R(f + g, \mathcal{P}^*) = R(f, \mathcal{P}^*) + R(g, \mathcal{P}^*).$$

Hence, in the sense of Theorem 7.3.5,

$$\lim_{\|\mathcal{P}\|\to 0} R(f + g, \mathcal{P}^*) = \lim_{\|\mathcal{P}\|\to 0} R(f, \mathcal{P}^*) + \lim_{\|\mathcal{P}\|\to 0} R(g, \mathcal{P}^*).$$

That is, $f + g$ is integrable over $[a, b]$, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$. ∎

Theorem 7.5.2 (*Algebra of the Integral, II—Preserving Inequalities*)

(a) *If f is integrable on $[a, b]$ and $\forall x \in [a, b]$, $f(x) \geq 0$, then $\int_a^b f \geq 0$.*

(b) *If f is integrable on $[a, b]$ and $\forall x \in [a, b]$, $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f \leq M(b - a)$.*

(c) *If f is integrable on $[a,b]$ and $\forall x \in [a,b]$, $|f(x)| \le M$, then* $\left|\int_a^b f\right| \le M(b-a)$.

(d) *If f and g are integrable on $[a,b]$ and $\forall x \in [a,b]$, $f(x) \le g(x)$, then* $\int_a^b f \le \int_a^b g$.

Proof. Exercise 5. ■

THE COMPOSITION THEOREM AND CONSEQUENCES

The composition of two continuous functions is continuous (Theorem 5.1.14) and the composition of two differentiable functions is differentiable (the chain rule). We might expect that the composition of two integrable functions is always integrable. That this is not the case, however, is demonstrated by the following example.

Example 7.5.3 Let T denote Thomae's function, defined in Example 5.1.12, and g denote the characteristic function of the half-open interval $(0,1]$. Then $T, g : [0,1] \to [0,1]$ are both integrable on $[0,1]$, but $g \circ T$ is **not** integrable on $[0,1]$.

Proof. Exercise 7. □

The following theorem shows that if f is integrable and g is continuous, the composite function $g \circ f$ is integrable. Its proof may seem a bit complicated, and you may be tempted to give up on it. But the theorem is quite powerful, yielding many significant results easily, such as the corollaries that follow.

Theorem 7.5.4 (***Algebra of the Integral, III—The Composition Theorem***) *Suppose $f{:}[a,b] \to \mathbb{R}$ is integrable on $[a,b]$. Further, suppose $f[a,b] \subseteq [c,d]$ and $g : [c,d] \to \mathbb{R}$ is continuous. Then the composition $g \circ f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$.*

Proof. Suppose $f{:}[a,b] \to \mathbb{R}$ is integrable on $[a,b]$, $f[a,b] \subseteq [c,d] = J$, and $g : J \to \mathbb{R}$ is continuous on J. Let $\varepsilon > 0$. Since g is continuous on the compact set $J = [c,d]$, it is bounded there, so $\exists\, A < B$ in $\mathbb{R} \ni$

$$\forall t \in J,\ A \le g(t) \le B.$$

Also, g is uniformly continuous on J, by Theorem 5.4.7. Hence, $\exists\, \delta > 0$ such that $\delta < \varepsilon$, and

$$\forall s, t \in J,\ |s - t| < \delta \Rightarrow |g(s) - g(t)| < \varepsilon. \tag{14}$$

By Riemann's condition for integrability (7.2.14) there is a partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ such that

$$\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \delta^2. \tag{15}$$

For each $i = 1, 2, \cdots, n$, we use the notation

$$m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\} \text{ and}$$
$$m_i(g \circ f) = \inf\{(g \circ f)(x) : x_{i-1} \le x \le x_i\}$$

and similarly for $M_i(f)$ and $M_i(g \circ f)$. Divide the set $\mathcal{N} = \{1, 2, \cdots, n\}$ into two subsets:

$$\mathcal{N}_1 = \{i \in \mathcal{N} : M_i(f) - m_i(f) < \delta\} \text{ and } \mathcal{N}_2 = \{i \in \mathcal{N} : M_i(f) - m_i(f) \ge \delta\}.$$

(a) Suppose $i \in \mathcal{N}_1$. Then

$$\begin{aligned} x, y \in [x_{i-1}, x_i] &\Rightarrow |f(x) - f(y)| \le M_i(f) - m_i(f) < \delta \\ &\Rightarrow |(g \circ f)(x) - (g \circ f)(y)| < \varepsilon \text{ by (14).} \end{aligned}$$

Thus, $\sum\limits_{i \in \mathcal{N}_1} (M_i(g \circ f) - m_i(g \circ f)) \triangle_i \le \sum\limits_{i \in \mathcal{N}_1} \varepsilon \triangle_i = \varepsilon \sum\limits_{i \in \mathcal{N}_1} \triangle_i$

$$\le \varepsilon(b - a). \tag{16}$$

(b) Suppose $i \in \mathcal{N}_2$. Then $M_i(g \circ f) - m_i(g \circ f) \le B - A$, so

$$\sum_{i \in \mathcal{N}_2} (M_i(g \circ f) - m_i(g \circ f)) \triangle_i \le (B - A) \sum_{i \in \mathcal{N}_2} \triangle_i. \tag{17}$$

Now $i \in \mathcal{N}_2 \Rightarrow M_i(f) - m_i(f) \ge \delta$, so $\dfrac{M_i(f) - m_i(f)}{\delta} \ge 1$, so

$$\begin{aligned} \sum_{i \in \mathcal{N}_2} \triangle_i \le \sum_{i \in \mathcal{N}_2} \frac{M_i(f) - m_i(f)}{\delta} \triangle_i &= \frac{1}{\delta} \sum_{i \in \mathcal{N}_2} [M_i(f) - m_i(f)] \triangle_i \\ &\le \frac{1}{\delta} \left[\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P})\right] < \frac{1}{\delta} \cdot \delta^2 = \delta \text{ by (15)} \\ &< \varepsilon \text{ by definition of } \delta. \end{aligned} \tag{18}$$

Putting (17) and (18) together, we have

$$\sum_{i \in \mathcal{N}_2} (M_i(g \circ f) - m_i(g \circ f)) \triangle_i < (B - A)\varepsilon. \tag{19}$$

(c) Finally, putting together the results of (16) and (19), we have

$$\sum_{i=1}^{n} (M_i(g \circ f) - m_i(g \circ f)) \,\Delta_i < \varepsilon(b-a) + (B-A)\varepsilon$$

$$= [(b-a) + (B-A)]\varepsilon.$$

Hence, $\overline{S}(g \circ f, \mathcal{P}) - \underline{S}(g \circ f, \mathcal{P}) < [(b-a) + (B-A)]\varepsilon$.

Therefore, by Riemann's criterion for integrability, $g \circ f$ is integrable on $[a, b]$. ■

Corollary 7.5.5 (***Algebra of the Integral, IV—Absolute Value***) *If f is integrable on $[a, b]$, then so is $|f|$. Moreover, $\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a)$, where M is any upper bound for $|f|$ on $[a, b]$.*

Proof. Suppose f is integrable on $[a, b]$. Then it is bounded there, so $\exists M > 0 \ni \forall x \in [a, b]$, $|f(x)| \le M$. Let $g(x) = |x|$. Since g is continuous everywhere, we can apply Theorem 7.5.4 to conclude that $|f|$ is integrable on $[a, b]$. By Theorem 7.5.1, so is $-|f|$, and $\int_a^b -|f| = -\int_a^b |f|$. Now, $\forall x \in [a, b]$,

$$-M \le -|f(x)| \le f(x) \le |f(x)| \le M.$$

Thus, by Theorems 7.5.2 (d) and 7.2.9,

$$-M(b-a) = \int_a^b -M \le -\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \le \int_a^b M = M(b-a)$$

$$\text{i.e., } \left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a). \quad ■$$

Corollary 7.5.6 (***Algebra of the Integral, V—Miscellany***) *If f is integrable on $[a, b]$, then*

(a) *$\forall n \in \mathbb{N}$, f^n is integrable on $[a, b]$.*

(b) *If f is positive and bounded away from 0 on $[a, b]$, then $1/f$ is integrable on $[a, b]$.*

(c) *For $n \in \mathbb{N}$, if $\sqrt[n]{f}$ exists $\forall x \in [a, b]$, then $\sqrt[n]{f}$ is integrable on $[a, b]$.*

(d) $\sin f(x)$, $\cos f(x)$, *and* $e^{f(x)}$ *are all integrable on $[a, b]$.*

(e) *If f is positive and bounded away from 0 on $[a, b]$, then $\ln f$ is integrable on $[a, b]$.*

Proof. Exercise 8. ■

Corollary 7.5.7 (*Algebra of the Integral, VI—Products and Max/Min*) *If f and g are integrable on $[a, b]$, then*

(a) *fg is integrable on $[a, b]$,*

(b) *$\max\{f, g\}$ is integrable on $[a, b]$, and*

(c) *$\min\{f, g\}$ is integrable on $[a, b]$.*

Proof. To prove (a), show that $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ and then apply Theorem 7.5.1 and Corollary 7.5.6 (a). To prove (b) and (c), recall from Exercise 1.2-B.6 that

$$\max\{f, g\} = \frac{f + g + |f - g|}{2} \quad \text{and} \quad \min\{f, g\} = \frac{f + g - |f - g|}{2}$$

and apply Corollaries 7.5.5 and 7.5.6. See Exercise 12. ■

EXERCISE SET 7.5

1. Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded on $[a, b]$.
   (a) Prove that $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g$.
   (b) Find functions f, g for which strict inequality holds in (a).
   (c) State and prove similar results for $\overline{\int_a^b}(f+g)$.
   (d) Use (a) and (c) to give an alternate proof of Theorem 7.5.1 (b).

2. Suppose $f_1, f_2, \cdots, f_n$ are all integrable on $[a, b]$, and $c_1, c_2, \cdots, c_n \in \mathbb{R}$. Prove that $\sum_{i=1}^{n} c_i f_i$ is integrable on $[a, b]$, and $\displaystyle\int_a^b \sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} c_i \int_a^b f_i$. [Use mathematical induction.]

3. A function $f:[-a, a] \to \mathbb{R}$ is an **even function** if $\forall x \in [-a, a]$, $f(-x) = f(x)$, and is an **odd function** if $\forall x \in [-a, a]$, $f(-x) = -f(x)$. Suppose f is integrable on $[-a, a]$. Prove that
   (a) if f is even, then $\int_{-a}^{a} f = 2\int_0^a f$.
   (b) if f is odd, then $\int_{-a}^{a} f = 0$.

4. Find a function $f:[0, 1] \to \mathbb{R}$ such that f^2 is integrable on $[0, 1]$ but f is **not**.

5. Prove Theorem 7.5.2.

6. Suppose f and g are continuous on $[a, b]$ and $\int_a^b f = \int_a^b g$. Prove that $\exists\, c \in [a, b] \ni f(c) = g(c)$. [See Exercise 5.1.26.] Find examples of discontinuous f, g for which this conclusion is not true.

7. Prove the claim made in Example 7.5.3. (See Exercise 7.4.18.)

8. Prove Corollary 7.5.6.

9. Find a function $f{:}[0, 1] \to \mathbb{R}$ such that f is integrable on $[0, 1]$ and $\forall x \in [0, 1]$, $f(x) > 0$, but $1/f$ is not integrable on $[0, 1]$. Does this contradict Corollary 7.5.6 (b)?

10. Prove that if $f{:}[a, b] \to \mathbb{R}$ is continuous and nonzero on $[a, b]$, then $1/f$ is integrable on $[a, b]$.

11. Find functions $f, g : [0, 1] \to \mathbb{R}$ such that f and $f \circ g$ are integrable on $[0, 1]$ but g is **not**.

12. Prove Corollary 7.5.7.

13. **Squeeze Principle:** Suppose $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$, and $\forall x \in [a, b]$, $f(x) \leq h(x) \leq g(x)$. Prove that if $\int_a^b f = \int_a^b g$, then h is integrable on $[a, b]$ and $\int_a^b h = \int_a^b f$.

14. Suppose $f{:}[a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and $k \in \mathbb{R}$. Define g on $[a + k, b + k]$ by $g(x) = f(x - k)$. Prove that g is integrable on $[a + k, b + k]$, and $\int_{a+k}^{b+k} g = \int_a^b f$. [That is, the integral is **translation invariant**.]

7.6 The Fundamental Theorem of Calculus

So far in our development of the integral we have ignored antidifferentiation. That is because the definition of the Riemann integral does not involve the antiderivative in any way. We have not even mentioned the possibility that there may be a connection between the integral and the derivative. But it is now time to show the connection between these two great pillars of the calculus. The fundamental theorem of calculus establishes that, in some sense, differentiation and integration are inverse processes.

The "fundamental theorem of calculus" exists in two forms. The first form is concerned with integrating derivatives. It is quite easily proved, and requires only what we already know from the beginning sections of this chapter and the definition of antiderivative. This form of the fundamental theorem is well known to all students of calculus, since it is the basic tool used to calculate integrals. Indeed, without this theorem, calculating integrals would be about as difficult as calculating derivatives without any derivative formulas.

INTEGRATING DERIVATIVES

Definition 7.6.1 A function F is an **antiderivative** of a function f over a set A if both f and F are defined over A and $\forall x \in A$, $F'(x) = f(x)$.

Theorem 7.6.2 (*Fundamental Theorem of Calculus, First Form*) *Suppose f is integrable over $[a,b]$. If F is any antiderivative of f over (a,b) that is continuous over $[a,b]$, then*

$$\boxed{\int_a^b f = F(b) - F(a).}$$

Proof. Suppose f is integrable over $[a,b]$, and F is an antiderivative of f over (a,b) that is continuous over $[a,b]$. By the definition of antiderivative, F is differentiable over (a,b), and $\forall x \in (a,b)$, $F'(x) = f(x)$.

Let $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ be **any** partition of $[a,b]$. Then the mean value theorem can be applied to F over each subinterval $[x_{i-1}, x_i]$, assuring us that $\exists\ x_i^* \in (x_{i-1}, x_i) \ni F'(x_i^*) = \dfrac{F(x_i) - F(x_{i-1})}{\triangle_i}$. Equivalently, for $i = 1, 2, \cdots n$,

$$F(x_i) - F(x_{i-1}) = f(x_i^*)\triangle_i, \text{ for some } x_i^* \in (x_{i-1}, x_i).$$

$$\begin{aligned}
\text{Thus, } F(b) - F(a) &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \\
&\qquad \text{[All terms cancel out except } F(x_n) \text{ and } -F(x_0).] \\
&= \sum_{i=1}^{n} f(x_i^*)\triangle_i \\
&= R(f, \mathcal{P}^*), \text{ a Riemann sum of } f \text{ over } \mathcal{P}^*.
\end{aligned}$$

By Lemma 7.3.4, all Riemann sums lies between $\underline{S}(f,\mathcal{P})$ and $\overline{S}(f,\mathcal{P})$, so

$$\underline{S}(f,\mathcal{P}) \le F(b) - F(a) \le \overline{S}(f,\mathcal{P}). \tag{20}$$

But $\mathcal{P}$ is an arbitrary partition of $[a,b]$. That means (20) is true for **all** partitions of $[a,b]$. By Theorem 7.2.15, there is only one number that lies between every lower sum and every upper sum of an integrable function over $[a,b]$, and that is $\int_a^b f$. Therefore, $\int_a^b f = F(b) - F(a)$. ■

Remarks 7.6.3 Some caution should be exercised to avoid jumping to conclusions not justified by the first form of the Fundamental Theorem of Calculus. Theorem 7.6.2 does **not** assert that a function integrable over $[a,b]$ has an antiderivative there, nor does it assert that a function with an antiderivative over $[a,b]$ is integrable there. Indeed, there exist functions integrable over $[a,b]$ that do not have antiderivatives there (Exercises 3, 4, and 19) and functions that have antiderivatives over $[a,b]$ but are not integrable there (Exercise 6).

To say this another way, Theorem 7.6.2 says that under certain conditions,

$$\int_a^b f' = f(b) - f(a).$$

The conditions are that the integrand f' be a derivative and this derivative must be integrable. As we have noted above, not every integrable function is a derivative and not every derivative is integrable (Exercises 3, 4, 6, and 19).

DIFFERENTIATING INTEGRALS

The second form of the fundamental theorem is concerned with differentiating integrals rather than integrating derivatives.

Up to now we have confined our attention to the integral of a function over an interval with fixed endpoints. Indeed, our notation $\int_a^b f$ suggests that only the function f and the interval $[a, b]$ are relevant, and that the integral is a *number*, not a function. To see the connection between integration and differentiation expressed by the second form of the fundamental theorem, we must allow one of the endpoints of the interval of integration to be variable. Specifically, we will be looking at the function $F(x) = \int_a^x f$.

So far, when we have written $\int_a^b f$ we have implicitly assumed that $a < b$. We will now allow $a \geq b$, but to do so requires a definition.

Definition 7.6.4 (a) $\forall\ a \in \mathbb{R}$, for any function f defined at a, we define $\int_a^a f = 0$.

(b) If f is integrable over $[a, b]$, we define $\int_b^a f = -\int_a^b f$.

Using these definitions, we are able to generalize Theorems 7.4.2 and 7.4.5, as follows.

Theorem 7.6.5 *If a, b, and c are any real numbers, then $\int_a^b f = \int_a^c f + \int_c^b f$, regardless of the relative positions of a, b, and c, in the sense that if any two of these integrals exist, then the third integral exists and this equation is satisfied.*

Proof. Consider the three numbers a, b, and c.

Case 1: If two or more of the numbers a, b, and c are equal, then $\int_a^b f = \int_a^c f + \int_c^b f$. (Exercise 1)

Case 2: Suppose a, b, and c are all different, and suppose f is integrable on two of the intervals $[a, b]$, $[a, c]$, and $[c, b]$. There are six different possible relative positions (permutations) of a, b, and c. We consider one of them, and leave the other five for Exercise 2. Suppose $c < b < a$, and f is integrable on two of the intervals $[c, a]$, $[c, b]$, and $[b, a]$. By Theorems 7.4.2 and 7.4.5, f is integrable on the third, and

$$\int_c^a f = \int_c^b f + \int_b^a f. \tag{21}$$

But $\int_c^a f = -\int_a^c f$ and $\int_b^a f = -\int_a^b f$. Thus, Equation (21) becomes

$$-\int_a^c f = \int_c^b f - \int_a^b f, \text{ which is equivalent to}$$

$$\int_a^b f = \int_a^c f + \int_c^b f. \qquad \blacksquare$$

Theorem 7.6.6 (***Continuity of the Integral***) *Suppose f is integrable on a compact interval I, and $a \in I$. Then the function $F : I \to \mathbb{R}$ defined by the formula $F(x) = \int_a^x f$ is (uniformly) continuous on I.*

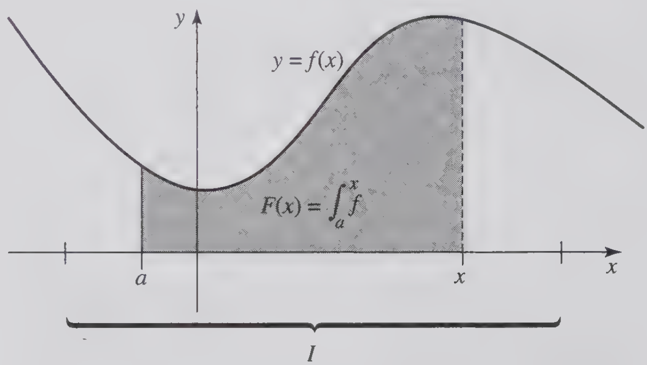


Figure 7.8

Proof. Suppose f is integrable on a compact interval I, and $a \in I$. Then $\forall x \in I$, f is integrable over $[a, x]$ if $a \leq x$ (or $[x, a]$ if $x \leq a$) by Corollary 7.4.4, and so the function $F(x) = \int_a^x f$ is defined $\forall x \in I$. Since f is integrable over I it is bounded there, and so $\exists\ M > 0 \ni \forall t \in I$, $|f(t)| \leq M$.

Then, $\forall x, y \in I$,

$$\begin{aligned}
|F(y) - F(x)| &= \left|\int_a^y f - \int_a^x f\right| \\
&= \left|\int_x^y f\right| && \text{by Theorem 7.6.5} \\
&\leq \begin{cases} \int_x^y |f| \text{ if } x \leq y \\ \int_y^x |f| \text{ if } x > y \end{cases} && \text{by Corollary 7.5.5} \\
&\leq \begin{cases} \int_x^y M \text{ if } x \leq y \\ \int_y^x M \text{ if } x > y \end{cases} && \text{by Corollary 7.5.5} \\
&= M|y - x|. && \text{(integral of a constant)}
\end{aligned}$$

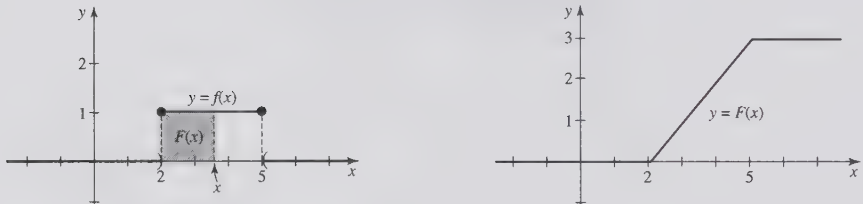
Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{M}$. Then $\delta > 0$, and $\forall x, y \in I$,

$$|y - x| < \delta \Rightarrow |F(y) - F(x)| \leq M|y - x| < M \cdot \delta = \varepsilon.$$

Therefore, F is uniformly continuous on I. ■

Example 7.6.7 Consider the function $f = \chi_{[2,5]}$, the characteristic function of $[2,5]$, defined by $f(x) = \begin{Bmatrix} 0 \text{ if } x < 2 \\ 1 \text{ if } 2 \leq x \leq 5 \\ 0 \text{ if } x > 5 \end{Bmatrix}$. Then

$$F(x) = \int_0^x f = \begin{Bmatrix} 0 & \text{if } x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 5 \\ 3 & \text{if } x > 5 \end{Bmatrix}.$$



(a) $y = f(x)$ (b) $y = F(x)$

Figure 7.9

The purpose of Example 7.6.7 is to illustrate that F is continuous even at points where f is not. Notice that even though f is discontinuous at 2 and 5, F is continuous on the entire interval $(-\infty, +\infty)$. (See Figure 7.9.)

We are now ready for the second form of the Fundamental Theorem of Calculus, which sheds more light on the connection between integration and differentiation. Specifically, it shows that under certain circumstances, these two processes are inverses of each other.

Theorem 7.6.8 (***Fundamental Theorem of Calculus, Second Form***) *Suppose f is integrable on a compact interval I, and $a \in I$. Define the function F on I by the formula $F(x) = \int_a^x f$. (See Figure 7.8.) Then F is* ***differentiable*** *at every point $x_0 \in I^\circ$ at which f is continuous; moreover, at any such x_0, $F'(x_0) = f(x_0)$.*

Proof. Suppose f is integrable on a compact interval I, and $a \in I$. Define the function F on I by the formula $F(x) = \int_a^x f$. Let x_0 be a point of I° at which f is continuous. Then, $\forall x \neq x_0$ in I,

$$\begin{aligned} F(x) - F(x_0) &= \textstyle\int_a^x f - \int_a^{x_0} f \\ &= \textstyle\int_{x_0}^x f \quad \text{by Theorem 7.6.5.} \end{aligned}$$

Hence, $\dfrac{F(x) - F(x_0)}{x - x_0} = \dfrac{1}{x - x_0} \displaystyle\int_{x_0}^x f.$

Moreover,
$$\begin{aligned} f(x_0) &= \frac{1}{x - x_0}(x - x_0) f(x_0) \\ &= \frac{1}{x - x_0} \int_{x_0}^x f(x_0) \text{ (integral of constant).} \end{aligned}$$

Thus,
$$\begin{aligned} \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) &= \frac{1}{x - x_0} \left[\int_{x_0}^x f - \int_{x_0}^x f(x_0) \right] \\ &= \frac{1}{x - x_0} \int_{x_0}^x [f(t) - f(x_0)]\, dt. \end{aligned} \tag{22}$$

Let $\varepsilon > 0$. Since f is continuous at x_0, $\exists\, \delta > 0 \ni$

$$|t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon. \tag{23}$$

Case 1 ($x > x_0$): Then, $|x - x_0| < \delta \Rightarrow x_0 < x < x_0 + \delta$, so

$$\begin{aligned} \left| \int_{x_0}^x [f(t) - f(x_0)]\, dt \right| &\leq \int_{x_0}^x |f(t) - f(x_0)|\, dt \quad \text{by Theorem 7.5.5} \\ &\leq \textstyle\int_{x_0}^x \varepsilon \quad \text{by (23) and Corollary 7.5.2} \\ &= \varepsilon(x - x_0) \quad \text{(integral of a constant).} \end{aligned}$$

Case 2 ($x < x_0$): Then, $|x - x_0| < \delta \Rightarrow x_0 - \delta < x < x_0$, so

$$\left| \int_{x_0}^x [f(t) - f(x_0)]\, dt \right| = \left| \int_x^{x_0} [f(t) - f(x_0)]\, dt \right| \leq \varepsilon(x_0 - x)$$

(following the same reasoning as in Case 1).

In either case, $|x - x_0| < \delta \Rightarrow \left| \int_{x_0}^x [f(t) - f(x_0)]\, dt \right| \leq \varepsilon |x - x_0|.$ (24)

Combining (22) and (24), we have $0 < |x - x_0| < \delta \Rightarrow$

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x [f(t) - f(x_0)]\, dt \right| \\ &\leq \frac{1}{|x - x_0|} \cdot \varepsilon |x - x_0| = \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$. That is, $F'(x_0) = f(x_0)$. ■

Example 7.6.9 Consider the function $f : [0, 10] \to \mathbb{R}$, where $f = \chi_{[2,5]}$ is the function defined in Example 7.6.7, $f(x) = \left\{\begin{matrix} 0 \text{ if } x < 2 \\ 1 \text{ if } 2 \leq x \leq 5 \\ 0 \text{ if } x > 5 \end{matrix}\right\}$. We showed there that $F(x) = \int_0^x f = \left\{\begin{matrix} 0 & \text{if } x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 5 \\ 3 & \text{if } x > 5 \end{matrix}\right\}$.

Note that $F'(x) = \left\{\begin{matrix} 0 \text{ if } x < 2 \\ 1 \text{ if } 2 < x < 5 \\ 0 \text{ if } x > 5 \end{matrix}\right\}$. Thus, F is differentiable everywhere in $[0, 10]$ except at 2 and 5, which are the points of $[0, 10]$ where f is not continuous. And, at every x where f is continuous, $F'(x) = f(x)$. □

Remark 7.6.10 The Fundamental Theorem of Calculus Second Form says that in any interval I on which f is integrable,

$$\boxed{\frac{d}{dx}\left(\int_a^x f\right) = f(x)}$$

at every point x in I where f is *continuous*. In other words, differentiation undoes integration of continuous functions: differentiation is a kind of inverse of integration, for continuous functions.

Remark 7.6.11 Hereafter, we shall use the following abbreviations:

> **FTC-I** will denote the Fundamental Theorem of Calculus, First Form, and
> **FTC-II** will denote the Fundamental Theorem of Calculus, Second Form.

INDEFINITE INTEGRALS, DIFFERENTIALS, AND SUBSTITUTION

Definition 7.6.12 The symbol $\int f(x)dx$ is used to represent an antiderivative of f over some domain. That is, it is a function $F(x)$ such that $F'(x) = f(x)$ for all x in that domain. $\int f(x)dx$ is called the **indefinite integral** of f for the given domain.

A few words about this notation are in order. The justification for using the "integral sign" for an antiderivative of f comes from **FTC-II**: on any interval $[a,b]$ where f is continuous, $\int_a^x f$ is one such antiderivative. The reason for using x and dx in the notation $\int f(x)dx$ lies in the suggestive power of Leibniz's notation for derivatives and differentials. In this notation, whenever $y = f(x)$ is differentiable we denote its derivative by $\frac{dy}{dx}$ or $\frac{df(x)}{dx}$, and its **differential** by

$$dy = \tfrac{dy}{dx}dx = f'(x)dx, \text{ or}$$

$$df = df(x) = \tfrac{df(x)}{dx}dx = f'(x)dx.$$

From this perspective, to find $\int f(x)dx$ means to find a function F whose differential is $f(x)dx$.

In elementary calculus, we learn to exploit differential notation in finding indefinite integrals. For example, to find $\int \tan^2 x \sec^2 x\, dx$ we notice that if we let $u = \tan x$ then $du = \sec^2 x dx$, so

$$\int \tan^2 x \sec^2 x\, dx = \int u^2 du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3 x + C.$$

We often call this the method of "u-substitution," or "change of variables." In this method, to find $\int f(x)dx$, we look for a differentiable function u and an integrable function g such that

$$f(x)dx = g(u)u'(x)dx = g(u)du. \text{ Then}$$

$$\int f(x)dx = \int g(u)du.$$

Of course, this procedure is most helpful if $\int g(u)du$ is "easier" to find than the original $\int f(x)dx$. Finally, note that in changing one differential into another we are using the chain rule. In the following theorem we give a formal proof of the validity of this procedure.

Theorem 7.6.13 (*Change of Variables, or Substitution*) *Suppose u is differentiable on $[a,b]$, u' is integrable on $[a,b]$, and g is continuous on $u[a,b]$. Then $(g \circ u)u'$ is integrable on $[a,b]$ and*

$$\int_a^b (g \circ u)u' = \int_{u(a)}^{u(b)} g.$$

In notation more familiar from elementary calculus,

$$\int_a^b (g(u(x))u'(x)\, dx = \int_{u(a)}^{u(b)} g(u)du.$$

Proof. Suppose u is differentiable on $[a,b]$, u' is integrable on $[a,b]$, and g is continuous on $u[a,b]$. Let $c = u(a)$ and $d = u(b)$. Since u is continuous

on $[a, b]$, $u[a, b]$ is a closed interval I containing c and d (see Corollary 5.3.12). Since g is continuous on I, we can define

$$G(x) = \int_c^x g(u)du, \text{ for } u \in I.$$

By FTC-II, G is differentiable and $G' = g$ on I. Consider the function $h = G \circ u$. Then $h : [a, b] \to \mathbb{R}$, and by the chain rule, $\forall t \in (a, b)$,

$$\begin{aligned} h'(x) &= G'(u(x))u'(x) \\ &= g(u(x))u'(x). \end{aligned}$$

Since the composition of continuous functions is continuous, h is continuous on $[a, b]$, and is an antiderivative of $g(u(x))u'(x)$ on (a, b). Therefore, by FTC-I,

$$\begin{aligned} \int_a^b (g(u(x))u'(x)\,dx &= h(b) - h(a) \\ &= G(u(b)) - G(u(a)) \\ &= \int_c^{u(b)} g \;-\; \int_c^{u(a)} g \\ &= \int_{u(a)}^{u(b)} g \quad \text{(by Theorem 7.6.5).} \quad \blacksquare \end{aligned}$$

Example 7.6.14 Evaluate $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2 + \frac{\pi}{2})dx$.

Solution: We let $u(x) = x^2 + \frac{\pi}{2}$ and $g(x) = \sin x$. Then $u'(x) = 2x$ and

$$\begin{aligned} \int_0^{\sqrt{\pi}} x\sin(x^2 + \frac{\pi}{2})dx &= \int_0^{\sqrt{\pi}} g\left(u(x)\right) \frac{1}{2}u'(x)dx \\ &= \tfrac{1}{2}\int_0^{\sqrt{\pi}} (g \circ u)(x)u'(x)dx. \end{aligned}$$

Now $u(0) = \frac{\pi}{2}$ and $u(\sqrt{\pi}) = \frac{3\pi}{2}$, so we want

$$\begin{aligned} \tfrac{1}{2}\int_{u(0)}^{u(\sqrt{\pi})} g(u)du &= \tfrac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin(u)du \\ &= \tfrac{1}{2}\left[-\cos\tfrac{3\pi}{2} + \cos\tfrac{\pi}{2}\right] = \tfrac{1}{2}[0 - 0] = 0. \end{aligned}$$

In elementary calculus, we would probably use a less formal approach. We would let $u = x^2 + \frac{\pi}{2}$. Then $du = 2xdx$ and we write simply

$$\int x\sin(x^2 + \tfrac{\pi}{2})dx = \int \sin u \cdot \tfrac{1}{2}du = \tfrac{1}{2}[-\cos u] + C.$$

From this we would write

$$\int_0^{\sqrt{\pi}} x\sin(x^2+\tfrac{\pi}{2})dx = \tfrac{1}{2}\left[-\cos u\right]_{\pi/2}^{3\pi/2} = \tfrac{1}{2}\left[-\cos\tfrac{3\pi}{2}+\cos\tfrac{\pi}{2}\right] = \tfrac{1}{2}[0-0] = 0. \ \square$$

Theorem 7.6.15 *(Integration by Parts) Suppose f and g are differentiable on $[a,b]$, and their derivatives f' and g' are integrable on $[a,b]$. Then both fg' and gf' are integrable on $[a,b]$, and*

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'.$$

Proof. Suppose f and g are differentiable on $[a,b]$, and their derivatives f' and g' are integrable on $[a,b]$. By the product rule for derivatives, gf is differentiable on $[a,b]$ and

$$(fg)' = fg' + gf'. \tag{25}$$

Since the product of integrable functions is integrable, fg' and gf' are integrable on $[a,b]$, and consequently, so is $(fg)'$. Moreover, fg is continuous on $[a,b]$. Thus, by FTC-I,

$$\int_a^b (fg)' = (fg)(b) - (fg)(a) \tag{26}$$

However, by Theorem 7.5.1 applied to (25),

$$\int_a^b (fg)' = \int_a^b fg' + \int_a^b gf'. \tag{27}$$

Putting (26) and (27) together,

$$f(b)g(b) - f(a)g(a) = \int_a^b fg' + \int_a^b gf'.$$

Therefore, $\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b gf'$. ∎

We can use integration by parts to develop an alternative formula for the remainder in Taylor's theorem.

Theorem 7.6.16 (*Taylor's Theorem,*[11] *with Remainder in Integral Form*) *Suppose f is n times differentiable on an open interval containing*

11. Taylor polynomials are defined in 6.5.1, $R_n(x)$ is defined in 6.5.9, and Taylor's theorem is stated in 6.5.11.

a and x, where $x \neq a$. Suppose $f^{(n+1)}(t)$ exists and is integrable on $[a,x]$ if $a < x$, or $[x,a]$ if $x < a$. Let $T_n(x)$ denote the n^{th} Taylor polynomial for f about a, and $R_n(x)$ denote the "remainder" $f(x) - T_n(x)$. Then,

$$R_n(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dt. \qquad (28)$$

Proof. Suppose f, x, a, and I satisfy the above hypotheses. The proof will proceed by mathematical induction. You will be asked to fill in the missing details in Exercise 15. $\forall n \in \mathbb{N}$, let $P(n)$ denote the statement that Equation (28) is true.

Part 1 (Prove $P(1)$): We apply integration by parts to the right side of Equation (28) with $n = 1$. Letting $u = x - t$ and $dv = f''(t)dt$, we have $du = -dt$ and $v = f'(t)$, so integration by parts yields

$$\begin{aligned}\frac{1}{1!}\int_a^x (x-t)f''(t)dt &= \frac{1}{1!}\left\{[(x-t)f'(t)]_a^x - \int_a^x f'(t)(-dt)\right\}\\ &\vdots \qquad \text{(fill in details)}\\ &= f(x) - T_1(x)\\ &= R_1(x).\end{aligned}$$

Part 2 (Prove $P(k) \Rightarrow P(k+1)$): Assume $P(k)$. That is,

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt.$$

We apply integration by parts to the right side of Equation (28) with $n = k+1$. Letting $u = (x-t)^{k+1}$ and $dv = f^{(k+2)}(t)dt$, we have $du = (k+1)(x-t)^k(-dt)$ and $v = f^{(k+1)}(t)$, so integration by parts yields

$$\begin{aligned}\frac{1}{(k+1)!}\int_a^x (x-t)^{k+1} f^{(k+2)}(t)dt &= \frac{1}{(k+1)!}\left\{\left[(x-t)^{k+1}f^{(k+1)}(t)\right]_a^x\right.\\ &\qquad \left. - \int_a^x f^{(k+1)}(t)(k+1)(x-t)^k(-dt)\right\}\\ &\vdots \qquad \text{(fill in details)}\\ &= -\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + R_k(x)\end{aligned}$$

$$\begin{aligned}
&= -\frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + [f(x) - T_k(x)] \\
&\quad \vdots \qquad \text{(fill in details)} \\
&= f(x) - T_{k+1}(x) \\
&= R_{k+1}(x).
\end{aligned}$$

Therefore, $P(k+1)$ is true, so $P(k) \Rightarrow P(k+1)$. ■

Theorem 7.6.17 (***First Mean Value Theorem for Integrals***) *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $\exists\, c \in (a, b) \ni \int_a^b f = f(c)(b - a)$.*

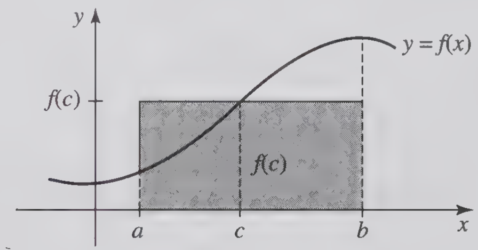


Figure 7.10

Proof. Exercise 16. ■

Theorem 7.6.18 (***Second Mean Value Theorem for Integrals***) *Suppose $f, g : [a, b] \to \mathbb{R}$ where $a < b$, f is continuous on $[a, b]$, and g is integrable and does not change sign on $[a, b]$. Then $\exists\, c \in (a, b) \ni \int_a^b fg = f(c) \int_a^b g$.*

Proof. Suppose $f, g : [a, b] \to \mathbb{R}$ where $a < b$, f is continuous on $[a, b]$, and g is integrable and does not change sign on $[a, b]$. Then fg is integrable on $[a, b]$, by Corollary 7.5.7. Since g does not change sign on $[a, b]$, either $g \geq 0$ throughout $[a, b]$, or $g \leq 0$ throughout $[a, b]$.

Case 1 ($g \geq 0$ throughout $[a, b]$): Then $\int_a^b g \geq 0$. By the extreme value theorem (5.3.7),

$$\exists\, m = \min f([a, b]) \quad \text{and} \quad \exists\, M = \max f([a, b]).$$

Then $\forall x \in [a, b]$, $mg(x) \leq f(x)g(x) \leq Mg(x)$, so by Theorems 7.5.2 (d) and 7.5.1 (a),

$$m \int_a^b g \leq \int_a^b fg \leq M \int_a^b g. \tag{29}$$

If $\int_a^b g = 0$, inequality (1) shows that $\int_a^b fg = 0$, in which case any $c \in (a, b)$ will do. So suppose $\int_a^b g \neq 0$. Then $\int_a^b g > 0$, so inequality (29) becomes

$$m \leq \frac{\int_a^b fg}{\int_a^b g} \leq M.$$

Thus, by the intermediate value theorem (5.3.9), $\exists\, c \in (a, b)$, such that

$$\frac{\int_a^b fg}{\int_a^b g} = f(c),$$

from which it follows that $\int_a^b fg = f(c) \int_a^b g$.

Case 2 ($g \leq 0$ throughout $[a, b]$): Assume this hypothesis and apply Case 1 to $-g$. ■

*IRRATIONALITY OF e^x AND π

We can use the Fundamental Theorem of Calculus to prove that e^x is irrational for all nonzero rational numbers x, and that π is irrational. It may come as a surprise that derivatives and integrals are involved in proving these facts. Since π can be defined and understood without reference to any mathematics beyond elementary algebra and geometry, one would not expect that proving the irrationality of π will require the concepts and techniques of real analysis.

Before we can proceed with the proofs, we need to develop a few properties of the following functions.

***Definition 7.6.19** In the remainder of this section, $\forall n \in \mathbb{N}$, define

$$\psi_n(x) = \frac{x^n(1-x)^n}{n!}.$$

***Theorem 7.6.20** *For all $n \in \mathbb{N}$, the function $\psi_n(x)$ has the following properties:*

(a) $\psi_n(x) = \dfrac{1}{n!} \displaystyle\sum_{k=n}^{2n} c_k x^k$ *for some* ***integers*** c_k.

(b) $\forall x \in (0,1),\ \ 0 < \psi_n(x) < \dfrac{1}{n!}.$

(c) $\psi_n(0) = 0$ *and the successive derivatives of* $\psi_n(x)$ *at* 0 *are:*

$$\psi_n^{(m)}(0) = \left\{ \begin{array}{ll} 0 & \text{if } 0 \le m < n, \\ \dfrac{m!}{n!} c_m & \text{if } n \le m \le 2n, \\ 0 & \text{if } m > 2n. \end{array} \right\}$$

In fact, $\psi_n^{(m)}(x) = 0$ *for all* x*, if* $m > 2n$.

(d) *Since* $\psi_n(1-x) = \psi_n(x),\ \ \psi_n^{(m)}(1) = \psi_n^{(m)}(0)$ *for all* $m \in \mathbb{N}$.

Proof. Exercise 19. ■

Theorem 7.6.21 *e^x is irrational, for all nonzero rational numbers x.*

***Proof.** Suppose[12] x is a nonzero rational number. If e^{-x} is rational, so is e^x. Thus, it is sufficient to prove the theorem for positive rational numbers $x = \dfrac{p}{q}$, where $p, q \in \mathbb{N}$. Further, if e^x is rational, so is $e^p = (e^x)^q$. Thus, it is sufficient to prove that e^p is irrational for all $p \in \mathbb{N}$.

For contradiction, suppose $\exists\, p \in \mathbb{N} \ni e^p$ is rational. Then $\exists\, a, b \in \mathbb{N} \ni e^p = \dfrac{a}{b}$. For each $n \in \mathbb{N}$, let $\psi_n(x)$ be defined as in 7.6.19, and define $F_n(x)$ by

$$\begin{aligned} F_n(x) &= p^{2n}\psi_n(x) - p^{2n}\psi_n'(x) + p^{2n-2}\psi_n''(x) - \cdots - p\psi^{(2n-1)}(x) + \psi^{(2n)}(x) \\ &= \sum_{k=0}^{2n} (-1)^k p^{2n-k} \psi_n^{(k)}(x). \end{aligned}$$

One can easily show (Exercise 19) that

$$\tfrac{d}{dx} F_n(x) = -pF_n(x) + p^{2n+1}\psi_n(x). \tag{30}$$

Thus, F_n satisfies the differential equation

$$F_n' + pF_n = p^{2n+1}\psi_n.$$

If we multiply both sides by e^{px}, we have

$$\begin{aligned} e^{px}F_n'(x) + pe^{px}F_n(x) &= e^{px}p^{2n+1}\psi_n(x) \\ \text{i.e., } \tfrac{d}{dx}\left[e^{px}F_n(x)\right] &= e^{px}p^{2n+1}\psi_n(x). \end{aligned}$$

12. See Theorem 8.8.7 for an easier proof using infinite series.

Thus, by the fundamental theorem of calculus,

$$\int_0^1 e^{px} p^{2n+1} \psi_n(x) dx = \Big[e^{px} F_n(x) \Big]_0^1 = e^p F_n(1) - F_n(0), \text{ so}$$

$$b \int_0^1 e^{px} p^{2n+1} \psi_n(x) dx = b e^p F_n(1) - b F_n(0) = a F_n(1) - b F_n(0), \tag{31}$$

which is an integer since $a, b, F_n(1)$, and $F_n(0)$ are integers.

By property (b) of $\psi_n(x)$ listed in Theorem 7.6.20 above, for all $x \in [0,1]$,

$$0 \le e^{px} p^{2n+1} \psi_n(x) < \frac{e^p p^{2n+1}}{n!}.$$

$$\begin{aligned} \text{So, } 0 < b \int_0^1 e^{px} p^{2n+1} \psi_n(x) dx &< b \int_0^1 \frac{e^p p^{2n+1}}{n!} dx \\ &= \frac{b e^p p^{2n+1}}{n!} \\ &= b e^p p \frac{(p^2)^n}{n!} = a p \frac{(p^2)^n}{n!}. \end{aligned}$$

By Corollary 2.3.11, $\lim\limits_{n \to \infty} \dfrac{(p^2)^n}{n!} = 0$. Hence we can find $n \in \mathbb{N}$ such that

$$0 < b \int_0^1 e^{px} p^{2n+1} \psi_n(x) dx < 1.$$

But we noted in (31) above that $b \int_0^1 e^{px} p^{2n+1} \psi_n(x) dx$ is an integer. Thus, we have an integer between 0 and 1. Since this is impossible, we must reject our assumption that e^x is rational. Therefore, e^x is irrational. ■

Theorem 7.6.22 *π is irrational.*

***Proof.** If π is rational, then π^2 is rational, so it suffices to prove that π^2 is irrational. For contradiction, suppose π^2 is rational. Then $\exists\, a, b \in \mathbb{N} \ni \pi^2 = \dfrac{a}{b}$. As in the proof of Theorem 7.6.21, we shall make use of the functions ψ_n defined in 7.6.19. This time, we define the functions F_n by

$$\begin{aligned} F_n(x) &= b^n \left[\pi^{2n} \psi_n(x) - \pi^{2n-2} \psi_n''(x) + \pi^{2n-4} \psi_n^{(4)}(x) - \cdots + (-1)^n \psi_n^{(2n)} \right] \\ &= b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} \psi_n^{(2k)}(x). \end{aligned}$$

It is straightforward to prove (Exercise 19) that

$$F_n''(x) = b^n \pi^{2n+2} \psi_n(x) - \pi^2 F_n(x) \tag{32}$$
$$= \pi^2 a^n \psi_n(x) - \pi^2 F_n(x). \tag{33}$$

Thus, F_n satisfies the differential equation

$$F_n''(x) + \pi^2 F_n(x) = \pi^2 a^n \psi_n(x). \tag{34}$$

Following the rules of elementary calculus, we have (Exercise 19),

$$\frac{d}{dx}\left[F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right] = \sin\pi x\left[F_n''(x) + \pi^2 F_n(x)\right]. \tag{35}$$

Combining Equations (34) and (35), we have

$$\tfrac{d}{dx}\left[F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\right] = \pi^2 a^n \psi_n(x)\sin\pi x.$$

Thus,

$$\begin{aligned}\int_0^1 \pi^2 a^n \psi_n(x)\sin\pi x\,dx &= \Big[\,F_n'(x)\sin\pi x - \pi F_n(x)\cos\pi x\,\Big]_0^1 \\ &= [F_n'(1)\sin\pi - \pi F_n(1)\cos\pi] - [F_n'(0)\sin 0 - \pi F_n(0)\cos 0] \\ &= \pi\,[F_n(1) + F_n(0)].\end{aligned}$$

Therefore, $\int_0^1 \pi a^n \psi_n(x)\sin\pi x dx = F_n(1) + F_n(0)$, an integer.

However, by property (b) of $\psi_n(x)$ given in Theorem 7.6.20,

$$0 < \int_0^1 \pi a^n \psi_n(x)\sin\pi x dx < \int_0^1 \frac{\pi a^n \sin\pi x}{n!} dx < \frac{\pi a^n}{n!}\int_0^1 dx = \frac{\pi a^n}{n!}.$$

Now, $\lim_{n\to\infty} \frac{\pi a^n}{n!} = 0$ (see Corollary 2.3.11). Thus, by taking n sufficiently large,

$$0 < \int_0^1 \pi a^n \psi_n(x)\sin\pi x dx < 1.$$

As noted above, the integral appearing here is an integer. But it is impossible to have an integer between 0 and 1. Therefore, π^2 cannot be rational; i.e., π^2 is irrational. Therefore, π is irrational. ■

Concluding Remarks: The irrationality of π was established in 1767 by the Swiss mathematician Johann Heinrich Lambert, whose proof was made more rigorous in 1794 by the French mathematician Adrien-Marie Legendre. Their proofs used "continued fractions," which would take us too far afield to discuss here.

The methods used here in proving Theorems 7.6.21 and 7.6.22 are attributed to the French mathematician Charles Hermite, 1873. These methods

have been extended and updated by Ivan Niven in his enlightening monograph, *Irrational Numbers*, [100]. In addition to proofs of our Theorems 7.6.21 and 7.6.22, Chapter 2 of Niven's book includes proofs of many additional interesting results, including:

(1) For every rational number $x \neq 0$, the trigonometric functions of x are all irrational.

(2) Any nonzero value of an inverse trigonometric function of x is irrational for all nonzero rational values of x.

(3) For every rational number $x \neq 0$, the hyperbolic functions of x are all irrational.

(4) Any nonzero value of an inverse hyperbolic function of x is irrational for all nonzero rational values of x.

(5) For all positive rational $x \neq 1$, $\ln x$ is irrational.

(6) For all positive rational $x \neq 1$, $\log_b x$ is irrational for any positive rational base $b \neq 1$ unless $x^m = b^n$ for some integers m and n.

In Chapter 9 of the same book, Niven also proves the deeper and more remarkable results that these numbers are "transcendental." [Recall[13] that a number a is **transcendental** if there is no polynomial $p(x)$ with rational coefficients such that $p(a) = 0$.] These results may help explain the title of the next section.

EXERCISE SET 7.6

1. Prove Case 1 of the proof of Theorem 7.6.5.

2. Select two of the relative positions of a, b, and c not treated in Theorems 7.4.2 or 7.6.5 and prove that theorem for those two cases.

3. Show that the signum function $sgn(x) = \left\{ \begin{array}{ll} \dfrac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is integrable on $[-1, 1]$, yet has no antiderivative on $[-1, 1]$. [Thus, a function can be integrable on an interval without having an antiderivative there. For a more complicated example, see Exercise 4.]

4. In Exercise 7.4.18 we showed that **Thomae's function** T is integrable on [0,1], even though it is discontinuous at every rational number. Prove that T has no antiderivative on any subinterval of $[0, 1]$. [Hint: Use Theorem 6.3.7.]

---

13. See Exercise 2.8.17.

5. In Chapter 6 we proved that the function $f(x) = \left\{ \begin{array}{ll} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is differentiable everywhere, but its derivative f' is discontinuous at 0. (See Exercise 6.2.17.) Prove that f' is integrable on $[0, 2/\pi]$ and find $\int_0^{2/\pi} f'$. [Explain why FTC-I cannot be used here.]

6. As noted in Exercise 5, the function $f(x) = \left\{ \begin{array}{ll} x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{array} \right\}$ is differentiable everywhere. Let $g(x) = f'(x)$, $\forall x \in \mathbb{R}$. Then g is a function that has an antiderivative everywhere. Nevertheless, show that g is **not** integrable on $[0, 1]$. (See Exercise 6.4.31.) Thus, a function can have an antiderivative over an interval and not be Riemann integrable there.[14]

7. Prove that $\forall x \in \mathbb{R}$, $\int_{-1}^{x} sgn = |x| - 1$. [Thus, $\int_a^x f$ can exist $\forall x \in [a, b]$, even when f has no antiderivative on $[a, b]$.]

8. Consider the function $f(x) = \left\{ \begin{array}{l} 1 \text{ if } x < 1 \\ x \text{ if } 1 \leq x < 2 \\ 3 \text{ if } x \geq 2 \end{array} \right\}$. Find a formula for $\int_0^x f$ that is valid for all $-\infty \leq x \leq +\infty$. (See Example 7.6.7.)

9. Suppose f is integrable on $[a, b]$. Prove that
   (a) if $f(x) \geq 0$ on $[a, b]$, then $F(x) = \int_a^x f$ is monotone increasing there.
   (b) if $f(x) \leq 0$ on $[a, b]$, then $F(x) = \int_a^x f$ is monotone decreasing there.

10. Suppose f and F are continuous on $[a, b]$ and $F(a) = 0$. Prove that the following are equivalent:
    (a) $F' = f$ on $[a, b]$.
    (b) $F(x) = \int_a^x f$, $\forall x \in [a, b]$.

11. Apply the second form of the Fundamental Theorem of Calculus and the chain rule to find a formula for each of the following, assuming that f is continuous and g, h are differentiable on the appropriate intervals:
    (a) $\dfrac{d}{dx} \displaystyle\int_x^a f$ (b) $\dfrac{d}{dx} \displaystyle\int_a^{g(x)} f$
    (c) $\dfrac{d}{dx} \displaystyle\int_{g(x)}^a f$ (d) $\dfrac{d}{dx} \displaystyle\int_{g(x)}^{h(x)} f$

14. For an example of a *bounded* function that has an antiderivative everywhere on $[a, b]$ but is not Riemann integrable there, see [49], p.107, Example 35. See also [131], Section 9.7.

12. Find each of the following (see Exercise 11):

(a) $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{3t^2+5}\,dt$
(b) $\dfrac{d}{dx}\displaystyle\int_{\pi/2}^x (\sin 3t + \cos 4t)\,dt$

(c) $\dfrac{d}{dx}\displaystyle\int_x^0 (2t^3+4)\,dt$
(d) $\dfrac{d}{dx}\displaystyle\int_x^{3.7} \sin 5t\,dt$

(e) $\dfrac{d}{dx}\displaystyle\int_0^{3x-2} \sqrt{1+t^2}\,dt$
(f) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \sin(t^3)\,dt$

(g) $\dfrac{d}{dx}\displaystyle\int_{3x-1}^0 \frac{dt}{t+4}$
(h) $\dfrac{d}{dx}\displaystyle\int_{2x^2+1}^4 \sqrt{t}\,dt$

(i) $\dfrac{d}{dx}\displaystyle\int_{x+1}^{x^2+1} \frac{dt}{t+2}$
(j) $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sqrt[3]{1+t}\,dt$

13. Use Theorem 7.6.13 to find

(a) $\int_0^{\pi/4} \sec^4 x \tan x\,dx$
(b) $\int_0^{\pi/2} \cos^5\,dx$

14. Use integration by parts to find each of the following:

(a) $\displaystyle\int \frac{x^3}{\sqrt{x^2-1}}\,dx$
(b) $\displaystyle\int x^5\sqrt{x^3-1}\,dx$

(c) $\displaystyle\int_2^4 \ln x\,dx$
(d) $\displaystyle\int_1^3 x^2 \ln\sqrt{x}\,dx$

(e) $\displaystyle\int_0^9 e^{\sqrt{x}}\,dx$
(f) $\displaystyle\int_1^4 \sqrt{x}e^{\sqrt{x}}\,dx$

(g) $\displaystyle\int_1^2 \sin(\ln x)\,dx$
(h) $\displaystyle\int \sin^{-1} x\,dx$

(i) $\displaystyle\int \tan^{-1} x\,dx$
(j) $\displaystyle\int x\tan^{-1} x\,dx$

15. Fill in the details requested in the proof of Theorem 7.6.16.

16. Prove the first mean value theorem for integrals (Theorem 7.6.17). [Hint: Use Theorem 7.5.2 and the intermediate value theorem.]

17. Prove the following variant of the first mean value theorem for integrals: If f is monotone on $[a,b]$, then $\exists\, c \in (a,b)$ such that $\int_a^b f = f(a)(c-a) + f(b)(b-c)$. [Hint: Consider the function $g(x) = f(a)(x-a) + f(b)(b-x)$. Show that $\int_a^b f$ falls between $g(a)$ and $g(b)$.]

18. Let $F(x) = \int_0^x T$, where T denotes **Thomae's function**. Prove that

    (a) F is differentiable everywhere on $[0, 1]$.

    (b) for all x in a dense subset of $[0, 1]$, $F'(x) \neq T(x)$.

19. Prove that $\forall a < b$, $\exists\, f{:}[a, b] \to \mathbb{R}$ such that f is integrable on $[a, b]$ but there is no nonempty subinterval $[c, d] \subseteq [a, b]$ on which f has an antiderivative. [Hint: Use the function given in Theorem 5.7.3. Show that this function has jump discontinuities on a dense subset of $[a, b]$ and apply Exercise 6.3.12.]

*20. **Monotone f Implies One-Sided Differentiability of $\int_a^x f$:** Suppose f is nonnegative and monotone increasing on $[a, b]$. Then f is integrable on $[a, b]$ and the function $F(x) = \int_a^x f$ is monotone increasing on $[a, b]$. (Justify.) Hence, at any point $x_0 \in (a, b)$, the four one-sided limits

$$\lim_{x \to x_0^-} f(x), \quad \lim_{x \to x_0^+} f(x), \quad \lim_{x \to x_0^-} F(x), \quad \lim_{x \to x_0^+} F(x)$$

exist (see Theorem 5.2.17). Prove that f is differentiable from the left at x_0, and differentiable from the right at x_0, and

$$F'_-(x_0) = \lim_{x \to x_0^-} f(x), \ \text{ and } F'_+(x_0) = \lim_{x \to x_0^+} f(x).$$

(These one-sided derivatives were defined in Definition 6.1.11.) [Hint: Revise the proof of Theorem 7.6.8.]

*21. **Application of the Previous Problem:**

    (a) Suppose f is nonnegative and monotone increasing on $[a, b]$, and $x_0 \in (a, b)$. Prove that if f is not continuous at x_0, then the function $F(x) = \int_a^x f$ is not differentiable at x_0.

    (b) In Theorem 5.7.3 we proved that there is a bounded, monotone increasing function $f{:}[a, b] \to \mathbb{R}$ that is continuous at every irrational number and discontinuous at every rational number in $[a, b]$. (See also Exercise 7.4.19.) Use this and the result of (a) above to prove that *there exists a continuous, monotone increasing function that is differentiable at every irrational number in* $[a, b]$ *and nondifferentiable at every rational number in* $[a, b]$.

*22. Prove Theorem 7.6.20.

*23. Prove Equation (30) of Theorem 7.6.21.

*24. Prove Equations (32), (33), and (35) of Theorem 7.6.22.

7.7 *Elementary Transcendental Functions

> In this section we define the exponential,[15] logarithmic,[15] and trigonometric functions. We call for rigorous proofs of their properties, well known from calculus and used routinely throughout real analysis. The section can be covered lightly in class, assigned as an independent reading project, or omitted completely. There is no exercise set in this section; instead, you are asked to fill in the proofs of results stated but left unproved in the text.
>
> For alternative approaches, see Sections 5.6 and 8.8.

You are surely familiar with the definition of e^x as the inverse of the function $\ln x$, which is defined in elementary calculus courses by an integral. But you may be unfamiliar with the definition of $\sin x$ as the inverse of a function defined by an integral. Indeed, you may be intrigued by the fact that in calculus courses we seem to accept the trigonometric functions *without definition.* That is, we seem to assume that they are defined somewhere *outside* of calculus. We are now about to remedy that situation.

In this section we show that, besides providing definitions of the logarithmic and exponential functions, the Riemann integral enables us to give rigorous definitions of the trigonometric functions. We shall use the integral to define these functions and outline proofs of their fundamental properties. We call these functions "transcendental," because their values cannot be calculated as roots of polynomial equations with rational coefficients.[16]

THE NATURAL LOGARITHM FUNCTION

Definition 7.7.1 The **natural logarithm** function $\ln x : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \tfrac{1}{t}\, dt \qquad (\text{for } x > 0).$$

Remarks 7.7.2 (a) $\ln x$ exists for all $x > 0$.
(b) $\ln x < 0$ if $0 < x < 1$; $\ln x = 0$ if $x = 1$; $\ln x > 0$ if $x > 1$.
(c) $\ln x$ is continuous and strictly increasing on $(0, \infty)$.
(d) $\ln x$ is differentiable on $(0, \infty)$, and $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$.

15. The exponential and logarithmic functions were defined differently in Section 5.6. This section is independent of that one, but the definitions are shown to be equivalent.

16. See Exercise 2.8.17 and the concluding remarks in Section 7.6.

Theorem 7.7.3 (***Laws of Logarithms***) *$\forall x, y \in (0, +\infty)$, and $\forall n \in \mathbb{N}$,*

(a) $\ln(xy) = \ln x + \ln y$ (b) $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$

(c) $\ln(x^n) = n \ln x$ (d) $\ln \sqrt[n]{x} = \frac{1}{n} \ln x$

(e) $\ln\left(\dfrac{1}{x}\right) = -\ln x$ (f) $\ln(x^r) = r \ln x,\ \forall r \in \mathbb{Q}$

Proof of (a): Let $y > 0$ be fixed. By Theorem 7.7.2, $\forall x > 0$,

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

and by the chain rule,

$$\frac{d}{dx} \ln(xy) = \frac{1}{xy} \frac{d}{dx}(xy) = \frac{1}{xy} \cdot y = \frac{1}{x}.$$

Since $\ln x$ and $\ln(xy)$ have the same derivative, they must differ by a constant. That is, $\exists\, C \in \mathbb{R}$ such that

$$\forall x > 0, \ln(xy) = \ln x + C. \tag{36}$$

Letting $x = 1$, we find that $\ln y = C$. Plugging this result into (36), we have $\ln(xy) = \ln x + \ln y$.

Proof of (f): Let $r \in \mathbb{Q}$ be fixed. By the chain rule, $\forall x > 0$,

$$\frac{d}{dx} \ln(x^r) = \frac{1}{x^r} \frac{d}{dx}(x^r) = \frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x} = \frac{d}{dx} r \ln x.$$

Since $\ln(x^r)$ and $r \ln x$ have the same derivative, they must differ by a constant. That is, $\exists\, C \in \mathbb{R}$ such that

$$\ln(x^r) = r \ln x + C. \tag{37}$$

Letting $x = 1$, we find that $0 = C$. Plugging this result into (37), we have $\ln(x^r) = r \ln x$. ■

THE NUMBER e

Remark 7.7.4 In calculus, we define e to be the number that satisfies the equation $\ln x = 1$.

In calculus courses we usually justify the existence of such a number by appealing to the intermediate value theorem. Since $\ln x$ is continuous on $(0, +\infty)$, and $\lim_{x \to 0^+} \ln x = -\infty$, and $\lim_{x \to +\infty} \ln x = +\infty$, we conclude from the intermediate value theorem that there must be a real number x such that $\ln x = 1$. To justify the uniqueness of such a number we note that $\ln x$ is strictly increasing, so must be 1-1.

However, we cannot take this approach here. That is because we have already defined e by a different procedure, in Definition 2.5.10. Thus, we need to take this value of e and *prove* that $\ln e = 1$. That is what we do next.

Theorem 7.7.5 *$\ln e = 1$, where $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ from Definition 2.5.10.*

Proof. By definition of e, $\ln e = \ln \left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n\right]$. The function $\ln x$ is continuous on $(0, +\infty)$, hence is continuous at e. The sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ converges to e. Thus, the sequential criterion for continuity implies that $\ln \left(1 + \frac{1}{n}\right)^n \to \ln e$. That is,

$$\begin{aligned} \ln e &= \lim_{n \to \infty} \ln \left(1 + \tfrac{1}{n}\right)^n \\ &= \lim_{n \to \infty} n \ln \left(1 + \tfrac{1}{n}\right) \\ &= \lim_{n \to \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}}. \end{aligned} \tag{38}$$

Now, by L'Hôpital's rule,

$$\begin{aligned} \lim_{x \to \infty} \frac{\ln \left(1 + \frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x \to \infty} \frac{\frac{x}{x+1} \cdot \frac{d}{dx}\left(\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)} \\ &= \lim_{x \to \infty} \frac{x}{x+1} \\ &= 1. \end{aligned}$$

Thus, by the sequential criterion for limits of functions at ∞ (See Exercise 4.4-B.8.)

$$\lim_{n \to \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = 1. \tag{39}$$

Plugging the result (39) into (38), we have $\ln e = 1$. ∎

Corollary 7.7.6 (a) $\forall r \in \mathbb{Q}$, $\ln(e^r) = r$;

(b) $\lim_{x \to +\infty} \ln x = +\infty$;

(c) $\lim_{x \to 0^+} \ln x = -\infty$;

(d) *The range of* ln: $(0, +\infty) \to \mathbb{R}$ *is* $\mathbb{R}$; *the graph is as shown in Figure 7.11.*

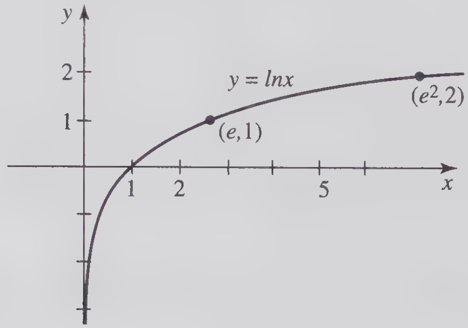


Figure 7.11

THE EXPONENTIAL FUNCTION

Definition 7.7.7 (The Function exp(x)): Since the function ln: $(0, +\infty) \to \mathbb{R}$ is continuous, strictly increasing, 1-1, and onto, Corollary 5.5.3 assures us that it has an inverse $\ln^{-1}\colon \mathbb{R} \to (0, +\infty)$ which is continuous, strictly increasing, 1-1, and onto. We denote that inverse (temporarily) by

$$\mathbf{exp}(\boldsymbol{x}) = \ln^{-1} x.$$

That is, $y = \exp(x)$ if and only if $x = \ln y$.

Remarks 7.7.8

(a) $\exp x > 1$, if $x > 0$.

$\exp x = 1$, if $x = 0$.

$0 < \exp x < 1$, if $x < 0$.

(b) $\forall r \in \mathbb{Q}, \exp(r) = e^r$.

(c) For any sequence $\{r_n\}$ of rational numbers converging to x,

$\exp(x) = \lim_{n\to\infty} \exp(r_n) = \lim_{n\to\infty} e^{r_n}$.

Definition 7.7.9 (The Function e^x)

We now come to the problem of defining e^x, for arbitrary real numbers x (in particular, for irrational numbers x) in such a way that our definition is consistent with previously agreed-upon definitions.

For those who skipped Section 5.6. The only previous definition we must be consistent with is e^r, for *rational* numbers r. By Remark 7.7.8 (b), $\exp(r) = e^r$ whenever r is a rational number. We extend this to all real numbers x by *defining*

$$\forall x \in \mathbb{R},\ e^x = \exp(x).$$

For those who studied Section 5.6. As we have noted above, for every rational number r, $e^r = \exp(r)$. Thus, e^x and $\exp(x)$ are continuous everywhere on $\mathbb{R}$ and agree on the dense set $\mathbb{Q}$. Therefore, by Exercise 5.1.29,

$$\forall x \in \mathbb{R},\ e^x = \exp(x).$$

Regardless of whether or not we studied Section 5.6, we see that the functions e^x and $\exp(x)$ are identical. Accordingly, we shall no longer use the notation $\exp(x)$; we shall use e^x exclusively.

The so-called "laws of exponents" described in the next theorem, were proved in Section 5.6. It is interesting, especially for those who skipped that section, to find that these laws can be derived directly from the definition of the exponential function as the inverse of the logarithm function. We do so in the next theorem.

Theorem 7.7.10 (***Laws of Exponents***) $\forall x, y \in \mathbb{R}$,

(a) $e^x e^y = e^{x+y}$ (b) $e^x / e^y = e^{x-y}$

(c) $e^0 = 1$ (d) $e^{-x} = 1/e^x$

(e) $\ln(e^x) = x$ (f) $e^{\ln x} = x$

Proof of (a): Let $u = e^x$ and $v = e^y$. Then $\ln u = x$ and $\ln v = y$. Thus, by the laws of logarithms (Theorem 7.7.3),

$$x + y = \ln u + \ln v = \ln(uv) = \ln(e^x e^y).$$

That is, by Definition 7.7.7, $e^{e+y} = e^x e^y$. ■

Theorem 7.7.11 (***Derivative of e^x***) *The function e^x is differentiable everywhere, and $\dfrac{d}{dx}e^x = e^x$.*

Proof. See the proof of Theorems 7.7.2 (d) and 6.2.9 (b). ■

Theorem 7.7.12 (***e^{kx} and the Differential Equation $f' = kf$***) *Let $k \in \mathbb{R}$. If $f : \mathbb{R} \to (0, \infty)$ satisfies the differential equation $f' = kf$ everywhere on $\mathbb{R}$, then $\exists\, c \in \mathbb{R} \ni \forall x \in \mathbb{R}$, $f(x) = ce^{kx}$. Moreover, $c = f(0)$.*

Proof. Suppose $f: \mathbb{R} \to (0,\infty)$ satisfies the differential equation $f' = kf$, $\forall x \in \mathbb{R}$. Define $F(x) = \ln f(x)$. Then by Theorem 7.7.2 (d) and the chain rule,

$$F'(x) = \frac{f'(x)}{f(x)} = \frac{kf(x)}{f(x)} = k.$$

By Corollary 6.4.5, this means there is a constant c such that $\forall x \in \mathbb{R}$,

$$\begin{aligned} F(x) &= kx + c; \text{ i.e.,} \\ \ln f(x) &= kx + c; \text{ i.e.,} \\ f(x) &= e^{kx+c} = e^c e^{kx}. \end{aligned}$$

Finish by regarding e^c as a constant, and evaluate it by letting $x = 0$. ∎

GENERAL EXPONENTIAL AND LOGARITHM FUNCTIONS

Definition 7.7.13 (General Exponential Functions) Suppose $a > 0$. Then $\forall x \in \mathbb{R}$, we define[17]

$$a^x = e^{x \ln a}.$$

Remarks 7.7.14 Let $a > 0$.

(a) Definition 7.7.13 yields the conventional meaning of a^x when x is a natural number, integer, or rational number, and is consistent with Definition 5.6.5.

(b) The "laws of exponents" given in Theorem 7.7.10 (a)–(d) hold for a^x. In addition, Definition 7.7.13 allows us to add the following laws of exponents and logarithms: $\forall\, x, y \in \mathbb{R}$, $(e^x)^y = e^{xy}$, $(a^x)^y = a^{xy}$, and $\ln(a^x) = x \ln a$.

(c) The function a^x is differentiable everywhere, and $\frac{d}{dx} a^x = a^x \ln a$.

(d) If $a > 1$, the function a^x is strictly increasing on $\mathbb{R}$, with range $(0, +\infty)$, and $\lim\limits_{x \to +\infty} a^x = +\infty$ while $\lim\limits_{x \to -\infty} a^x = 0$.

(e) If $0 < a < 1$, the function a^x is strictly decreasing on $\mathbb{R}$, with range $(0, +\infty)$, and $\lim\limits_{x \to +\infty} a^x = 0$ while $\lim\limits_{x \to -\infty} a^x = +\infty$.

Definition 7.7.15 (General Logarithmic Functions) Suppose $a > 0$ and $a \neq 1$. Then $\forall x \in \mathbb{R}$, we define the function $\log_a x$ to be the inverse of the function a^x.

Remarks 7.7.16 Suppose $a > 0$ and $a \neq 1$. Then

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17. For students who studied Section 5.6, this is a theorem rather than a definition.

(a) $\forall x \in (0, \infty)$, $\log_a(a^x) = x$ and $a^{\log_a x} = x$.

(b) If $a > 1$, the function $\log_a x$ is strictly increasing on $(0, +\infty)$ with range $\mathbb{R}$.

(c) If $a > 1$, then $\lim\limits_{x \to +\infty} \log_a x = +\infty$ and $\lim\limits_{x \to 0^+} \log_a x = -\infty$.

(d) $\log_{1/a} x = -\log_a x$.

(e) If $0 < a < 1$, $\log_a x$ is strictly decreasing on $(0, +\infty)$ with range $\mathbb{R}$.

(f) If $0 < a < 1$, then $\lim\limits_{x \to +\infty} \log_a x = -\infty$ and $\lim\limits_{x \to 0^+} \log_a x = +\infty$.

(g) The "laws of logarithms" given in Theorem 7.7.3 hold for $\log_a x$.

Remarks 7.7.17 Suppose $a > 0$ and $a \neq 1$. Then

(a) $\log_a x = \dfrac{\ln x}{\ln a}$.

(b) $\forall b > 0$ and $b \neq 1$, $\log_b x = \dfrac{\log_a x}{\log_a b}$.

(c) The function $\log_a x$ is differentiable on $(0, \infty)$, and $\dfrac{d}{dx} \log_a x = \dfrac{1}{x \ln a}$.

THE ARCSINE FUNCTION

We now direct our attention to a rigorous development of the trigonometric functions. We would prefer to begin by defining the sine and cosine functions, because we know that from them we can derive all the remaining trigonometric functions and their interrelationships. That approach will be taken in Chapter 8, but it requires the theory of power series. To take advantage of the Riemann integral, we begin with the inverse sine function and use it as a foundation for defining the sine function.[18]

Definition 7.7.18 (The Arcsine Function) We define the function

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1 - t^2}}, \quad \text{for } -1 < x < 1.$$

(Recall from your calculus course why this is a reasonable definition.)

Theorem 7.7.19 Arcsin x *has the following properties on* $(-1, 1)$:

(a) $\arcsin x$ *is strictly increasing on* $(-1, 1)$.

(b) $\arcsin x$ *is continuous on* $(-1, 1)$.

18. For an alternative development of the trigonometric functions based on the inverse tangent function, see Giaquinta and Modica [53], pp. 170–173.

(c) $\arcsin x$ *is differentiable on* $(-1,1)$, *and* $\dfrac{d}{dx} = \dfrac{1}{\sqrt{1-x^2}}$.

(d) $\arcsin x$ *is an odd*[19] *function on* $(-1,1)$.

Remarks 7.7.20 (a) $\arcsin(-1,1)$ is an interval. [See Theorem 5.3.8.]

(b) $\arcsin x$ is bounded on $(-1,1)$. [Show that $|\arcsin x| < 2$.]

(c) Since $\arcsin x$ is continuous, strictly increasing, and bounded on $(-1,1)$, Corollary 5.3.14 assures us that we can extend $\arcsin x$ to a continuous, strictly increasing function f on the closed interval $[-1,1]$, and

$$\arcsin[-1,1] = [c,d]$$

where $c = \inf\{\arcsin x : -1 < x < 1\} = \lim\limits_{x\to -1^+} \arcsin x = \lim\limits_{x\to -1^+} \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$

and $d = \sup\{\arcsin x : -1 < x < 1\} = \lim\limits_{x\to 1^-} \arcsin x = \lim\limits_{x\to 1^-} \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Definition 7.7.21 (Definition of π)

$$\begin{aligned}\pi = 2\arcsin 1 &= 2\sup\{\arcsin x : -1 < x < 1\}\\ &= 2\lim_{x\to 1^-} \arcsin x\\ &= 2\lim_{x\to 1^-} \int_0^x \frac{dt}{\sqrt{1-t^2}}.\end{aligned}$$

THE SINE FUNCTION[20]

Definition 7.7.22 (of $\sin x$, for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$)

Since $\arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and is 1-1 and onto, it has an inverse function, which we call the sine function. Thus,

$$\sin : \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right] \to [-1,1].$$

Note that:

(a) By Definition 7.7.21, $\frac{\pi}{2} = \arcsin 1$; that is, $\sin\frac{\pi}{2} = 1$.

(b) By Corollary 5.5.3, $\sin x$ is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(c) $\sin x$ is an odd function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In fact, the inverse of any invertible odd function is odd.

19. For definitions of **even** and **odd** functions, see Exercise 6.2.5.

20. For a definition based on infinite series, see Section 8.8.

Definition 7.7.23 (of $\sin x$, for All Real Numbers)

(a) For $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, we define $\sin x = -\sin(x - \pi)$.

Note that when $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, $-\frac{\pi}{2} \leq x - \pi \leq \frac{\pi}{2}$, and that when $x = \frac{\pi}{2}$, both sides of the defining equation are the same.

(b) The function $\sin\colon \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to \mathbb{R}$ is continuous on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and $\sin\left(-\frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$. Thus, by Exercise 6.2.19, we can extend $\sin x$ to a function that is continuous and periodic on $\mathbb{R}$, with period $\frac{3\pi}{2} - \left(-\frac{\pi}{2}\right) = 2\pi$.

(c) Show that $\sin x$ is an odd function on $\mathbb{R}$. [First show it is odd on $[-\pi, \pi]$, then show it is odd on $\mathbb{R}$.]

DIFFERENTIABILITY OF THE SINE FUNCTION

Lemma 7.7.24 *The sine function is differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and $\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\dfrac{d}{dx} \sin x = \sqrt{1 - \sin^2 x}$.*

Proof. The function $y = \arcsin x$ is differentiable on $(-1, 1)$, and $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$. Thus, by the inverse function theorem for differentiable functions (6.2.4), the function $x = \sin y$ is differentiable at every $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and

$$\frac{dx}{dy} = \frac{d}{dy} \sin y = \frac{1}{\dfrac{1}{\sqrt{1 - x^2}}} = \sqrt{1 - x^2} = \sqrt{1 - \sin^2 y}. \qquad \blacksquare$$

Lemma 7.7.25 *The sine function is differentiable on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\forall x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\dfrac{d}{dx} \sin x = -\sqrt{1 - \sin^2 x}$.*

Proof. Apply Definition 7.7.22 and the chain rule on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. $\blacksquare$

Lemma 7.7.26 *The sine function is differentiable at $\frac{\pi}{2}$, and when $x = \frac{\pi}{2}$, $\dfrac{d}{dx} \sin x = 0 = \sqrt{1 - \sin^2 \frac{\pi}{2}}$.*

Proof.
$$\begin{aligned}
\lim_{x \to \pi/2^-} \frac{d}{dx} \sin x &= \lim_{x \to \pi/2^-} \sqrt{1 - \sin^2 x} \\
&= 0, \text{ by continuity of } \sin x \text{ on } \mathbb{R} \\
&= \lim_{x \to \pi/2^+} \left(-\sqrt{1 - \sin^2 x}\right) \\
&= \lim_{x \to \pi/2^+} \frac{d}{dx} \sin x.
\end{aligned}$$

Apply Theorems 6.1.13 and 6.1.14. ∎

Corollary 7.7.27 *The sine function is differentiable on* $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, *and*

$$\frac{d}{dx}\sin x = \left\{ \begin{array}{ll} \sqrt{1-\sin^2 x} & \text{if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ -\sqrt{1-\sin^2 x} & \text{if } x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{array} \right\}.$$

Corollary 7.7.28 *The sine function is differentiable everywhere, and*

$$\frac{d}{dx}\sin x = \left\{ \begin{array}{ll} \sqrt{1-\sin^2 x} & \text{if } x - 2n\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ for some integer } n \\ -\sqrt{1-\sin^2 x} & \text{if } x - 2n\pi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ for some integer } n \end{array} \right\}.$$

Proof. Prove differentiability at $-\frac{\pi}{2}$ and apply periodicity (see Exercise 6.2.19). ∎

THE COSINE FUNCTION[21]

Definition 7.7.29 (The Cosine Function) (a) On $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, define $\cos x$ by

$$\cos x = \left\{ \begin{array}{ll} \sqrt{1-\sin^2 x} & \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ -\sqrt{1-\sin^2 x} & \text{if } x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \end{array} \right\}.$$

(b) Note that $\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$, and $\cos 0 = 1$.

(c) To extend the cosine function periodically, with period 2π, to $(-\infty, +\infty)$, we appeal to Exercise 6.2.19.

(d) Note that cosine is an *even* function. [See Exercise 6.2.5.]

(e) Note that $\forall x \in \mathbb{R}$, $\sin^2 x + \cos^2 x = 1$.

DERIVATIVES OF SINE AND COSINE

Theorem 7.7.30 *The sine and cosine functions are differentiable everywhere, and* $\forall x \in \mathbb{R}$, $\quad \frac{d}{dx}\sin x = \cos x \quad$ *and* $\quad \frac{d}{dx}\cos x = -\sin x$.

Proof. First prove these results true on $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, and then use the chain rule and periodicity (see Exercise 6.2.20) to extend them to all of $(-\infty, +\infty)$. ∎

Finally, we come to the matter of the trigonometric identities. The following theorem establishes the crucial ones.

21. For a definition based on infinite series, see Section 8.8.

Theorem 7.7.31 *The sine and cosine functions obey the following identities:*

$\forall x, y \in \mathbb{R},$ (a) $\sin(x+y) = \sin x \cos y + \cos x \sin y;$

(b) $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ *and* $\cos\left(\frac{\pi}{2} - x\right) = \sin x;$

(c) $\cos(x+y) = \cos x \cos y - \sin x \sin y.$

Proof of (a): Let x, y be fixed real numbers, and let $z = x + y$. Then $\forall t \in \mathbb{R}, \dfrac{d}{dt}[\sin t \cos(z-t) + \cos t \sin(z-t)]$

$= (\sin t)[-\sin(z-t)(-1)] + \cos(z-t)(\cos t) + (\cos t)[-\cos(z-t)] + \sin(z-t)(-\sin t)$

$= 0.$

Thus, $\sin t \cos(z-t) + \cos t \sin(z-t) = K$, a constant. (40)

Letting $t = 0$ in (40), we have $0 + 1 \sin z = K$. That is, $K = \sin z$. Letting $t = x$ in (40), we have

$$\sin x \cos(z-x) + \cos x \sin(z-x) = \sin z. \tag{41}$$

But $z = x + y$, so Equation (41) becomes

$$\sin x \cos y + \cos x \sin y = \sin(x+y).$$

The proofs of (b) and (c) are easy consequences of (a) and previous identities. ■

We stop with these identities, because from them and previous identities, we can derive all the remaining trigonometric identities in the familiar manner.

Exercise 7.7.32 State and prove the derivative formulas for the remaining trigonometric functions.

Exercise 7.7.33 State and prove the integral (antiderivative) formulas for all six trigonometric functions.

Theorem 7.7.34 (***Characterization of the Sine Function***) *The only function $F : \mathbb{R} \to \mathbb{R}$ such that*

(a) $\forall x \in \mathbb{R}, F''(x) = -F(x),$

(b) $F(0) = 0,$ *and*

(c) $F'(0) = 1$

is the function $F(x) = \sin x$.

Proof. Suppose $F : \mathbb{R} \to \mathbb{R}$ has properties (a)–(c). By (b), $F(0) = \sin 0$. So, suppose $x \neq 0$. Define $H(x) = F(x) - \sin x$. Then H has derivatives of all orders at x, and by Taylor's theorem, $\forall n \in \mathbb{N}$, $\exists\, c_n$ between 0 and $x \ni$

$$H(x) = H(0) + \tfrac{H'(0)}{1}x + \tfrac{H''(0)}{2!}x^2 + \cdots + \tfrac{H^{(n)}(0)}{n!}x^n + \tfrac{H^{(n+1)}(c_n)}{(n+1)!}x^{n+1} \qquad (42)$$

for some c_n between 0 and x. Now,

$$\begin{aligned}
H(0) &= F(0) - \sin 0 = 0 - 0 = 0\\
H'(0) &= F'(0) - \cos 0 = 1 - 1 = 0\\
H''(0) &= F''(0) + \sin 0 = -F(0) + 0 = 0\\
H'''(0) &= F'''(0) + \cos 0 = -F'(0) + 1 = -1 + 1 = 0\\
H^{(4)}(0) &= F^{(4)}(0) - \sin 0 = (F'')''(0) = (-F)''(0) = -F''(0) = F(0) = 0\\
&\vdots\\
H^{(n)}(0) &= 0.
\end{aligned}$$

Thus, by (42), $\exists\, c_n$ between 0 and x such that

$$H(x) = \frac{H^{(n+1)}(c_n)}{(n+1)!}x^{n+1}. \qquad (43)$$

Now, we take a closer look at $H^{(n+1)}(c_n)$, first for even values of $n+1$ and then for odd values. From (a) we get $F^{(2n)}(x) = (-1)^n F(x)$. We also know that $\cos^{(2n)} x = (-1)^n \sin x$. Thus,

$$\begin{aligned}
\left|H^{(2n)}(c_n)\right| &= |(-1)^n F(c_n) - (-1)^n \sin(c_n)| = |F(c_n) - \sin(c_n)|\\
&= |H(c_n)|.
\end{aligned}$$

Differentiating H one more time, we see that

$$\left|H^{(2n+1)}(c_n)\right| = |H'(c_n)|.$$

Now, both H and H' are differentiable over the closed interval between 0 and x, hence they are bounded there. Thus, $\exists\, B > 0 \ni \forall n \in \mathbb{N}$, in the expression (43) we have

$$\left|H^{(2n+1)}(c_n)\right| \leq B. \qquad (44)$$

Putting together (43) and (44), we have

$$\forall n \in \mathbb{N},\ |H(x)| \leq \frac{B|x^{n+1}|}{(n+1)!}.$$

Now, $\displaystyle\lim_{n\to\infty} \frac{|x^{n+1}|}{(n+1)!} = 0$ by Corollary 2.3.11. Thus, since limits preserve inequalities, $|H(x)| \leq 0$, so

$$H(x) = 0.$$

Therefore, $F(x) = \sin x$. ■

Theorem 7.7.35 (*Characterization of the Cosine Function*) *The only function $F : \mathbb{R} \to \mathbb{R}$ such that*

(a) $\forall x \in \mathbb{R},\ F''(x) = -F(x)$,

(b) $F(0) = 1$, *and*

(c) $F'(0) = 0$

is the function $F(x) = \cos x$.

Proof. Exercise. ■

7.8 *Improper Riemann Integrals

In our definition of the Riemann integral of a function over an interval, we require (1) that the function be bounded on the interval, and (2) that the interval of integration be bounded. Indeed, the theorems we have given, and their proofs, depend on these boundedness assumptions for their validity. There are times, however, when we wish to extend the notion of integration to situations in which one or both of these boundedness assumptions are not met. For example, since $\arcsin 1 = \frac{\pi}{2}$, we would expect that, in some sense,

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2} \qquad \text{(See Definition 7.7.18.)}$$

even though the function $1/\sqrt{1-t^2}$ is not Riemann integrable on $[0, 1]$. As we shall see below, this is an improper integral of "type I."

IMPROPER INTEGRALS OF TYPE I

Definition 7.8.1 Suppose $a < b$ and f is integrable on every closed subinterval of the form $[c, b]$, where $a < c < b$, but f is not integrable on $[a, b]$. Then we call $\int_a^b f$ an **improper integral of type I**. If $\lim\limits_{c \to a^+} \int_c^b$ exists (as a real number) then we say that the improper integral **converges**, and we write

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f.$$

If this limit does not exist, then we say that the improper integral **diverges**.

Definition 7.8.2 Suppose $a < b$ and f is integrable on every closed subinterval of the form $[a, c]$, where $a < c < b$, but f is not integrable on $[a, b]$. Then we call

$\int_a^b f$ an **improper integral of type I**. If $\lim_{c \to b^-} \int_a^c$ exists (as a real number) then we say that the improper integral **converges**, and we write

$$\int_a^b f = \lim_{c \to b^-} \int_a^c f.$$

If this limit does not exist, then we say that the improper integral **diverges**.

In Definitions 7.8.1 and 7.8.2, $\exists \varepsilon > 0$ such that either f is unbounded on $[a, a+\varepsilon)$ or f is unbounded on $(b-\varepsilon, b]$, but not both (see Exercise 1). We can extend the notion of improper integral to include cases in which both of these are true, or f is unbounded in some neighborhood of an interior point of $[a, b]$. The following definition covers these possibilities.

Definition 7.8.3 Suppose that, for some $a < c < b$, one or both of $\int_a^c f$ and $\int_c^b f$ are improper integrals in a sense previously defined. If the improper integrals (both) converge, then we say that $\int_a^b f$ **converges**, and write

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Otherwise, we say that $\int_a^b f$ **diverges**.

Example 7.8.4 Show that each of the following is an improper integral, and determine their convergence or divergence.

$$\text{(a)} \int_0^1 \frac{1}{x^2}\, dx \qquad\qquad \text{(b)} \int_0^1 \frac{1}{\sqrt{x}}\, dx$$

Solution. (a) The function $f(x) = \dfrac{1}{x^2}$ is continuous and bounded on any closed interval $[c, 1]$, for $0 < c < 1$. Thus, f is integrable on $[c, 1]$, but is not integrable on $[0, 1]$ since it is not bounded there (it is not even defined at $x = 0$). Thus, $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$ is an improper integral.

Now, for $0 < c < 1$, $\displaystyle\int_c^1 \frac{1}{x^2}\, dx = \left[-x^{-1}\right]_1^c = \frac{1}{c} - 1.$

Thus, $\displaystyle\lim_{c \to 0^+} \int_c^1 \frac{1}{x^2}\, dx = \lim_{c \to 0^+} \left(\frac{1}{c} - 1\right) = +\infty$. Therefore, $\displaystyle\int_0^1 \frac{1}{x^2}\, dx$ diverges.

(b) The function $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous and bounded on any closed interval $[c, 1]$, for $0 < c < 1$. Thus, f is integrable on $[c, 1]$, but is not integrable on $[0, 1]$ since it is not bounded there (it is not even defined at $x = 0$). Thus, $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$ is an improper integral.

Now, for $0 < c < 1$,

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\,\right]_1^c = 2(1-\sqrt{c}).$$

Thus, $\displaystyle\lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx = 2$, so $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges, and $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$. □

Theorem 7.8.5 (***The Comparison Test, I***) *Suppose $a < b$, and for all $a < x < b$, $0 \le f(x) \le g(x)$, and f, g are integrable over $[a,x]$. If $\int_a^b g$ converges, then so does $\int_a^b f$, and $\int_a^b f \le \int_a^b g$. (Of course, if $\int_a^b f$ diverges, then so does $\int_a^b g$.)*

Proof. Suppose $a < b$, and $\forall a < x < b$, $0 \le f(x) \le g(x)$, and f, g are integrable over $[a,x]$. Suppose $\int_a^b g$ converges. Since $g(x) \ge 0$ on $[a,b]$, the function $G(x) = \int_a^x g$ is monotone increasing on $[a,b)$. Since $\int_a^b g$ converges, $\lim_{x\to b^-} \int_a^x g$ exists and

$$\int_a^b g = \lim_{x\to b^-} \int_a^x g = \sup\left\{\int_a^x g : a < x < b\right\}.$$

(See Theorem 5.2.17 and Exercise 5.2.18.)

Similarly, the function $F(x) = \int_a^x f$ is monotone increasing on $[a,b)$ and $\forall x \in [a,b)$,

$$\int_a^x f \le \int_a^x g \le \int_a^b g \quad \text{(see above)}.$$

Thus, by Theorem 5.2.17, $\lim_{x\to b^-} \int_a^x f$ exists and is $\le \int_a^b g$. That is, $\int_a^b f$ converges and $\int_a^b f \le \int_a^b g$. ■

Remark 7.8.6 The comparison test remains true if we replace "$0 \le f(x) \le g(x)$" by "$0 \ge f(x) \ge g(x)$" or replace "$[a,x]$" by "$[x,b]$."

Example 7.8.7 $\displaystyle\int_0^1 \frac{1}{x^2+x}\,dx$ converges because $\forall x \in (0,1]$, $0 \le \dfrac{1}{x^2+x} \le \dfrac{1}{x^2}$ and $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ converges. □

Theorem 7.8.8 (***Absolute Convergence***) *Suppose $\int_a^b f$ is an improper integral of Type I. If $\int_a^b |f|$ converges, then so does $\int_a^b f$. [In this case, we say that $\int_a^b f$* ***converges absolutely****.]*

Proof. Suppose $\int_a^b f$ is an improper integral of Type I. We consider the case in which f is integrable over every subinterval $[c, b]$, where $a < c < b$. The other cases have similar proofs. Now, $\forall x \in (a, b]$,

$$-|f(x)| \leq f(x) \leq |f(x)|, \text{ so}$$

$$0 \leq f(x) + |f(x)| \leq 2|f(x)|. \tag{45}$$

But $\int_a^b |f|$ converges, by hypothesis, so $\int_a^b 2|f|$ also converges. Hence, by (45) and the comparison test, $\int_a^b (f(x) + |f(x)|)$ converges.

Now, $\forall a < c < b$,

$$\int_c^b f = \int_c^b [(f + |f|) - |f|] = \int_c^b (f + |f|) - \int_c^b |f|.$$

Then $\lim\limits_{c \to a^+} \int_c^b f$ exists, since both $\lim\limits_{c \to a^+} \int_c^b (f + |f|)$ and $\lim\limits_{c \to a^+} \int_c^b |f|$ exist. Therefore, $\int_a^b f$ converges. ∎

EXERCISE SET 7.8-A

1. Suppose the hypotheses of Definition 7.8.1 are met: $a < b$ and f is integrable on every closed subinterval of the form $[c, b]$, where $a < c < b$. But suppose that f is not integrable on $[a, b]$. Prove that $\exists \varepsilon > 0 \ni f$ is not bounded on $[a, a + \varepsilon)$. State a similar result about Definition 7.8.2.

2. Prove that if f is integrable on $[a, b]$, where $a < b$, then $\lim\limits_{c \to a^+} \int_c^b f = \int_a^b f$ and $\lim\limits_{c \to b^-} \int_a^c f = \int_a^b f$.

3. In each of the following, determine whether $\int_a^b f$ is an improper integral and, if so, determine its convergence or divergence. When possible, find the values of those improper integrals that converge.

   (a) $\displaystyle\int_0^1 \frac{1}{x}\, dx$ (b) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} dx$ (c) $\displaystyle\int_0^1 \frac{1}{e^x}\, dx$

   (d) $\displaystyle\int_1^2 \frac{1}{x \ln x}\, dx$ (e) $\displaystyle\int_0^1 \ln x \, dx$ (f) $\displaystyle\int_1^3 \frac{x}{\sqrt{x^2 - 1}}\, dx$

(g) $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx$ (h) $\displaystyle\int_{2}^{5} \frac{1}{(x-3)^2}\,dx$ (i) $\displaystyle\int_{-2}^{2} \frac{1}{x-1}\,dx$

(j) $\displaystyle\int_{-1}^{2} \frac{dx}{x^2+e^x}$ (k) $\displaystyle\int_{0}^{1} \frac{e^x\,dx}{\sqrt{x}}$ (l) $\displaystyle\int_{0}^{1} x\ln x\,dx$

(m) $\displaystyle\int_{0}^{\pi/2} \frac{\sin x}{\sqrt[3]{x}}\,dx$ (n) $\displaystyle\int_{0}^{1} \frac{1}{x\ln x}\,dx$

4. Without using the arcsine function, use the comparison test to prove that $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ converges.

5. Determine the convergence or divergence of $\displaystyle\int_0^1 \frac{dx}{x(\ln x)^2}$.

6. Prove that $\displaystyle\lim_{c\to 1^-}\int_{-c}^{c} \frac{x}{(x^2-1)^2}\,dx = 0$. Does this imply that $\displaystyle\int_{-1}^{1} \frac{x}{(x^2-1)^2}\,dx$ converges to 0? Graph the function $f(x) = \dfrac{x}{(x^2-1)^2}$ over the interval $(-1,1)$ and explain what is going on.

IMPROPER INTEGRALS OF TYPE II

Definition 7.8.9 Suppose $a \in \mathbb{R}$ and $\forall\, b > a$, f is integrable on $[a,b]$. Then we call $\int_a^{+\infty} f$ an **improper integral of type II**. If $\lim\limits_{b\to+\infty} \int_a^b f$ exists, we say that $\int_a^{+\infty} f$ **converges**, and write

$$\int_a^{+\infty} f = \lim_{b\to+\infty} \int_a^b f.$$

Otherwise, we say that $\int_a^{+\infty} f$ **diverges**.

Definition 7.8.10 Suppose $b \in \mathbb{R}$ and $\forall\, a < b$, f is integrable on $[a,b]$. Then we call $\int_{-\infty}^{b} f$ an **improper integral of type II**. If $\lim\limits_{a\to-\infty} \int_a^b f$ exists, we say that $\int_{-\infty}^{b} f$ **converges**, and write

$$\int_{-\infty}^{b} f = \lim_{a\to-\infty} \int_a^b f.$$

Otherwise, we say that $\int_{-\infty}^{b} f$ **diverges**.

Definition 7.8.11 Suppose that $\forall\, a < b$, f is integrable on $[a, b]$. Then we call $\int_{-\infty}^{+\infty} f$ an **improper integral of type II**. If for some $c \in \mathbb{R}$, both $\int_{-\infty}^{c} f$ and $\int_{c}^{+\infty} f$ converge, we say that $\int_{-\infty}^{+\infty} f$ **converges**, and write

$$\int_{-\infty}^{+\infty} f = \int_{-\infty}^{c} f + \int_{c}^{+\infty} f.$$

Otherwise, we say that $\int_{-\infty}^{+\infty} f$ **diverges**.

Examples 7.8.12 Determine the convergence or divergence of each of the following improper integrals.

$$\text{(a)} \int_1^{+\infty} \frac{1}{x^2}\,dx \qquad \text{(b)} \int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx \qquad \text{(c)} \int_{-\infty}^{+\infty} e^x\,dx$$

Solution. (a) The function $f(x) = \dfrac{1}{x^2}$ is continuous on the closed interval $[1, b]$ for all $b > 1$, hence it is integrable there. Moreover,

$$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left[-x^{-1}\right]_1^b = \lim_{b\to\infty} \left[1 - \frac{1}{b}\right] = 1.$$

Therefore, $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, and $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx = 1.$

(b) The function $f(x) = \dfrac{1}{\sqrt{x}}$ is continuous on the closed interval $[1, b]$ for all $b > 1$, hence it is integrable there. Moreover,

$$\lim_{b\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \left[2\sqrt{x}\,\right]_1^b = \lim_{b\to\infty} \left[2\sqrt{b} - 1\right] = +\infty.$$

Therefore, $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}}\,dx$ diverges.

(c) To investigate $\displaystyle\int_{-\infty}^{+\infty} e^x\,dx$ we must consider two separate integrals, say $\displaystyle\int_{-\infty}^{0} e^x\,dx$ and $\displaystyle\int_0^{+\infty} e^x\,dx$. First, $\displaystyle\lim_{a\to-\infty} \int_a^0 e^x\,dx = \lim_{a\to-\infty} \left[e^x\right]_a^0 = \lim_{a\to-\infty} \left[1 - e^a\right] =$ $1 - 0 = 1$. Thus, $\displaystyle\int_{-\infty}^{0} e^x\,dx$ converges (to 1).

Secondly, $\displaystyle\lim_{b\to+\infty} \int_0^b e^x\,dx = \lim_{b\to+\infty} \left[e^x\right]_0^b = \lim_{b\to+\infty} \left[e^b - 1\right] = +\infty$. Thus, $\displaystyle\int_0^{+\infty} e^x\,dx$ diverges. Since one of these two improper integrals diverges, we must say that $\displaystyle\int_{-\infty}^{+\infty} e^x\,dx$ diverges. $\square$

Theorem 7.8.13 (*Comparison Test, IIa*) *Suppose that* $\forall x \geq a$, $0 \leq f(x) \leq g(x)$. *If* $\int_a^{+\infty} g$ *converges, then so does* $\int_a^{+\infty} f$, *and* $\int_a^{+\infty} f \leq \int_a^{+\infty} g$.

(Of course, if $\int_a^{+\infty} f$ diverges, then so does $\int_a^{+\infty} g$.)

Proof. Exercise 8. ■

Theorem 7.8.14 (*Comparison Test, IIb*) *Suppose that $\forall x \leq b$, $0 \leq f(x) \leq g(x)$. If $\int_{-\infty}^{b} g$ converges, then so does $\int_{-\infty}^{b} f$, and $\int_{-\infty}^{b} f \leq \int_{-\infty}^{b} g$. (If $\int_{-\infty}^{b} f$ diverges, then so does $\int_{-\infty}^{b} g$.)*

Proof. Exercise 9. ■

Example 7.8.15 $\displaystyle\int_1^{+\infty} \frac{\sqrt{x}}{x^3+5}\,dx$ converges, since $\forall x \geq 1$, $\dfrac{\sqrt{x}}{x^3+5} \leq \dfrac{x}{x^3} = \dfrac{1}{x^2}$, and $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ was shown to converge in Example 7.8.12. □

Theorem 7.8.16 (*Absolute Convergence*) *Suppose $\int_a^b f$ (where $a = \infty$ or $b = -\infty$) is an improper integral of Type II. If $\int_a^b |f|$ converges, then so does $\int_a^b f$. [In this case, we say that $\int_a^b f$ **converges absolutely**.]*

Proof. Exercise 10. ■

Example 7.8.17 $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$ converges absolutely, since $\forall x \geq 1$, $\left|\dfrac{\sin x}{x^2}\right| \leq \dfrac{1}{x^2}$, and $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ was shown to converge in Example 7.8.12. □

Note: For an example showing that convergence does not imply absolute convergence, see Exercise 5.

IMPROPER INTEGRALS OF MIXED TYPES

Improper integrals can be of mixed types, as the following example shows. In such cases, we split the improper integral into two or more improper integrals and employ the appropriate methods from each type as needed.

Example 7.8.18 Determine the convergence or divergence of $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x}e^x}\,dx$.

Solution. We consider two separate improper integrals: $\int_0^1 \frac{1}{\sqrt{x e^x}}\, dx$ and $\int_1^{+\infty} \frac{1}{\sqrt{x e^x}}\, dx$.

(a) Consider $\int_0^1 \frac{1}{\sqrt{x e^x}}\, dx$. For $0 < x < 1$, $e^x > 1$ so $\sqrt{x e^x} > \sqrt{x}$, so $\frac{1}{\sqrt{x e^x}} < \frac{1}{\sqrt{x}}$. In Example 7.8.4 we proved that $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges, so by the comparison test, $\int_0^1 \frac{1}{\sqrt{x e^x}}\, dx$ converges.

(b) Consider $\int_1^{+\infty} \frac{1}{\sqrt{x e^x}}\, dx$. For $x > 1$, $\sqrt{x} > 1$, so $\sqrt{x e^x} > e^x$, so $\frac{1}{\sqrt{x e^x}} < \frac{1}{e^x}$. Now, $\lim_{b \to \infty} \int_1^b e^{-x} dx = \lim_{b \to \infty} \left[-e^{-x}\right]_1^b = \lim_{b \to \infty} \left[\frac{1}{e} - \frac{1}{b^b}\right] = 1/e$. Thus, $\int_1^{+\infty} \frac{1}{\sqrt{x e^x}}\, dx$ converges.

(c) By (a) and (b) together, $\int_0^{+\infty} \frac{1}{\sqrt{x e^x}}\, dx$ converges. □

FINAL CAUTION ON IMPROPER INTEGRALS

Care must be taken not to conclude that improper integrals behave algebraically like ordinary integrals. They do not. For example, we know that if f and g are integrable over $[a, b]$, then so is their product fg. But it is *not* true that if the *improper* integrals $\int_a^b f$ and $\int_a^b g$ converge, then so does $\int_a^b fg$. For example, $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges, but $\int_0^1 \left(\frac{1}{\sqrt{x}}\right)^2 dx$ does not. For further examples showing that improper integrals do not obey all the algebraic properties of ordinary integrals, see Exercises 4 and 5 below.

EXERCISE SET 7.8-B

1. Determine the convergence or divergence of each of the following improper integrals. Find the values of those that converge.

(a) $\displaystyle\int_1^\infty \frac{dx}{x^3}$ (b) $\displaystyle\int_0^\infty e^{-x}\,dx$

(c) $\displaystyle\int_{-\infty}^0 e^{-x}\,dx$ (d) $\displaystyle\int_0^\infty \frac{dx}{\sqrt{e^x}}$

(e) $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx$ (f) $\displaystyle\int_0^\infty \sin x\,dx$

(g) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$ (h) $\displaystyle\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

(i) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ (j) $\displaystyle\int_0^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

(k) $\displaystyle\int_0^\infty xe^{-x}\,dx$ (l) $\displaystyle\int_1^\infty \frac{x}{\sqrt{x^2-1}}\,dx$

(m) $\displaystyle\int_1^\infty \frac{x}{(x^2-1)^3}\,dx$ (n) $\displaystyle\int_1^\infty \frac{1}{x(\ln x)^2}\,dx$

2. For each of the following, determine the values of r for which the integral exists or converges, and determine the values of the integral in those cases.

   (a) $\int_1^\infty x^{-r}\,dx$ (b) $\int_0^1 x^{-r}\,dx$ (c) $\int_0^\infty x^{-r}\,dx$

3. Use the comparison test to determine whether the following improper integrals converge:

   (a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x^3+x}}$ (b) $\displaystyle\int_1^\infty \frac{x\,dx}{\sqrt{x^3+x}}$

   (c) $\displaystyle\int_0^\infty \frac{2x\,dx}{\sqrt{x+1}}$ (d) $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x+1}}$

   (e) $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+4x+6}$ (f) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+4x+6}$

4. Find a function f such that $\int_1^\infty f$ converges, but $\int_1^\infty \sqrt{f}$ does not.

5. A classic example of an improper integral that converges, but not absolutely, is $\displaystyle\int_\pi^\infty \frac{\sin x}{x}\,dx$. Verify this, as follows:

   (a) Prove that $\displaystyle\int_\pi^\infty \frac{\cos x}{x^2}\,dx$ converges (absolutely).

   (b) Use integration by parts and (a) to prove that $\displaystyle\int_\pi^\infty \frac{\sin x}{x}\,dx$ converges.

(c) Show that $\forall n \in \mathbb{N}$, $\displaystyle\int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}\,dx.$

(d) Show that $\forall k \in \mathbb{N}, \forall x \in \left[k\pi + \frac{\pi}{6},\ (k+1)\pi - \frac{\pi}{6}\right]$, $\displaystyle\frac{|\sin x|}{x} \geq \frac{1}{2(k+1)\pi}$.

(e) Use (c) and (d) to prove that $\displaystyle\lim_{n\to\infty} \int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx = +\infty$. [Recall the harmonic series, Example 2.5.16.]

6. One might expect that if $\int_0^\infty f$ converges, then $\forall x > 0$, $\lim_{x\to\infty} f(x) = 0$, but this is not necessarily true, as shown by $\int_0^\infty \cos x^2\,dx$.

   (a) Use integration by parts to obtain $\int \cos x^2\,dx = \dfrac{\sin x^2}{2x} + \displaystyle\int \frac{\sin x^2}{2x^2}\,dx$.

   (b) Use the result of (a) to prove that $\int_0^\infty \cos x^2\,dx$ converges. Show that $\lim_{x\to\infty} \cos x^2$ does not exist.

7. Prove that in Definition 7.8.11 the choice of c affects neither the convergence of $\int_{-\infty}^{+\infty} f$ nor its value.

8. (a) Prove[22] Theorem 7.8.13.

   (b) State and prove a modification of Theorem 7.8.13 in which "$0 \leq f(x) \leq g(x)$" is replaced by "$0 \geq f(x) \geq g(x)$." Illustrate graphically.

9. (a) Prove Theorem 7.8.14.

   (b) State and prove a modification of Theorem 7.8.14 in which "$0 \leq f(x) \leq g(x)$" is replaced by "$0 \geq f(x) \geq g(x)$." Illustrate graphically.

10. Prove Theorem 7.8.16.

7.9 *Lebesgue's Criterion for Riemann Integrability

> This section can be skipped or assigned as independent reading. The concepts are abstract and the proofs are challenging.

In this section we present what is perhaps the most celebrated criterion for the integrability of a bounded function on a compact interval. It is a straightforward criterion, involving only the set of points of discontinuity of the function. This truly remarkable criterion is easy to state but far from easy to prove. We begin by stating the criterion as a theorem. Then we develop its proof in stages.

22. Exercise 4.4-B.15 will be helpful in this and the following exercise.

Unfortunately, the proof is somewhat complicated. You may even decide the proof is indigestible, and skip it. But I recommend that you at least read the statement of the theorem and contemplate its significance. Even without the proof, simply knowing the statement of the theorem will enrich your understanding of Riemann integrability. At the point where you decide you have had enough of the proof, go directly to the examples and applications that follow.

Theorem 7.9.1 (***Lebesgue's Criterion for Riemann Integrability***) *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if the set of points of discontinuity of f in $[a, b]$ has measure zero.*

Obviously, to understand the statement of Lebesgue's criterion one must understand what is meant by "measure zero." This concept was defined in Section 3.4. We repeat the definition here.

Definition 7.9.2 A set A of real numbers has **measure zero** if $\forall \varepsilon > 0$, A can be covered by a countable collection of open intervals of total length less than ε. That is, A has measure zero iff $\forall \varepsilon > 0$, $\exists$ collection $\{I_n : n \in \mathbb{N}\}$ of open intervals $I_n = (a_n, b_n)$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) < \varepsilon$, where $l(I_n) = \mathit{length}(I_n) = (b_n - a_n)$.

Countable sets were defined and discussed in Section 2.8. In Theorem 3.4.20 we proved that every countable set has measure zero. Hence, "measure 0" is, in a sense, a generalization of "countable." In Section 3.4, we showed that some uncountable sets also have measure 0. In fact, the Cantor set has measure 0, even though it is uncountable.

Two additional concepts, treated earlier in (optional) sections of the book, are also essential in proving Theorem 7.9.1. First, we shall need to use the topological definition of "compact" set given in Section 3.3. We shall not repeat that definition here, but advise you to review Definition 3.3.2 through Corollary 3.3.12. Secondly, given a bounded function $f:\mathcal{D}(f) \to \mathbb{R}$, a nonempty set $A \subseteq \mathcal{D}(f)$, and a point $x_0 \in \mathcal{D}(f)$, we shall use the following concepts:

(a) the **oscillation of** f on A : $\omega_f(A) = \sup f(A) - \inf f(A)$;
(b) the **oscillation of** f at x_0 : $\omega_f(x_0) = \lim\limits_{\varepsilon \to 0^+} \omega_f\left(N_\varepsilon(x_0) \cap \mathcal{D}(f)\right)$.

The function $\omega_f:(0, +\infty) \to (0, +\infty)$ is called the **saltus function of** f, and $\omega_f(x)$ is called the **saltus of** f at x. These concepts were defined and discussed in Section 5.7. (See Definition 5.7.4 through Theorem 5.7.11.)

Now, we begin our proof of Lebesgue's criterion, in stages.

Remarks 7.9.3 Let $a > 0$.

(a) $\forall$ nonempty $A \subseteq \mathcal{D}(f)$, $\omega_f(A) \geq 0$.

(b) $\forall$ nonempty $A, B \subseteq \mathcal{D}(f)$, $A \subseteq B \Rightarrow \omega_f(A) \leq \omega_f(B)$.

(c) $\forall x \in [a, b]$, $\omega_f(x) \geq 0$.

(d) f is continuous at x_0 if and only if $\omega_f(x) = 0$.

(e) $\forall \delta > 0$, the set $S_\delta(f) = \{x \in \mathcal{D}(f) : \omega_f(x) \geq \delta\}$ is closed, hence compact.

(f) If S denotes the set of points of $[a, b]$ where f is discontinuous, then

$$S = \{x \in \mathcal{D}(f) : \omega_f(x) > 0\} = \bigcup_{\delta > 0} S_\delta(f) = \bigcup_{n=1}^{\infty} S_{1/n}(f).$$

Theorem 7.9.4 *Suppose f is Riemann integrable on $[a, b]$, and let $\delta > 0$. Then, $\forall \varepsilon > 0$, $S_\delta(f)$ can be covered by a* ***finite*** *collection of open intervals of total length less than ε. [Hence, $S_\delta(f)$ has measure 0.]*

Proof. Suppose f is Riemann integrable on $[a, b]$, and let $\delta > 0$ be fixed. Let $\varepsilon > 0$. Since f is integrable on $[a, b]$, there is a partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ such that

$$\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \frac{\delta\varepsilon}{2}.$$

Since $\mathcal{P}$ is a finite set, we can easily cover it by a finite collection of open intervals of total length less than $\varepsilon/2$; just use $\{N_{\varepsilon/5n}(x_i) : i = 1, 2, \cdots, n\}$. So, the proof will be complete if we can show that $S_\delta(f) - \mathcal{P}$ can be covered by a finite collection of open intervals of total length less than $\varepsilon/2$.

Let $\mathcal{N} = \{i : (x_{i-1}, x_i) \cap S_\delta(f) \neq \varnothing\}$. Then $\{(x_{i-1}, x_i) : i \in \mathcal{N}\}$ is a finite collection of open intervals covering $S_\delta(f) - \mathcal{P}$ with total length $\sum_{i \in \mathcal{N}} \Delta_i$, where $\Delta_i = x_i - x_{i-1}$. The proof will be complete if we can show that $\sum_{i \in \mathcal{N}} \Delta_i < \varepsilon/2$.

$$\begin{aligned} \text{Now, } i \in \mathcal{N} &\Rightarrow \exists x \in (x_{i-1}, x_i) \cap S_\delta(f) \\ &\Rightarrow \omega_f(x) \geq \delta \\ &\Rightarrow M_i(f) - m_i(f) \geq \delta \end{aligned}$$

where $M_i(f) = \sup f[x_{i-1}, x_i]$ and $m_i(f) = \inf f[x_{i-1}, x_i]$.

$$\text{Thus, } \sum_{i \in \mathcal{N}} [M_i(f) - m_i(f)]\Delta_i \geq \delta \sum_{i \in \mathcal{N}} \Delta_i. \tag{46}$$

$$\text{On the other hand, } \sum_{i \in \mathcal{N}} [M_i(f) - m_i(f)]\Delta_i \leq \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \frac{\delta\varepsilon}{2}. \tag{47}$$

Putting (46) and (47) together, $\delta \sum_{i \in \mathcal{N}} \triangle_i < \frac{\delta\varepsilon}{2}$. Therefore,

$$\sum_{i \in \mathcal{N}} \triangle_i < \frac{\varepsilon}{2}. \qquad \blacksquare$$

Theorem 7.9.5 (***Lebesgue's Criterion Is Necessary for Integrability***) *If f is bounded and integrable on $[a, b]$, then the set of discontinuities of f in $[a, b]$ has measure* 0.

Proof. Suppose f is bounded and integrable on $[a, b]$. Let $\varepsilon > 0$. As noted in Remark 7.9.3 (f), the set of discontinuities of f in $[a, b]$, is $S = \bigcup_{n=1}^{\infty} S_{1/n}(f)$. By Theorem 7.9.4, each $S_{1/n}(f)$ can be covered by a collection $\mathcal{C}_n$ of *finitely many* open intervals; i.e., $\mathcal{C}_n = \{I_{n1}, I_{n2}, \cdots, I_{nk_n}\}$ of total length less than $\frac{\varepsilon}{2^n}$. Then

(a) S can be covered by the collection $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ of all the intervals in all of these collections. That is, $S \subseteq \cup\, \mathcal{C} = \bigcup_{n=1}^{\infty} \left(\bigcup_{i=1}^{k_n} I_{ni} \right)$.

(b) $\mathcal{C}$ is a countable collection of open intervals, since it is the union of *countably many* finite collections of open intervals.

(c) The total length of all the intervals in $\mathcal{C}$ is less than $\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.

Therefore, S has measure zero. ■

Lemma 7.9.6 *Suppose f:$[a, b] \to \mathbb{R}$ is bounded, and $\varepsilon > 0$. If $\forall x \in [a, b]$, $\omega_f(x) < \varepsilon$, then $\exists\, \delta > 0 \ni \forall$ closed intervals $I \subseteq [a, b]$ of length $l(I) < \delta$, $\omega_f(I) < \varepsilon$.*

That is, if the oscillation of f at every point of $[a, b]$ is less than ε, then there is some $\delta > 0$ such that the oscillation of f is less than ε on all closed subintervals of $[a, b]$ with length less than δ.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and $\exists\, \varepsilon > 0 \ni \forall x \in [a, b]$, $\omega_f(x) < \varepsilon$. Keep ε fixed throughout the remainder of the proof. Now $\forall x \in [a, b]$, $\omega_f(x) = \lim_{\delta \to 0^+} \omega_f(N_\delta(x) \cap [a, b])$. Hence $\forall x \in [a, b]$, $\exists\, \delta_x > 0 \ni$

$$\omega_f(N_{\delta_x}(x) \cap [a, b]) < \varepsilon. \tag{48}$$

To simplify notation, $\forall x \in [a, b]$, let $U_x = N_{\delta_x/2}(x)$, the neighborhood of x of radius $\frac{1}{2}\delta_x$.

The collection $\{U_x : x \in [a,b]\}$ is an open cover of $[a,b]$, which is compact. Thus, it has a finite subcover, say

$$U_{x_1}, U_{x_2}, \cdots, U_{x_n}.$$

Let δ be the minimum radius of these neighborhoods; i.e.,

$$\delta = \tfrac{1}{2}\min\{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_n}\}.$$

Now, suppose I is any closed subinterval of $[a,b]$ such that $l(I) < \delta$. Since the open sets U_{xi} cover $[a,b]$, I has nonempty intersection with at least one of them; say $\exists\, x_0 \in I \cap U_{x_i}$.

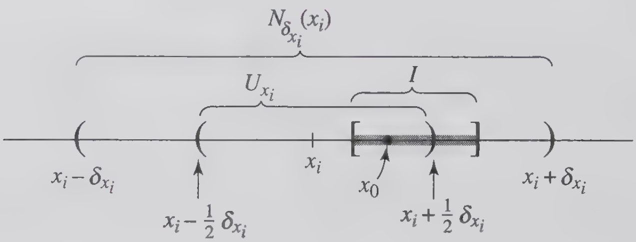

Figure 7.12

Note that $\delta \le \frac{1}{2}\delta_{x_i}$. Thus, I is a closed interval of length $l(I) < \frac{1}{2}\delta_{x_i}$ containing a point x_0 of the open interval U_{x_i}. So $I \subseteq N_{\delta_{x_i}}(x_i)$ (since U_{x_i} has radius $\frac{1}{2}\delta_{x_i}$). Therefore,

$$\begin{aligned} \omega_f(I) &\le \omega_f\left(N_{\delta_{x_i}}(x_i) \cap [a,b]\right) \quad \text{by Remark 7.9.3 (b)} \\ &< \varepsilon. \qquad \text{by (48)} \qquad \blacksquare \end{aligned}$$

Theorem 7.9.7 (***Lebesgue's Criterion Is Sufficient for Integrability***) *Suppose $f : [a,b] \to \mathbb{R}$ satisfies Lebesgue's criterion: the set of discontinuities of f in $[a,b]$ has measure 0. Then f is Riemann integrable on $[a,b]$.*

Proof. Suppose $f : [a,b] \to \mathbb{R}$ and the set of discontinuities of f in $[a,b]$ has measure 0. If f is constant on $[a,b]$, then f is integrable there, so suppose f is not constant on $[a,b]$. Let

$$m = \inf f[a,b] \quad \text{and} \quad M = \sup f[a,b].$$

Then $m < M$, so $M - m > 0$.

Let $S = \{x \in [a,b] : f \text{ is not continuous at } x\}$. Recall that

$$S = \{x \in [a,b] : \omega_f(x) > 0\} = \bigcup_{\delta > 0} S_\delta(f)$$

where $S_\delta(f) = \{x \in [a,b] : \omega_f(x) \ge \delta\}$. (See Remarks 7.9.3.)

Let $\varepsilon > 0$. To prove that f is integrable on $[a, b]$, it suffices to prove that there is a partition $\mathcal{P}$ of $[a, b]$ such that $\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$ (Riemann's condition, 7.2.14). Toward that end, let

$$\delta = \frac{\varepsilon}{2(b-a)}.$$

Since S has measure 0, and $S_\delta(f) \subseteq S$, $S_\delta(f)$ also has measure 0. Hence, there exists a countable collection of open intervals $\{I_1, I_2, \cdots I_n, \cdots\}$ such that

$$S_\delta(f) \subseteq \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} l(I_i) < \frac{\varepsilon}{2(M-m)} \tag{49}$$

where $l(I_i)$ denotes the length of I_i.

Now, from Remark 7.9.3 (e), $S_\delta(f)$ is compact, so it can be covered by finitely many of the open intervals I_i, say

$$S_\delta(f) \subseteq I_1 \cup I_2 \cup \cdots \cup I_N.$$

By de Morgan's law, $[a, b] - \bigcup_{i=1}^{N} I_i = \bigcap_{i=1}^{N} ([a, b] - I_i)$, which is the intersection of a collection of closed intervals, say

$$[a, b] - \bigcup_{i=1}^{N} I_i = \bigcap_{i=1}^{M} J_i, \tag{50}$$

where each J_i is a closed subinterval of $[a, b]$.

Note that $x \in J_i \Rightarrow x \notin S_\delta(f) \Rightarrow \omega_f(x) < \delta$. Thus, by Lemma 7.9.6, each J_i can be further subdivided into closed intervals J_{ij} of length less than δ_i (for some $\delta_i > 0$), such that

$$\omega_f(J_{ij}) < \delta \quad \text{and} \quad J_i = \bigcup_{j=1}^{K_i} J_{ij}. \tag{51}$$

Let $\mathcal{P} = \{$all endpoints of all the intervals I_i and J_{ij} described above$\}$ $\cup$ $\{a, b\}$. Say $\mathcal{P} = \{x_0, x_1, \cdots x_n\}$. Then

$$\begin{aligned} \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) &= \sum_{i=1}^{n} (M_i - m_i)\triangle_i \\ &= \sum_{i \in \mathcal{N}_1} (M_i - m_i)\triangle_i + \sum_{i \in \mathcal{N}_2} (M_i - m_i)\triangle_i \end{aligned} \tag{52}$$

where $\mathcal{N}_1 = \left\{ i : [x_{i-1}, x_i] \cap \bigcup_{i=1}^{N} I_i \neq \varnothing \right\}$ and $\mathcal{N}_2 = \left\{ i : [x_{i-1}, x_i] \cap \bigcup_{i=1}^{N} I_i = \varnothing \right\}$.

$$\begin{aligned}
\text{Now, } \sum_{i\in\mathcal{N}_1}(M_i-m_i)\triangle_i &\le \sum_{i\in\mathcal{N}_1}(M-m)\triangle_i \\
&= (M-m)\sum_{i\in\mathcal{N}_1} l(I_i) \\
&< (M-m)\frac{\varepsilon}{2(M-m)} = \frac{\varepsilon}{2}. \qquad (53)\\
&\text{[by (49)]}
\end{aligned}$$

$$\begin{aligned}
\text{Note that, } i\in\mathcal{N}_2 \Rightarrow [x_{i-1},x_i] &\subseteq \bigcap_{i=1}^{M} J_i \quad \text{by (50)} \\
&\text{and } \omega_f\left([x_{i-1},x_i]\right) < \delta \quad \text{by (51)} \\
&\Rightarrow M_i - m_i < \delta
\end{aligned}$$

$$\text{Thus, } \sum_{i\in\mathcal{N}_2}(M_i-m_i)\triangle_i < \sum_{i\in\mathcal{N}_2}\delta\triangle_i \le \delta(b-a) = \frac{\varepsilon}{2}. \qquad (54)$$

Putting together (52)–(54), we have $\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < \varepsilon$. Therefore, by Riemann's condition for integrability (7.2.14) f is integrable on $[a,b]$. ■

Examples 7.9.8

Lebesgue's criterion is extremely powerful. It can be applied to virtually every bounded function we have encountered in this course to determine whether that function is integrable on a given interval. In the following examples, remember that countable sets have measure 0.

(a) Bounded, piecewise continuous functions have only finitely many discontinuities, so they are integrable on every closed interval. Likewise, monotone functions are integrable on every closed interval since they have at most countably many discontinuities (see Theorem 5.2.20). The same conclusion carries over to piecewise monotone functions.

(b) The function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous except at $x = 0$, so it is integrable on every closed interval. Similar conclusions hold for other functions related to it.

(c) **Thomae's function**, defined in Example 5.1.12, is integrable on $[0,1]$ because the set of its discontinuities in $[0,1]$ is the set of rational numbers in $[0,1]$, which is a countable set. (See also Exercise 7.4.17.)

(d) The function $f : [0,1] \to \mathbb{R}$ defined in Example 7.4.10 by

$$f(x) = \left\{\begin{matrix} 1 & \text{if } x = 1/n \text{ for some } n\in\mathbb{N}, \\ 0 & \text{otherwise} \end{matrix}\right\}$$

is integrable on $[0,1]$. The set of its discontinuities in $[0,1]$ is $\left\{\frac{1}{n} : n\in\mathbb{N}\right\}$, which is countable.

(e) **Dirichlet's function**, defined in Example 5.1.11, is not integrable on any proper interval because it is discontinuous everywhere. Its set of discontinuities in any proper interval does not have measure 0. □

Example 7.9.9 (Integrating the Characteristic Function of a Set)

Let $A = \{x_1, x_2, \cdots, x_n\}$ where $a < x_1 < x_2 < \cdots < x_n < b$. Then the characteristic function χ_A has n discontinuities on $[a, b]$; namely, the points of A. By Theorem 7.4.9, χ_A is integrable on $[a, b]$, and

$$\int_a^b \chi_A = 0.$$

Thus, the integral of the characteristic function of a finite set is always 0. The integral of the characteristic function of an *infinite* set can also be 0. In Example 7.4.10 we showed that

$$\int_0^1 \chi_{\{\frac{1}{n}:n\in\mathbb{N}\}} = 0.$$

The integral of the characteristic function of an infinite set can also be any real number x; for example, if $x \geq 0$ and $[0, x] \subseteq [a, b]$, then

$$\int_a^b \chi_{[0,x]} = x.$$

In fact, the integral of the characteristic function of a proper interval is its length: if $[a, b] \subseteq [c, d]$, then

$$\int_c^d \chi_{[a,b]} = b - a.$$

On the other hand, the characteristic function of an infinite set is not necessarily integrable. For example, $\chi_{\mathbb{Q}}$ is the Dirichlet function, which is not integrable on any proper interval.

Now, let $\mathbf{C}$ denote the Cantor set, defined in Section 3.4. Recall that

$$C = \bigcap_{n=1}^{\infty} C_n$$

where C_n is the union of 2^n disjoint closed intervals of total length $\left(\frac{2}{3}\right)^n$ (see Definition 3.4.1). Note also that each χ_{C_n} is a step function and that $\forall n \in \mathbb{N}$,

$$0 \leq \chi_C \leq \chi_{C_n}$$

since $C \subseteq C_n$. This inequality shows that χ_C can be squeezed between two step functions, 0 and χ_{C_n} and

$$\int_0^1 (\chi_{C_n} - 0) = \int_0^1 \chi_{C_n} = \left(\tfrac{2}{3}\right)^n \to 0.$$

Hence, $\forall\varepsilon > 0$, χ_C can be squeezed between two step functions, 0 and $\chi_{C_{n_0}}$ such that

$$\textstyle\int_0^1 (\chi_{C_{n_0}} - 0) < \varepsilon. \tag{55}$$

By Theorem 7.4.14, this means that χ_C is integrable on $[0,1]$. Moreover, by inequality (55) and the forcing principle,

$$\textstyle\int_0^1 \chi_C = 0. \quad \square$$

CONDITIONS HOLDING "ALMOST EVERYWHERE"

Definition 7.9.10 A condition P is said to hold **almost everywhere** in a set A if $\{x \in A : x \text{ does not satisfy condition } P\}$ has measure 0.

The term "almost everywhere" is sometimes useful in describing a situation in analysis. For example, Lebesgue's criterion could be rephrased, "f is Riemann integrable on $[a,b]$ iff f is continuous almost everywhere in $[a,b]$." As another example, recall the Cantor function φ defined in Section 5.5. There we proved that φ is continuous and monotone increasing on $[0,1]$, $\varphi(0) = 0$ and $\varphi(1) = 1$. Yet in Exercise 6.3.13 we saw that $\varphi'(x) = 0$ almost everywhere in $[0,1]$. We shall see further uses of this language in the exercises below.

EXERCISE SET 7.9

1. Prove that
   (a) a subset of a set of measure 0 has measure 0.
   (b) the union of two sets of measure 0 has measure 0.
   (c) the union of finitely many sets of measure 0 has measure 0.
   (d) the union of countably many sets of measure 0 has measure 0.
2. Suppose $f(x) \geq 0$ on $[a,b]$, and $\int_a^b f = 0$. Prove that
   (a) $\forall c > 0$, $\{x \in [a,b] : f(x) \geq c\}$ has measure 0. (See Exercise 7.2.12.)
   (b) $f(x) = 0$ almost everywhere on $[a,b]$.
3. Suppose f and g are Riemann integrable on $[a,b]$, and $\int_a^b |f-g| = 0$. Prove that $f(x) = g(x)$ almost everywhere on $[a,b]$.
4. Suppose f is Riemann integrable on $[a,b]$, and $f(x) = g(x)$ almost everywhere on $[a,b]$. Must g also be integrable on $[a,b]$? (Compare with Theorem 7.4.9.)
5. Suppose f is Riemann integrable on $[a,b]$, and define F on $[a,b]$ by $F(x) = \int_a^x f$. Prove that $F'(x) = f(x)$ almost everywhere on $[a,b]$.

6. An example[23] of a bounded open set A such that χ_A is not integrable on any closed interval containing A:

   Let $0 < \varepsilon < \frac{1}{4}$, $\{r_n : n \in \mathbb{N}\}$ denote the set of rational numbers in $[0, 1]$, $J_n = \left(r_n - \frac{\varepsilon}{2^n}, r_n + \frac{\varepsilon}{2^n}\right)$, and $A = \bigcup_{n=1}^{\infty} J_n$. Let $a = \inf A$ and $b = \sup A$. By considering $\underline{S}(\chi_A, \mathcal{P})$ and $\overline{S}(\chi_A, \mathcal{P})$ for partitions $\mathcal{P}$ of $[a, b]$, show that A and χ_A have the desired properties.

7. Show[23] that the set A of Exercise 6 can be written as the union of *pairwise disjoint* open intervals, $A = \bigcup_{n=1}^{\infty} I_n$ (see Exercise 3.1.23). Then $\forall n \in \mathbb{N}$, let $f_n = \chi_{I_1} + \chi_{I_2} + \cdots + \chi_n$. Prove that $\{f_n\}$ is a sequence of Riemann integrable functions on $I = [a, b]$ such that $\forall x \in I$, the sequence of numbers $\{f_n(x)\}$ converges, but the limit function $f(x) = \lim\limits_{n \to \infty} f_n(x)$ is *not* Riemann integrable on I.

23. Exercises 6 and 7 were suggested by an anonymous reviewer of an early version of the manuscript.

Chapter 8

Infinite Series of Real Numbers

Sections 8.1 and 8.2 cover basic concepts and the standard convergence tests. Section 8.3 covers absolute and conditional convergence, alternating series, and rearrangements. Section 8.4 explores Cauchy products of series. Section 8.5 explores Abel's summation by parts, Dirichlet's test, and series of products. Sections 8.6–8.8 give a standard introduction to power series and analytic functions. Uniform convergence is left to Chapter 9.

8.1 Basic Concepts and Examples

Definition 8.1.1 If $\{a_n\}$ is a sequence of real numbers, the formal notation

$$\sum_{n=1}^{\infty} a_n \tag{1}$$

is called an **infinite series**, with **nth term** a_n. Corresponding to each infinite series (1) there is a related sequence $\{S_n\}$ called its **sequence of partial sums**,

$$S_n = \sum_{k=1}^{n} a_k. \tag{2}$$

We say that an infinite series (1) **converges** to a real number S (or **diverges**) if the corresponding sequence of partial sums (2) converges to S (or diverges). If $\lim_{n\to\infty} S_n = S$, we say that $\sum_{n=1}^{\infty} a_n$ has **sum** S, and write

$$\sum_{n=1}^{\infty} a_n = S. \tag{3}$$

If $\sum\limits_{n=1}^{\infty} a_n = +\infty$ (or $-\infty$) we do **not** say that $\sum\limits_{n=1}^{\infty} a_n$ converges, but rather say that it diverges to $+\infty$ (or $-\infty$).

We occasionally forego the sigma notation, and write the series (1) as

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots.$$

In this notation,

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n,$$

and in case of convergence,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \lim_{n\to\infty} (a_1 + a_2 + a_3 + \cdots + a_n).$$

Thus, it makes sense to write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

whenever this limit exists. The symbol $\sum\limits_{n=1}^{\infty} a_n$ can be used to represent either the series or its sum; we depend on the context to make clear which we mean.

As with sequences, we may begin the subscripts in a series with an integer other than 1. Thus, we may consider series of the form

$$\sum_{n=n_0}^{\infty} a_n = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots$$

as long as each term a_n is defined for all $n \geq n_0$. Furthermore, when it will not lead to confusion, we often denote a series merely by the more generic symbol, $\sum a_n$.

Example 8.1.2 The **geometric series** $\sum\limits_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$.

In Exercise 1.3.12 we saw that the n^{th} partial sum of this series is

$$\begin{aligned} S_n &= \sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1-r} \\ &= \frac{a}{1-r}\left[1 - r^{n+1}\right], \end{aligned}$$

and in Theorem 2.4.6 we noted that $\lim\limits_{n\to\infty} r^n = \left\{\begin{array}{ll} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ +\infty & \text{if } r > 1. \end{array}\right\}$. Thus, if

$a \neq 0$, the geometric series converges iff $|r| < 1$, and for all such r,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}. \quad \square$$

Examples 8.1.3 (a) The nonzero "constant" series, $\sum c = c + c + c + \cdots$, diverges if $c \neq 0$, since its sequence of partial sums is $\{S_n\} = \{nc\}$, which diverges.

(b) The nonzero "alternating constant" series $\sum_{n=0}^{\infty} (-1)^n c = c - c + c - c + \cdots$ diverges if $c \neq 0$, since its sequence of partial sums is $\{c, 0, c, 0, c, 0, \cdots\}$. $\square$

You will recall from your first-year calculus course that much attention was given there to various tests that can be used to determine whether a given series converges or diverges. The next theorem is the most basic of these tests.

Theorem 8.1.4 (*General Term Test*) *If $\sum a_n$ converges, then $a_n \to 0$. [Equivalently, if $a_n \not\to 0$, then $\sum a_n$ diverges.]*

Proof. Suppose $\sum a_n$ converges, say to S. Let $S_n = \sum_{k=1}^{n} a_k$. Then $S_n \to S$. Notice that $\forall n \in \mathbb{N}$,

$$\begin{aligned} S_{n+1} &= S_n + a_{n+1}, \text{ so} \\ a_{n+1} &= S_{n+1} - S_n. \end{aligned}$$

Thus, by the algebra of limits for sequences,

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} S_{n+1} - \lim_{n\to\infty} S_n \\ &= S - S = 0. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = 0$. ■

Example 8.1.5 A divergent series: $\sum_{n=1}^{\infty} \frac{n}{5n+11}$.

The general term test (Theorem 8.1.4) shows that this series diverges, since $\frac{n}{5n+11} \to \frac{1}{5} \neq 0$. $\square$

The general term test gives a **necessary condition for convergence**, not a sufficient condition. It cannot be used to prove that a series converges; it can only be used to prove that a series diverges. Thus, it is best viewed as a test for divergence, as in Example 8.1.5. The following example shows that a series can diverge even though its general term converges to 0.

Example 8.1.6 The **harmonic series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$

In Example 2.5.16 we showed that this series diverges; in fact, $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$ □

Example 8.1.7 The ***p*-series** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots.$

Convergence or divergence of this series depends on the value of p. We shall have more to say about this series in the next section. □

Example 8.1.8 Given any sequence $\{b_n\}$, the series $\sum_{n=1}^{\infty} (b_n - b_{n+1})$ is called a **telescoping series** because of the telescoping (or collapsing) nature of its partial sums:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} (b_k - b_{k+1}) \qquad (4) \\ &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_n - b_{n+1}) \\ &= b_1 - b_{n+1}. \end{aligned}$$

Thus, a telescoping series (4) converges if and only if the sequence $\{b_n\}$ converges. (See Exercises 7–14.) □

Definition 8.1.9 (Grouping by Inserting Parentheses) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series, and let $\{n_k\}$ be a strictly increasing sequence of positive integers. Define the sequence $\{b_k\}$ by

$$\begin{aligned} b_1 &= a_1 + a_2 + a_3 + \cdots + a_{n_1} \\ b_2 &= a_{n_1+1} + a_{n_2+2} + a_{n_3+3} + \cdots + a_{n_2} \\ b_3 &= a_{n_2+1} + a_{n_2+2} + a_{n_2+3} + \cdots + a_{n_3} \\ &\vdots \\ b_{k+1} &= a_{n_k+1} + a_{n_k+2} + a_{n_k+3} + \cdots + a_{n_{k+1}} \\ &\vdots \end{aligned}$$

Then $\sum_{n=1}^{\infty} b_n$ is said to be a **grouping** of the series $\sum_{n=1}^{\infty} a_n$ **by inserting parentheses.**

Theorem 8.1.10 (*Grouping by Inserting Parentheses*)

(a) *If $\sum a_n$ converges, then any grouping $\sum b_n$ formed from $\sum a_n$ by inserting parentheses also converges, and has the same sum.*

(b) *If $\sum a_n$ has a grouping that diverges, then $\sum a_n$ diverges.*

(c) *Some divergent series can be grouped by inserting parentheses to form a convergent series.*

Proof. Exercise 15. ■

Actually, we have already seen an application of this theorem in Chapter 2. Our proof that the harmonic series diverges, given in Section 2.5, used the method of inserting parentheses.

The remaining two theorems of this section are adaptations of theorems about sequences to the context of series. The first of these theorems is a straightforward adaptation of the Cauchy criterion of sequences.

Theorem 8.1.11 (*Cauchy Criterion for Convergence of Series*) *A series $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \ni n > m \geq n_0 \Rightarrow \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon$.*

Proof. Exercise 16. ■

The next theorem is a straightforward adaptation of the algebra of limits of sequences to the context of series.

Theorem 8.1.12 (*Linearity of Sums of Series*) *If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, and $c \in \mathbb{R}$, then both $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} ca_n$ converge, and*

(a) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(b) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$.

Proof. Exercise 17. ■

EXERCISE SET 8.1

1. Prove that $\sum_{n=0}^{\infty} \frac{9}{10^n}$ converges, and find its limit. What does this tell you about the infinite, nonterminating decimal $0.99999999\cdots$?

2. Find the sum of each of the following series, if it converges:
   (a) $0.0101010101\cdots$
   (b) $0.987698769876\cdots$
   (c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$
   (d) $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$
   (e) $\sum_{n=0}^{\infty} \frac{3^n + 4^n}{5^n}$
   (f) $\sum_{n=0}^{\infty} \frac{3^n + 5^n}{4^n}$
   (g) $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$
   (h) $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
   (i) $\sum_{n=1}^{\infty} \sin(n\pi)$
   (j) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

3. Prove that $\sum_{n=1}^{\infty} \cos nx$ diverges $\forall x \in \mathbb{R}$, and $\sum_{n=1}^{\infty} \sin nx$ diverges unless x is an integral multiple of π.

4. Define the m-tail of a series $\sum_{n=1}^{\infty} a_n$ to be the series $\sum_{n=m}^{\infty} a_n$. Show that $\forall m \in \mathbb{N}$, a given series converges if and only if its m-tail converges. Show that in case of convergence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{m-1} a_n + \sum_{n=m}^{\infty} a_n$. How does the behavior of m-tails of series differ from the behavior of m-tails of sequences? [See Definition 2.2.15 and Theorem 2.2.16.]

5. Prove that altering or deleting a finite number of terms of an infinite series does not affect its convergence or divergence.

6. Prove that every sequence $\{a_n\}$ is the sequence of partial sums of some series $\sum x_k$.

7. Prove the claim made in Example 8.1.8: A telescoping series $\sum_{n=1}^{\infty} (b_n - b_{n+1})$ converges if and only if the sequence $\{b_n\}$ converges; in fact, if $b_n \to B$, then $\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - B$.

8. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$ is a telescoping series, and find its sum. [Hint: separate $\frac{1}{n^2 + n}$ into two fractions, using the method of "partial fractions."]

9. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$ converges, and find its sum.

10. Prove that if x is not a negative integer, $\sum_{n=1}^{\infty} \frac{1}{(x+n)(x+n+1)} = \frac{1}{1+x}$.

11. Use the method of telescoping series to find $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

12. Use partial fractions and other results to find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

13. Prove that any series can be rewritten as a telescoping series.

14. Let $\{a_n\}$ be an arbitrary sequence of positive real numbers. Find a formula for the n^{th} partial sum of the series $\sum_{n=1}^{\infty} \ln\left(\frac{a_n}{a_{n+1}}\right)$. For what sequences $\{a_n\}$ does this series converge? In the case of convergence, what is the sum of the given series?

15. Prove Theorem 8.1.10.

16. Prove Theorem 8.1.11, the **Cauchy criterion** for series.

17. Prove Theorem 8.1.12.

8.2 Nonnegative Series

In this section we develop tests for convergence of series having all nonnegative terms. The assumption of nonnegativity greatly facilitates our investigation of convergence, and leads to techniques that are applicable to more general series as well.

Definition 8.2.1 A series $\sum a_n$ is said to be a **nonnegative series** (or a **series of nonnegative terms**) if, $\forall n$, $a_n \geq 0$.

Theorem 8.2.2 *A nonnegative series converges if and only if its sequence of partial sums is bounded above.*

Proof. Exercise 1. ■

Theorem 8.2.3 (*The Integral Test*) *Suppose* $\sum_{n=n_0}^{\infty} a_n$ *is a nonnegative series, and suppose* $f : [n_0, +\infty) \to \mathbb{R}$ *is a continuous, monotone decreasing*

function such that $\lim_{x\to\infty} f(x) = 0$. *Then the series* $\sum_{n=n_0}^{\infty} a_n$ *converges* $\Longleftrightarrow$ *the improper integral* $\int_{n_0}^{\infty} f$ *converges.*

Proof. Suppose $\sum_{n=n_0}^{\infty} a_n$ and f are as described in the hypotheses. Since f is monotone on $[n_0, +\infty)$, f is integrable on any compact subinterval of $[n_0, +\infty)$.

Let n be any integer $> n_0$, and consider the partition

$$\mathcal{P} = \{n_0, n_0 + 1, n_0 + 2, \cdots, n\}$$

of $[n_0, n]$. Since f is decreasing on $[n_0, +\infty)$, we see from Figure 8.1(a) that

$$\sum_{k=n_0}^{n-1} a_k = \sum_{k=n_0}^{n-1} f(k) = \overline{S}(f, \mathcal{P}) \geq \int_{n_0}^{n} f.$$

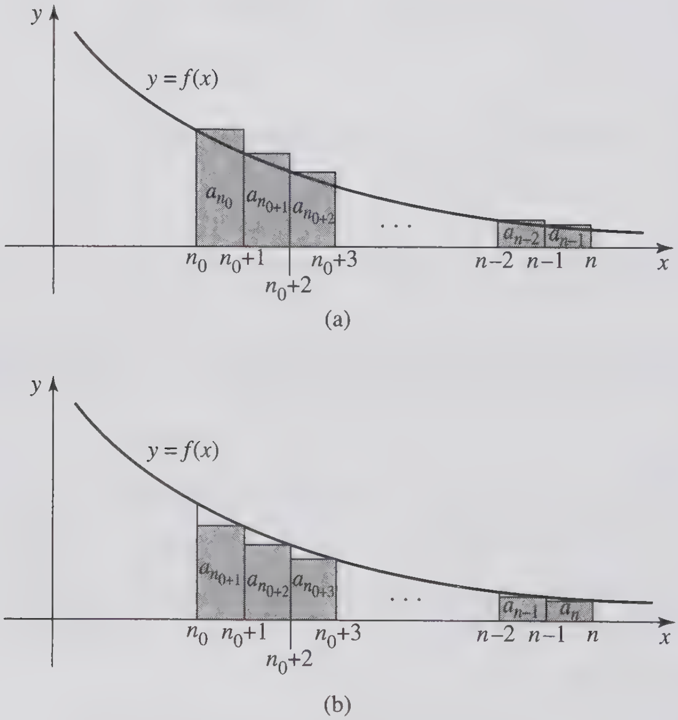

Figure 8.1

Thus, for all integers $n > n_0$,

$$\int_{n_0}^{n} f \le \left(\sum_{k=n_0}^{n} a_k \right) - a_n. \tag{5}$$

With the help of Figure 8.1(b) we see that

$$\int_{n_0}^{n} f \ge \underline{S}(f, \mathcal{P}) = \sum_{k=n_0+1}^{n} f(k) = \sum_{k=n_0+1}^{n} a_k.$$

Thus,

$$\int_{n_0}^{n} f \ge \left(\sum_{k=n_0}^{n} a_k \right) - a_{n_0},$$

so,

$$\sum_{k=n_0}^{n} a_k \le \int_{n_0}^{n} f + a_{n_0}. \tag{6}$$

Part 1 ($\Rightarrow$): Suppose $\sum_{n=n_0}^{\infty} a_n$ converges. Then its sequence of partial sums is bounded. So, by (5), the monotone increasing sequence $\left\{ \int_{n_0}^{n} f \right\}_{n=n_0}^{\infty}$ is bounded. Thus, $\lim_{n \to \infty} \int_{n_0}^{n} f$ exists, from which we can easily prove that the improper integral $\int_{n_0}^{\infty} f$ converges.

Part 2 ($\Leftarrow$): Suppose the improper integral $\int_{n_0}^{\infty} f$ converges. Then the sequence $\left\{ \int_{n_0}^{n} f \right\}_{n=n_0}^{\infty}$ converges, so it is bounded. Thus, by (6), the monotone increasing sequence of partial sums $\left\{ \sum_{k=n_0}^{n} a_n \right\}_{n=n_0}^{\infty}$ is bounded, hence converges. That is, $\sum_{n=n_0}^{\infty} a_n$ converges. ■

Example 8.2.4 Use the integral test to prove that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+10}$ diverges.

Solution. Let $f(x) = \frac{x}{x^2+10}$. Then $f'(x) = \frac{10-x^2}{(x^2+10)^2}$, so $f'(x) < 0$ when $x \ge 4$. Thus, f is continuous and monotone decreasing on $[4, \infty)$ and $\lim_{x \to \infty} \frac{x}{x^2+10} = 0$. Moreover, the improper integral $\int_4^{\infty} \frac{x}{x^2+10} dx$ diverges, since

$$\lim_{b \to \infty} \int_4^b \frac{x}{x^2+10} dx = \lim_{b \to \infty} \left[\frac{1}{2} \ln(x^2+10) \right]_4^b = +\infty.$$

Thus, by the integral test, $\sum_{n=4}^{\infty} \frac{n}{n^2+10}$ diverges. Therefore, $\sum_{n=1}^{\infty} \frac{n}{n^2+10}$ diverges. □

Corollary 8.2.5 (*p*-series) *Let p be a fixed real number. The* ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if $p > 1$ and diverges if $p \leq 1$.*

Proof. Let p be a fixed real number.

Case 1 ($p \leq 0$): By Theorem 5.6.15, $\frac{1}{n^p} \not\to 0$, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by the general term test.

Case 2 ($p > 0$): The function $f(x) = \frac{1}{x^p}$ is continuous and monotone decreasing on $[1, \infty)$, and $\lim_{x\to\infty} \frac{1}{x^p} = 0$, by Theorem 5.6.15. Hence, we can apply the integral test to the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Subcase 2a ($p \neq 1$): Then

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{b\to\infty} \int_1^b x^{-p} dx \\ &= \lim_{b\to\infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b \\ &= \frac{1}{1-p} \left[\lim_{b\to\infty} b^{1-p} - 1 \right]. \end{aligned} \tag{7}$$

If $p > 1$, $\lim_{b\to\infty} b^{1-p} = 0$ by Theorem 5.6.15. Then by (7), $\int_1^{\infty} \frac{1}{x^p} dx$ converges, so by the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

If $p < 1$, $\lim_{b\to\infty} b^{1-p} = +\infty$ by Theorem 5.6.14. Then by (7), $\int_1^{\infty} \frac{1}{x^p} dx$ diverges, so by the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Subcase 2b ($p = 1$): Then

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x} dx \\ &= \lim_{b\to\infty} \ln b = +\infty. \end{aligned}$$

In this case, $\int_1^{\infty} \frac{1}{x^p} dx$ diverges, so by the integral test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges. ■

COMPARISON TESTS

Theorem 8.2.6 (*Comparison Test*) *Suppose $\sum a_n$ and $\sum b_n$ are nonnegative series, and $\exists n_0 \in N \ni n \geq n_0 \Rightarrow a_n \leq b_n$. (When this holds we say that $\sum b_n$* **dominates** *$\sum a_n$.) Then*

(a) *if $\sum b_n$ converges, then so does $\sum a_n$.*

(b) *If $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof. Exercise 2. ■

Examples 8.2.7 Use the comparison test to prove that the following series converge (or diverge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n+8}{n^3-5n+7}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+6}{\sqrt{n^3+2}}$

Solution.

(a) It appears that as n gets large, the terms of the series (a) are something like $\frac{1}{n^2}$. Thus, we will try to show that this series converges by comparing it with $\sum \frac{1}{n^2}$, which is a convergent p-series $(p = 2)$. Observe that $\forall n \geq 8$,

$$\frac{n+8}{n^3-5n+7} < \frac{n+n}{n^3-5n} = \frac{2}{n^2-5} < \frac{2}{n^2-n^2/2} = \frac{4}{n^2}.$$

[Note: $n \geq 8 \Rightarrow n^2 > 10 \Rightarrow n^2/2 > 5 \Rightarrow n^2 - 5 > n^2 - n^2/2$.]

Since $\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges, the comparison test assures us that $\sum_{n=1}^{\infty} \frac{n+8}{n^3-5n+7}$ converges.

(b) As n gets large, the terms of the series (b) are something like $\frac{1}{\sqrt{n}}$. Thus, we will try to prove that the series (b) diverges by comparing it with $\sum \frac{1}{\sqrt{n}}$, which is a divergent p-series $\left(p = \frac{1}{2}\right)$. Observe that $\forall n \geq 2$,

$$\frac{n+6}{\sqrt{n^3+2}} > \frac{n}{\sqrt{n^3+n^2}} = \frac{1}{\sqrt{n+1}}.$$

Now, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, which diverges. Therefore, by the comparison test, $\sum_{n=1}^{\infty} \frac{n+6}{\sqrt{n^3+2}}$ diverges. □

Theorem 8.2.8 (***Limit Comparison Test***) *Suppose* $\sum a_n$ *and* $\sum b_n$ *are nonnegative series. Let* $\lim_{n\to\infty} \frac{a_n}{b_n} = \rho$ *(possibly* $+\infty$*).*

(a) *If* $0 < \rho < \infty$, *then either both series converge or both series diverge.*

(b) *If* $\rho = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
(Equivalently, if $\rho = 0$ *and* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.)*

(c) *If* $\rho = +\infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*
(Equivalently, if $\rho = +\infty$ *and* $\sum a_n$ *converges, then* $\sum b_n$ *converges.)*

Proof. Suppose $\sum a_n$ and $\sum b_n$ are nonnegative series, and let $\lim_{n\to\infty} \frac{a_n}{b_n} = \rho$ (possibly $+\infty$). The three statements of this theorem can be proved in two parts, as follows:

Part 1 ($0 \le \rho < \infty$, and $\sum b_n$ converges): Since $\left\{\frac{a_n}{b_n}\right\}$ converges to a finite number, it is bounded. Thus, $\exists M > 0 \ni \forall n \in \mathbb{N}$, $\frac{a_n}{b_n} \le M$. Then

(a) $\forall n \in \mathbb{N}$, $a_n \le M b_n$, and
(b) $\sum M b_n$ converges.

Thus, by the comparison test, $\sum a_n$ converges.

Part 2 ($0 < \rho \le \infty$, and $\sum b_n$ diverges): Since $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, $\exists M > 0$ and $\exists n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow \frac{a_n}{b_n} > M$. Then,

(a) $n \ge n_0 \Rightarrow a_n > M b_n$, and
(b) $\sum M b_n$ diverges.

Thus, by the comparison test, $\sum a_n$ diverges. ■

Examples 8.2.9 Use the limit comparison test to prove that the following series converge (or diverge):

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{5n} - 10}{3n + \sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution.

(a) As n gets large, the terms of the series (a) are something like $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. Thus, we will try to show that this series diverges by comparing it with $\sum \frac{1}{\sqrt{n}}$, which is a divergent p-series ($p = \frac{1}{2}$). Let $a_n = \frac{\sqrt{5n} - 10}{3n + \sqrt{n}}$ and $b_n =$

$\dfrac{1}{\sqrt{n}}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\left(\dfrac{\sqrt{5n}-10}{3n+\sqrt{n}}\cdot\dfrac{\sqrt{n}}{1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{\sqrt{5}\,n-10\sqrt{n}}{3n+\sqrt{n}}\right) =$ $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{5}-10/\sqrt{n}}{3+1/\sqrt{n}}\right) = \dfrac{\sqrt{5}}{3}$. Thus, $\rho = \dfrac{\sqrt{5}}{3}$, and so by the limit comparison test, the series (a) diverges.

(b) In Exercise 6.4.19 we proved that $\ln n < n$, so for large n the terms of the series (b) are somewhere between $\dfrac{1}{n^2}$ and $\dfrac{1}{n}$. Thus, it might be a good idea to compare $\dfrac{\ln n}{n^2}$ with $\dfrac{1}{n^{3/2}}$. Letting $a_n = \dfrac{\ln n}{n^2}$ and $b_n = \dfrac{1}{n^{3/2}}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{\ln n}{n^2}\cdot\frac{n^{3/2}}{1}\right) = \lim_{n\to\infty}\left(\frac{\ln n}{n^{1/2}}\right).$$

By L'Hôpital's rule,

$$\lim_{x\to\infty}\left(\frac{\ln x}{x^{1/2}}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}\right) = \lim_{x\to\infty}\left(\frac{2\sqrt{x}}{x}\right) = \lim_{x\to\infty}\left(\frac{2}{\sqrt{x}}\right) = 0.$$

In this case $\rho = 0$ and $\sum \dfrac{1}{n^{3/2}}$ is a convergent p-series $\left(p = \frac{3}{2}\right)$. Thus, Part (b) of the limit comparison test tells us that $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^2}$ converges. $\square$

THE RATIO TEST

Theorem 8.2.10 (***Ratio Test, Basic Form***) *Let $\sum a_n$ be a series with strictly positive terms.*

(a) *If $\exists\ 0 < r < 1$ such that $\dfrac{a_{n+1}}{a_n} \le r$ for all but finitely many n, then $\sum a_n$ converges.*

(b) *If $\dfrac{a_{n+1}}{a_n} \ge 1$ for all but finitely many n, then $\sum a_n$ diverges.*

Proof. Let $\sum a_n$ be a series with strictly positive terms.

(a) Suppose $\exists\ 0 < r < 1$ such that $\dfrac{a_{n+1}}{a_n} \le r$ for all but finitely many n. That is, $\exists\ n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow \dfrac{a_{n+1}}{a_n} \le r$. Then

$$\begin{aligned}
a_{n_0+1} &\le r a_{n_0} \\
a_{n_0+2} &\le r a_{n_0+1} \le r^2 a_{n_0} \\
&\vdots \\
a_{n_0+k} &\le r^k a_{n_0} \\
&\vdots
\end{aligned}$$

Apply the comparison test to the series $\sum_{n=n_0+1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{n_0} r^n$. Since the second series is a geometric series with $0 < r < 1$, it converges. Therefore, the above inequalities together with the comparison test show that $\sum_{n=n_0+1}^{\infty} a_n$ converges. Therefore, $\sum_{n=1}^{\infty} a_n$ converges.

(b) Suppose $\frac{a_{n+1}}{a_n} \geq 1$ for all but finitely many n. That is, $\exists\ n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \frac{a_{n+1}}{a_n} \geq 1 \Rightarrow a_{n+1} \geq a_n$. Then $n \geq n_0 \Rightarrow a_n \geq a_{n_0} > 0$. But then $a_n \not\to 0$. Therefore, by the general term test, $\sum_{n=1}^{\infty} a_n$ diverges. ■

Theorem 8.2.10 states the ratio test in basic form. However, the student may remember the ratio test by the more familiar limit form in which it appears in elementary calculus. The familiar form is frequently preferred since it suggests a direct procedure that can be used whenever a certain limit exists and is not 1. It is easily proved as a corollary to Theorem 8.2.10.

Corollary 8.2.11 (***Ratio Test, Limit Form***) *If $\sum a_n$ is a nonnegative series and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ (possibly $+\infty$) then*

(a) *if $L < 1$, the series $\sum a_n$ converges, and*

(b) *if $L > 1$, the series $\sum a_n$ diverges.*

Proof. Exercise 36. ■

Note that the ratio test gives us no information about the convergence or divergence of $\sum a_n$ when $L = 1$. It is customary to say that the ratio test "fails" when $L = 1$, although it would be more correct to say that it is "inconclusive" in this case. There are nonnegative series $\sum a_n$ with $L = 1$ that converge and others that diverge. For example, $L = 1$ for both $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$, yet the former diverges and the latter converges.

Examples 8.2.12 Use the ratio test to test the following series for convergence or divergence:

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$ (b) $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ (c) $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

Solution.

(a) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^2+1}{2^{n+1}}\cdot\frac{2^n}{n^2+1}=\lim_{n\to\infty}\frac{n^2+2n+2}{n^2+1}\cdot\frac{2^n}{2^{n+1}}$$
$$=\lim_{n\to\infty}\left(\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}\cdot\frac{1}{2}\right)=\frac{1}{2}.$$

In this case, $L=\frac{1}{2}<1$, so by the ratio test, $\sum_{n=1}^{\infty}\frac{n^2+1}{2^n}$ converges.

(b) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}=\lim_{n\to\infty}\frac{n+1}{3}=+\infty.$$

In this case, $L=+\infty$, so by the ratio test, $\sum_{n=1}^{\infty}\frac{n!}{3^n}$ diverges.

(c) $$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n+1}{(n+1)^3+1}\cdot\frac{n^3+1}{n}$$
$$=\lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{n^3+1}{n^3+3n^2+3n+2}=1.$$

In this case, $L=1$, so the ratio test gives no insight into the convergence or divergence of $\sum_{n=1}^{\infty}\frac{n}{n^3+1}$. However, $\frac{n}{n^3+1}<\frac{1}{n^2}$ so the comparison test tells us that this series converges. □

The limit form of the ratio test cannot be used in cases where $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ fails to exist. This shortcoming can be overcome by using upper and lower limits, defined in Section 2.9. The upper and lower limits always exist (but may be infinite).[1]

Theorem 8.2.13 (***Ratio Test, Upper and Lower Limit Form***) *Suppose* $\sum a_n$ *is a nonnegative series and let* $\underline{L}=\varliminf_{n\to\infty}\frac{a_{n+1}}{a_n}$ *and* $\overline{L}=\varlimsup_{n\to\infty}\frac{a_{n+1}}{a_n}$ *(possibly* $+\infty$*). Then*

(a) *if* $\overline{L}<1$, *the series* $\sum a_n$ *converges, and*

(b) *if* $\underline{L}>1$, *the series* $\sum a_n$ *diverges.*

Proof. Exercise 37. ■

1. The upper limit $\varlimsup_{k\to\infty}x_k$ and lower limit $\varliminf_{k\to\infty}x_k$ of a sequence $\{x_k\}$ are defined in Section 2.9, where it is also proved that, whenever $\{x_k\}$ converges, $\varlimsup_{k\to\infty}x_k=\lim_{k\to\infty}x_k=\varliminf_{k\to\infty}x_k$.

THE ROOT TEST

Theorem 8.2.14 (***Root Test, Basic Form***) *Let $\sum a_n$ be a nonnegative series.*

(a) *If $\exists\ 0 < r < 1 \ni \sqrt[n]{a_n} \le r$ for all but finitely many n, then $\sum a_n$ converges.*

(b) *If $\sqrt[n]{a_n} \ge 1$ for infinitely many n, then $\sum a_n$ diverges.*

Proof. Let $\sum a_n$ be a nonnegative series.

(a) Suppose $\exists\ 0 < r < 1 \ni \sqrt[n]{a_n} \le r$ for all but finitely many n. That is, $\exists\ n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow \sqrt[n]{a_n} \le r$. Then $n \ge n_0 \Rightarrow a_n \le r^n$. Since $0 < r < 1$, the geometric series $\sum r^n$ converges. Thus, by the comparison test, $\sum a_n$ converges.

(b) Suppose $\sqrt[n]{a_n} \ge 1$ for infinitely many n. Then $a_n \ge 1$ for infinitely many n, so $a_n \not\to 0$. Thus, by the general term test, $\sum a_n$ diverges. ■

The root test may be more familiar to you in its limit form, which can easily be proved as a corollary to Theorem 8.2.14.

Corollary 8.2.15 (***Root Test, Limit Form***) *If $\sum a_n$ is a nonnegative series, and $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = R$ (possibly $+\infty$) then*

(a) *if $R < 1$, the series $\sum a_n$ converges, and*

(b) *if $R > 1$, the series $\sum a_n$ diverges.*

Proof. Exercise 38. ■

As with the ratio test, the limit form of the root test cannot be used in cases where $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ fails to exist. However, this can be overcome by using the upper limit, which is guaranteed to exist (but may be infinite).

Theorem 8.2.16 (***Root Test, Upper Limit Form***)[2] *Suppose $\sum a_n$ is a nonnegative series and let $\overline{R} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{a_n}$ (possibly $+\infty$). Then*

(a) *if $\overline{R} < 1$, the series $\sum a_n$ converges, and*

(b) *if $\overline{R} > 1$, the series $\sum a_n$ diverges.*

Proof. Exercise 39. ■

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2. Compare with Theorem 8.2.13, in which both upper and lower limits are required.

Examples 8.2.17 Use the root test to test the following series for convergence or divergence:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^2+1}{2^n} \qquad (b) \sum_{n=1}^{\infty} \frac{n!}{3^n} \qquad (c) \sum_{n=1}^{\infty} \frac{n}{n^3+1}$$

Solution. These examples are the same as those considered in Example 8.2.12. We shall see here how using the root test differs from using the ratio test.

(a) $\lim_{n\to\infty} \sqrt[n]{\dfrac{n^2+1}{2^n}} = \lim_{n\to\infty} \dfrac{\sqrt[n]{n^2+1}}{2} = \frac{1}{2} \lim_{n\to\infty} \sqrt[n]{n^2+1}$. Now, $(\sqrt[n]{n})^2 = \sqrt[n]{n^2} < \sqrt[n]{n^2+1} < \sqrt[n]{2n^2} = \sqrt[n]{2}\,(\sqrt[n]{n})^2$. By Examples 2.3.8 and 2.3.9, $\sqrt[n]{n} \to 1$ and $\sqrt[n]{2} \to 1$. Thus, by the squeeze theorem, $\sqrt[n]{n^2+1} \to 1$. Therefore, in the notation of Theorem 8.2.15, $R = \lim_{n\to\infty} \sqrt[n]{\dfrac{n^2+1}{2^n}} = \frac{1}{2}$. Since $L < 1$, the limit form of the root test guarantees that $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$ converges.

(b) In the notation of Theorem 8.2.15, $R = \lim_{n\to\infty} \sqrt[n]{\dfrac{n!}{3^n}} = \lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{3}$. Now, $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$ [see Exercise 2.6.22]. Thus, $R = +\infty$. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{3^n}$ diverges (to $+\infty$).

(c) In the notation of Theorem 8.2.15, $R = \lim_{n\to\infty} \sqrt[n]{\dfrac{n}{n^3+1}}$. Now,
$\dfrac{1}{\sqrt[n]{2}}\dfrac{1}{(\sqrt[n]{n})^2} = \sqrt[n]{\dfrac{n}{2n^3}} < \sqrt[n]{\dfrac{n}{n^3+1}} < \sqrt[n]{\dfrac{n}{n^3}} = \sqrt[n]{\dfrac{1}{n^2}} = \dfrac{1}{(\sqrt[n]{n})^2}$. Since $\sqrt[n]{2} \to 1$ and $\sqrt[n]{n} \to 1$, these inequalities along with the squeeze principle tell us that $R = 1$, so we must use another test. As noted in Example 8.2.12 (c), this series converges by the comparison test. $\square$

The alert reader surely noticed that the ratio test and the root test produced identical limits ($L = R$) in all three series in Example 8.2.17. This does not always happen, but it happens often enough for us to look for a reason. In fact, if the ratio test proves that a nonnegative series converges (or diverges), the root test will also. But the converse relation is not true. The following theorem and example show what is going on.

Theorem 8.2.18 *Given any sequence $\{a_n\}$ of positive real numbers,*

$$\varliminf_{n\to\infty} \frac{a_{n+1}}{a_n} \le \varliminf_{n\to\infty} \sqrt[n]{a_n} \le \varlimsup_{n\to\infty} \sqrt[n]{a_n} \le \varlimsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Thus, if $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ exists, then $\lim_{n\to\infty} \sqrt[n]{a_n}$ also exists and is equal to it.

Proof. Let $\{a_n\}$ be a sequence of positive real numbers.

Part 1: We prove first that $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$.

Let $L = \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}$. If $L = +\infty$, there is nothing to prove. So, we suppose $L < +\infty$. Choose any $M > L$. Then by the ε-criterion for upper limits (see Theorem 2.9.7) $\exists n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow \frac{a_{n+1}}{a_n} < M$. Thus, $\forall k \in \mathbb{N}$,

$$a_{n_0+k} < M a_{n_0+(k-1)} < M^2 a_{n_0+(k-2)} < \cdots < M^k a_{n_0}.$$

That is,

$$n > n_0 \Rightarrow a_n < M^{n-n_0} a_{n_0} = M^n \frac{a_{n_0}}{M^{n_0}}$$

$$\Rightarrow \sqrt[n]{a_n} < M \sqrt[n]{c}, \text{ where } c = \frac{a_{n_0}}{M^{n_0}}.$$

Now, $\lim_{n\to\infty} \sqrt[n]{c} = 1$ (see Example 2.3.9). Thus, since upper limits preserve inequalities,

$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le M \overline{\lim_{n\to\infty}} \sqrt[n]{c} = M.$$

Hence, by the forcing principle (1.5.9), $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le L$.

Part 2: The proof that $\underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} \le \underline{\lim_{n\to\infty}} \sqrt[n]{a_n}$ is left as Exercise 40. ■

Theorem 8.2.18 shows that the root test is at least as strong as the ratio test, in the sense that if the ratio test proves that a nonnegative series converges (or diverges), the root test will also. The next example shows that the root test is actually "stronger" than the ratio test. That is, there are nonnegative series for which the ratio test is inconclusive but the root test works.

Example 8.2.19 Let $\sum_{n=0}^{\infty} a_n = 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$. That is, $\forall k \in \mathbb{N}$, $a_{2k} = \frac{1}{2^k}$, while $a_{2k+1} = \frac{1}{3^k}$.

When we attempt to use the ratio test on this series, we get

$$\frac{a_{n+1}}{a_n} = \left\{ \begin{array}{ll} \frac{1}{3^k} \Big/ \frac{1}{2^k} = \left(\frac{2}{3}\right)^k & \text{if } n = 2k, \\ \frac{1}{2^{k+1}} \Big/ \frac{1}{3^k} = \left(\frac{3}{2}\right)^k \cdot \frac{1}{2} & \text{if } n = 2k+1 \end{array} \right\}.$$

Thus, the subsequence consisting of even-numbered terms of $\left\{\frac{a_{n+1}}{a_n}\right\}$ converges to 0, while the subsequence consisting of odd-numbered terms diverges to $+\infty$. So for the ratio test, $\underline{L} = 0$ and $\overline{L} = +\infty$, and the ratio test is inconclusive.

We now test this series using the root test.

If $n = 2k$, $\sqrt[n]{a_n} = \sqrt[2k]{\frac{1}{2^k}} = \left(2^{-k}\right)^{1/2k} = 2^{-1/2} = \frac{1}{\sqrt{2}}$.

If $n = 2k+1$,

$$\sqrt[n]{a_n} = \sqrt[2k+1]{\frac{1}{3^k}} = \left(3^{-k}\right)^{1/(2k+1)} = 3^{\frac{-k}{2k+1}} = 3^{\frac{1}{2}\frac{1}{2k+1}-\frac{1}{2}} = \frac{(\sqrt{3})^{\frac{1}{2k+1}}}{\sqrt{3}} \to \frac{1}{\sqrt{3}}$$

by Example 2.3.9. Thus, $\underline{R} = \dfrac{1}{\sqrt{3}}$ and $\overline{R} = \dfrac{1}{\sqrt{2}}$. Since $\overline{R} < 1$, the root test proves that the given series converges. $\square$

*RAABE'S TEST

Several sophisticated tests for convergence of a nonnegative series $\sum a_k$ have been developed for use when the ratio and root tests are inconclusive. We include Raabe's test here because it is reasonably straightforward. Other, more complicated, tests such as Kummer's test and Gauss' test can be found in references such as [39], [47], [73], [85], [86], [87] [102], [111], [130], and [131]. To understand where Raabe's test (and the others) come from it is helpful to look at the situation in which the ratio test is inconclusive: that is, when $\dfrac{a_{k+1}}{a_k} \to 1$. It is plausible to expect $\sum a_k$ to diverge if $\left\{\dfrac{a_{k+1}}{a_k}\right\}$ converges to 1 "too rapidly," and $\sum a_k$ to converge if $\left\{\dfrac{a_{k+1}}{a_k}\right\}$ converges to 1 "slowly enough." Raabe's test captures this idea in mathematical precision.

Theorem 8.2.20 (***Raabe's Test***) *Suppose $\sum a_k$ is a series of positive terms.*

(a) *If $\exists\ r > 1 \ni \dfrac{a_{k+1}}{a_k} \leq 1 - \dfrac{r}{k}$ for all but finitely many k, then $\sum a_k$ converges.*

(b) *If $\exists\ 0 < r \leq 1 \ni \dfrac{a_{k+1}}{a_k} \geq 1 - \dfrac{r}{k}$ for all but finitely many k, then $\sum a_k$ diverges.*

Proof. (a) Suppose the hypotheses of (a) hold. Then $\exists\ r > 1$, $n_0 \in \mathbb{N} \ni$

$$\begin{aligned} k \geq n_0 \Rightarrow \frac{a_{k+1}}{a_k} &\leq 1 - \frac{r}{k} \\ &\Rightarrow k a_{k+1} \leq k a_k - r a_k = (k-1)a_k - (r-1)a_k \\ &\Rightarrow (k-1)a_k - k a_{k+1} \geq (r-1)a_k > 0. \end{aligned}$$

Thus, for $n \geq n_0$,

$$\sum_{k=n_0}^{n} [(k-1)a_k - k a_{k+1}] \geq \sum_{k=n_0}^{n} (r-1)a_k.$$

*An asterisk with a theorem, proof, or other material in this chapter indicates that the material is challenging and can be omitted.

Since the sum on the left is telescoping, this says

$$(n_0-1)a_{n_0} - na_{n+1} \geq (r-1)\sum_{k=n_0}^{n} a_k, \text{ so}$$
$$\sum_{k=n_0}^{n} a_k \leq \frac{(n_0-1)a_{n_0}}{r-1}.$$

Thus, the partial sums of $\sum a_k$ are bounded, so it converges.

(b) Suppose the hypotheses of (b) hold. Then $\exists\ 0 < r \leq 1,\ n_0 \in \mathbb{N} \ni$

$$k \geq n_0 \Rightarrow \frac{a_{k+1}}{a_k} \geq 1 - \frac{r}{k}$$
$$\Rightarrow ka_{k+1} \geq ka_k - ra_k = (k-r)a_k \geq (k-1)a_k.$$

Thus, $\{ka_{k+1}\}$ is monotone increasing for $k \geq n_0$. Thus, from these inequalities,

$$k \geq n_0 \Rightarrow ka_{k+1} \geq (n_0-1)a_{n_0}.$$

So, $\exists\ c > 0 \ni$

$$k \geq n_0 \Rightarrow a_{k+1} \geq \frac{c}{k}.$$

But $\sum \frac{c}{k}$ diverges, so by the comparison test, $\sum a_k$ diverges. ■

Corollary 8.2.21 (***Raabe's Test, Limit Form***) *Suppose $\sum a_k$ is a series of positive terms, and suppose $R = \lim_{k\to\infty} k\left(1 - \frac{a_{k+1}}{a_k}\right)$ exists.*

(a) *If $R > 1$, then $\sum a_k$ converges.*
(b) *If $0 < R < 1$, then $\sum a_k$ diverges.*
(c) *The test is inconclusive if $R = 1$.*

Proof. Exercise 42. ■

Example 8.2.22 Both the ratio test and the root test are inconclusive for the p-series $\sum \frac{1}{k^p}$ (Exercise 35). To apply Raabe's test to the p-series we first note that

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^{-p}}{k^{-p}} = \frac{k^p}{(k+1)^p}.$$

To find $\lim_{k\to\infty} k\left(1-\frac{a_{k+1}}{a_k}\right) = \lim_{k\to\infty} k\left[1-\frac{k^p}{(k+1)^p}\right]$, we use L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to\infty} x\left[1-\frac{x^p}{(x+1)^p}\right] &= \lim_{x\to\infty} x\left[\frac{(x+1)^p - x^p}{(x+1)^p}\right]\\
&= \lim_{x\to\infty} x\left[\frac{\left(1+\frac{1}{x}\right)^p - 1}{1}\cdot\frac{1}{\left(1+\frac{1}{x}\right)^p}\right]\\
&= \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^p - 1}{\frac{1}{x}} \cdot \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)^p} = \lim_{x\to\infty} \frac{\left(1+\frac{1}{x}\right)^p - 1}{\frac{1}{x}}\\
&= \lim_{x\to\infty} \frac{p\left(1+\frac{1}{x}\right)^{p-1}\left(-x^{-2}\right)}{-x^{-2}} \qquad \text{(using L'Hôpital's rule)}\\
&= \lim_{x\to\infty} p\left(1+\frac{1}{x}\right)^{p-1} = p.
\end{aligned}$$

Thus, Raabe's test tells us that the p-series converges if $p > 1$ and diverges if $p < 1$. It is inconclusive if $p = 1$. □

EXERCISE SET 8.2

1. Prove Theorem 8.2.2.

2. Prove Theorem 8.2.6. Also, express this theorem using the language of one series "dominating" another.

   In Exercises 3–12, write the given series in the form $\sum a_n$ and use tests given in Sections 8.1 and 8.2 to determine whether the series converges or diverges.

3. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots$

4. $\frac{1}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{3}{9\cdot 11} + \frac{4}{13\cdot 15} + \cdots$

5. $\frac{1}{3} + \frac{\sqrt{2}}{5} + \frac{\sqrt{3}}{7} + \frac{\sqrt{4}}{9} + \frac{\sqrt{5}}{11} + \frac{\sqrt{6}}{13} + \cdots$

6. $\frac{1}{1\cdot 3} + \frac{\sqrt{2}}{2\cdot 4} + \frac{\sqrt{3}}{3\cdot 5} + \frac{\sqrt{4}}{4\cdot 6} + \frac{\sqrt{5}}{5\cdot 7} + \cdots$

7. $\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \frac{1}{\sqrt{4\cdot 5}} + \frac{1}{\sqrt{5\cdot 6}} + \cdots$

8. $\dfrac{\sqrt{2}}{3\cdot 5}+\dfrac{\sqrt{4}}{5\cdot 7}+\dfrac{\sqrt{6}}{7\cdot 9}+\dfrac{\sqrt{8}}{9\cdot 11}+\dfrac{\sqrt{10}}{11\cdot 13}+\cdots$

9. $\dfrac{1}{3}+\dfrac{1\cdot 2}{3\cdot 5}+\dfrac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\dfrac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}+\cdots$

10. $\dfrac{1}{\sqrt{1\cdot 2\cdot 3}}+\dfrac{2}{\sqrt{2\cdot 3\cdot 4}}+\dfrac{3}{\sqrt{3\cdot 4\cdot 5}}+\dfrac{4}{\sqrt{4\cdot 5\cdot 6}}+\dfrac{5}{\sqrt{5\cdot 6\cdot 7}}+\cdots$

11. $\dfrac{1}{e}+\dfrac{4}{e^2}+\dfrac{27}{e^3}+\dfrac{64}{e^4}+\dfrac{3,125}{e^5}+\cdots$

12. $\dfrac{1}{\ln 2}+\dfrac{1}{(\ln 3)^2}+\dfrac{1}{(\ln 4)^3}+\dfrac{1}{(\ln 5)^4}+\cdots$

In Exercises 13–30, use tests given in Sections 8.1 and 8.2 to determine whether the series converges or diverges.

13. $\displaystyle\sum_{n=1}^{\infty}\frac{3n}{n^2-5}$

14. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3n^2-10n}$

15. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$

16. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{5n}$

17. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln n}{n^3}$

18. $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{n}}{\ln n}$

19. $\displaystyle\sum_{n=1}^{\infty}\frac{1+\cos n}{\sin^2 n}$

20. $\displaystyle\sum_{n=1}^{\infty}\frac{7\sqrt{n}}{n^2+6\sqrt[3]{n}}$

21. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$

22. $\displaystyle\sum_{n=1}^{\infty}\frac{n^3}{3^n}$

23. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{e^n}$

24. $\displaystyle\sum_{n=2}^{\infty}\frac{\ln(n^3)}{n^2}$

25. $\displaystyle\sum_{n=1}^{\infty}\sin^n\left(\frac{1}{n}\right)$

26. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3n+5}{2n+1}\right)^{n/2}$

27. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-1}{3n}\right)^{2n}$

28. $\displaystyle\sum_{n=1}^{\infty}\cos^n\left(\frac{n+1}{2n}\right)$

29. $1+\dfrac{1}{\sqrt[3]{3}}+\dfrac{1}{\sqrt[3]{2}}+\dfrac{1}{\sqrt[3]{5}}+\dfrac{1}{\sqrt[3]{4}}+\dfrac{1}{\sqrt[3]{7}}+\dfrac{1}{\sqrt[3]{6}}+\dfrac{1}{\sqrt[3]{9}}+\dfrac{1}{\sqrt[3]{8}}+\cdots$

30. $\dfrac{1}{2^2}+1+\dfrac{1}{4^2}+\dfrac{1}{3^2}+\dfrac{1}{6^2}+\dfrac{1}{5^2}+\dfrac{1}{8^2}+\dfrac{1}{7^2}+\dfrac{1}{10^2}+\dfrac{1}{9^2}+\cdots$

31. Prove that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converges if and only if $p>1$.

32. Prove that $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)[\ln(\ln n)]^p}$ converges if and only if $p > 1$.

33. Prove that $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$ converges if and only if $p > 1$.

34. Determine whether the series $\frac{\sqrt{2}\ln 4}{1 \cdot 2} + \frac{\sqrt{5}\ln 7}{2 \cdot 3} + \frac{\sqrt{8}\ln 10}{3 \cdot 4} + \frac{\sqrt{11}\ln 13}{4 \cdot 5} + \cdots$ converges or diverges. [Hint: Use Exercise 33.]

35. Prove that both the ratio test and the root test are inconclusive for the p-series, $\sum \frac{1}{k^p}$.

36. Prove Theorem 8.2.11.

37. Prove Theorem 8.2.13. [Use Theorems 2.9.7 and 8.2.10.]

38. Prove Theorem 8.2.15. [See proof of Exercise 39.]

39. Prove Theorem 8.2.16 [Use Theorems 2.9.7 and 8.2.14.]

40. Complete Part 2 of the proof of Theorem 8.2.18.

41. **Euler's constant** $\left(\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n\right)$: Although the harmonic series $\sum \frac{1}{k}$ diverges, there is an interesting relationship between its partial sums and $\ln n$. In fact, we shall show that as $n \to \infty$ their difference converges to a constant, denoted γ, called **Euler's constant**. Define the sequence $\{\gamma_n\}$ by

$$\gamma_n = \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}\right) - \ln n.$$

Using techniques used in the proof of the integral test (8.2.3) show that $\gamma_n > 0$, and hence $\{\gamma_n\}$ is bounded below. To see that $\{\gamma_n\}$ is monotone decreasing, start by showing that

$$\gamma_{n+1} - \gamma_n = \tfrac{1}{n+1} - \ln(n+1) + \ln n = \tfrac{1}{n+1} - \tfrac{\ln(n+1) - \ln n}{(n+1) - n},$$

and then apply the mean value theorem. Conclude that $\gamma = \lim_{n\to\infty} \gamma_n$ exists. [γ is approximately 0.557215665, to nine decimal places. It is not known whether this number is rational or irrational.]

42. Prove Corollary 8.2.21.

43. Use Raabe's test to prove that $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k)}$ diverges.

44. Determine whether the series $\sum_{k=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k+2)}$ converges.

45. Use Raabe's test to find values of p for which $\sum_{k=1}^{\infty} \left[\dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)}\right]^p$ converges and values of p for which it diverges.

46. For $a, b, c > 0$, use Raabe's test to prove that the **hypergeometric series**

$$1 + \frac{ab}{1!c} + \frac{a(a+1)b(b+1)}{2!c(c+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} + \cdots$$

converges when $c > a + b$ and diverges when $c < a + b$. (Assume c is not 0 or a negative integer.)

*47. **Comparison of $\sum a_k$ and $\sum \ln(1 - a_k)$:** Suppose that $\forall k \in \mathbb{N}$, $0 < a_k < 1$. Prove that

(a) $a_k \to 0 \Leftrightarrow \ln(1 - a_k) \to 0$;

(b) $\sum a_k$ converges $\Leftrightarrow \sum \ln(1-a_k)$ converges. [Use the limit comparison test, keeping L'Hôpital's rule in mind.]

(c) $\sum a_k$ diverges $\Leftrightarrow \lim_{k\to\infty} (1 - a_1)(1 - a_2) \cdots (1 - a_k) = 0$. [Apply (b).]

*48. **A sometimes useful test to prove $a_k \to 0$:** Suppose $\{a_k\}$ is monotone decreasing and let $b_k = 1 - \dfrac{a_{k+1}}{a_k}$. Prove that $a_k \to 0 \Leftrightarrow \sum b_k = +\infty$. [Apply Exercise 47.]

*49. Apply Exercise 48 to prove that $\dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdot \cdots \cdot (2k)} \to 0$. [Compare with Exercise 43.]

8.3 Series with Positive and Negative Terms

There is only one way a nonnegative series can diverge, and that is to $+\infty$. Similarly, the only way a nonpositive series can diverge is to $-\infty$. As we have seen in Section 8.2, convergence of such a series is equivalent to boundedness of its sequence of partial sums. In this section we shall consider convergence of more general series, whose terms may be positive, negative, or zero. As we shall see, the convergence behavior of such series can be much more complicated than that of nonnegative series.

ALTERNATING SERIES

Definition 8.3.1 If $\{a_n\}$ is a sequence of positive numbers, then both of the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ and $\sum_{n=1}^{\infty}(-1)^{n}a_n$ are called **alternating series.**

Since it is more common to write alternating series starting with a positive term, we shall usually use the first of these two forms to represent a generic alternating series. It will be clear in what follows that our results can be made to apply to alternating series of the second form as well. There is a simple test for determining whether certain alternating series converge.

Theorem 8.3.2 (***Alternating Series Test***) *If $\{a_n\}$ is a monotone decreasing sequence of positive numbers, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if and only if $a_n \to 0$.*

Proof. Let $\{a_n\}$ be a monotone decreasing sequence of positive numbers.

Part 1: Suppose $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. Then, by Theorem 8.1.4 (the general term test), $(-1)^{n+1}a_n \to 0$, so $a_n \to 0$.

Part 2: Suppose $a_n \to 0$. Consider the even-numbered partial sums, $S_{2n} = \sum_{k=1}^{2n}(-1)^{k+1}a_k$. Observe that $\{S_{2n}\}$ is monotone increasing, since

$$\begin{aligned} S_{2(n+1)} - S_{2n} &= [S_{2n} + a_{2n+1} - a_{2n+2}] - S_{2n} \\ &= a_{2n+1} - a_{2n+2} \\ &\geq 0, \text{ because } \{a_n\} \text{ is monotone decreasing.} \end{aligned}$$

Next, observe that $\{S_{2n}\}$ is bounded above, since

$$\begin{aligned} S_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} + a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \\ &\leq a_1, \text{ since } \{a_n\} \text{ is monotone decreasing and positive.} \end{aligned}$$

Therefore, $\{S_{2n}\}$ converges, by the monotone convergence theorem (2.5.3).

Now, $S_{2n+1} = S_{2n} - a_{2n+1}$. Thus, by the algebra of limits of sequences, $\{S_{2n+1}\}$ converges, and

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} S_{2n}.$$

Therefore, $\{S_n\}$ converges (see Exercise 2.6.7). That is, $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges. ■

Example 8.3.3 The **alternating harmonic series**

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\cdots$$

converges, by the alternating series test.

In the case of an alternating series, we can do more than tell whether it converges. If it converges, we can calculate the actual sum of the series to any specified degree of accuracy. The following theorem tells how this can be done.

Theorem 8.3.4 (***Sum of Alternating Series***) *If $\{a_n\}$ is a monotone decreasing sequence of positive numbers with $a_n \to 0$, and $S = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$, then, $\forall n \in \mathbb{N}$,*

(a) $S_{2n} < S < S_{2n+1}$, *and*

(b) $|S - S_n| < a_{n+1}$.

Proof. Suppose $\{a_n\}$ is a monotone decreasing sequence of positive numbers with $a_n \to 0$. By Theorem 8.3.2, $S = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$ exists. Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} S &= S_{2n} + a_{2n+1} - a_{2n+2} + a_{2n+3} - a_{2n+4} + \cdots \\ &= S_{2n} + (a_{2n+1} - a_{2n+2}) + (a_{2n+3} - a_{2n+4}) + \cdots \\ &> S_{2n}. \qquad (8) \end{aligned}$$

From these relations we also have

$$\begin{aligned} 0 < S - S_{2n} &= a_{2n+1} - a_{2n+2} + a_{2n+3} - a_{2n+4} + a_{2n+5} \cdots \\ &= a_{2n+1} - (a_{2n+2} - a_{2n+3}) - (a_{2n+4} - a_{2n+5}) - \cdots \\ &< a_{2n+1}. \qquad (9) \end{aligned}$$

Similarly,

$$\begin{aligned} S &= S_{2n+1} - (a_{2n+2} - a_{2n+3}) - (a_{2n+4} - a_{2n+5}) - \cdots \\ &< S_{2n+1}, \qquad (10) \end{aligned}$$

and

$$\begin{aligned}0 < S_{2n+1} - S &= a_{2n+2} - a_{2n+3} + a_{2n+4} - a_{2n+5} + a_{2n+6} - \cdots \\ &= a_{2n+2} - (a_{2n+3} - a_{2n+4}) - (a_{2n+5} - a_{2n+6}) - \cdots \\ &< a_{2n+2}. \qquad (11)\end{aligned}$$

Putting (8) and (10) together, we have $S_{2n} < S < S_{2n+1}$, and putting (9) and (11) together, we have $|S - S_m| < a_{m+1}$ regardless of whether m is odd or even. ■

Example 8.3.5 Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+5n+2}$ converges, and find its sum to three decimal place accuracy.

Solution. The sequence $\left\{\frac{1}{n^2+5n+2}\right\}$ is a monotone decreasing sequence of positive numbers converging to 0 (show). So, by Theorem 8.3.2, we may let $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+5n+2}$. By Theorem 8.3.4, we know that $\forall n \in \mathbb{N}$,

$$|S - S_n| < \frac{1}{(n+1)^2 + 5(n+1) + 2} = \frac{1}{n^2+7n+8}.$$

To find S to three decimal place accuracy it is sufficient to calculate S_n for n large enough that $|S - S_n| < 0.0005$. For this it will be sufficient to take any n such that

$$\frac{1}{n^2+7n+8} < \frac{5}{10,000} = \frac{1}{2000}$$

$$\text{i.e., } n^2 + 7n + 8 > 2000$$
$$n(n+7) > 1992.$$

By direct calculation we find that this inequality is satisfied when $n \geq 42$. Thus, S_{42} will have the desired accuracy. Resorting to a calculator, we obtain (to nine decimal places):

$$S_{42} = \sum_{n=1}^{42} \frac{(-1)^{n+1}}{n^2+5n+2} = 0.085371590.$$

Thus, to three decimal place accuracy,

$$S = 0.085.$$

In fact, according to Theorem 8.3.4, $S_{42} < S < S_{43}$. Using a calculator, we find $S_{43} = 0.085855618$ (to nine decimal places). Rounding off, Theorem 8.3.4 (a) assures us that

$$0.0853 < S < 0.0859. \quad \square$$

CAUTION: In using the alternating series test, students often ask whether it is necessary to include the hypothesis that $\{a_n\}$ is monotone decreasing. Indeed, students will often disregard this hypothesis when deciding whether the alternating series test is applicable to a particular series. The following example is offered to show that this test cannot be used when $\{a_n\}$ is not monotone decreasing.

Example 8.3.6 The alternating series

$$1 - 2 + \tfrac{1}{2} - 1 + \tfrac{1}{3} - \tfrac{2}{3} + \tfrac{1}{4} - \tfrac{2}{4} + \tfrac{1}{5} - \tfrac{2}{5} + \cdots + \tfrac{1}{n} - \tfrac{2}{n} + \cdots$$

diverges, even though $a_n \to 0$.

Proof. Exercise 1. □

ABSOLUTE CONVERGENCE

Definition 8.3.7 A series $\sum a_n$ is said to **converge absolutely** (or, be **absolutely convergent**) if $\sum |a_n|$ converges. A series that converges, but not absolutely, is said to **converge conditionally**.[3]

For example, the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges conditionally, as seen in Examples 2.5.16 and 8.3.3.

As we shall see, absolute convergence is stronger than convergence. We shall show in what follows that an absolutely convergent series must converge, but the converse is not true. To gain an understanding of this concept, and closely related facts, it will be helpful to define two series intimately related to a given series.

Definition 8.3.8 Given a sequence $\{a_n\}$ of real numbers, we define the sequences $\{a_n^+\}$ and $\{a_n^-\}$ by

$$a_n^+ = \max\{a_n, 0\} \text{ and } a_n^- = \max\{-a_n, 0\}.$$

Lemma 8.3.9 *For any $a \in \mathbb{R}$,*

(a) $a^+ \geq 0$ *and* $a^- \geq 0$;

(b) $a = a^+ - a^-$;

(c) $|a| = a^+ + a^-$.

Proof. Exercise 2. ■

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3. The term "conditionally" convergent will be explained more fully later in this section.

Theorem 8.3.10 *Given any series $\sum a_n$ of real numbers,*

(a) *$\sum a_n$ converges absolutely $\iff$ both $\sum a_n^+$ and $\sum a_n^-$ converge.*

(b) *If $\sum a_n$ converges absolutely then it converges.*

(c) *If $\sum a_n$ converges conditionally then both $\sum a_n^+$ and $\sum a_n^-$ diverge to $+\infty$.*

(d) *If one of the series $\sum a_n^+$, $\sum a_n^-$ converges and the other diverges, then $\sum a_n$ diverges.*

Proof. First note that because of Lemma 8.3.9,

$$(*) \quad \sum a_n = \sum(a_n^+ - a_n^-) \text{ and } (**) \quad \sum |a_n| = \sum(a_n^+ + a_n^-).$$

(a) If $\sum |a_n|$ converges, then from the inequalities $0 \leq a_n^+ \leq |a_n|$ and $0 \leq a_n^- \leq |a_n|$, together with the comparison test for nonnegative series, both $\sum a_n^+$ and $\sum a_n^-$ converge.

Conversely, if both $\sum a_n^+$ and $\sum a_n^-$ converge, then (**) together with Theorem 8.1.12 assures us that $\sum |a_n|$ converges.

(b) If $\sum |a_n|$ converges, then by Part (a) both $\sum a_n^+$ and $\sum a_n^-$ converge, so (*) together with Theorem 8.1.12 assures us that $\sum a_n = \sum(a_n^+ - a_n^-)$ converges.

(c) Suppose $\sum a_n$ converges, but not absolutely. For contradiction, suppose that one of the series $\sum a_n^+$, $\sum a_n^-$ converges. Without loss of generality, suppose the second of these two converges. That is,

$$\sum(a_n^+ - a_n^-) \text{ and } \sum a_n^- \text{ converge.}$$

Then, by Theorem 8.1.12, $\sum[(a_n^+ - a_n^-) + a_n^-]$ converges; i.e., $\sum a_n^+$ converges. But then, by (a), $\sum a_n$ converges absolutely. Contradiction.

(d) Exercise 3. ∎

Theorem 8.3.11 (*Generalized Triangle Inequality*) *If $\sum a_n$ converges absolutely, then $|\sum a_n| \leq \sum |a_n|$.*

Proof. Suppose $\sum a_n$ converges absolutely. Then, by Theorem 8.3.10 (a), both $\sum a_n^+$ and $\sum a_n^-$ converge, and using Lemma 8.3.9,

$$\begin{aligned}
|\sum a_n| &= |\sum(a_n^+ - a_n^-)| = |\sum a_n^+ - \sum a_n^-| \\
&\leq |\sum a_n^+| + |\sum a_n^-| \quad \text{(by the ordinary triangle inequality)} \\
&= \sum a_n^+ + \sum a_n^- = \sum(a_n^+ + a_n^-) = \sum |a_n|. \quad \blacksquare
\end{aligned}$$

REARRANGEMENTS AND SUBSERIES

We shall now explore some deeper consequences of absolute convergence, and thereby see more clearly the difference between absolute and conditional convergence.

Definition 8.3.12 A **rearrangement** of a series $\sum a_n$ is a series of the form $\sum a_{\sigma_n}$, where σ is a permutation[4] of the set of natural numbers and $\sigma_n = \sigma(n)$.

Theorem 8.3.13 *Every rearrangement of an absolutely convergent series converges absolutely, and has the same sum.*

Proof. (a) Let $\sum a_n$ be a convergent *nonnegative* series, with rearrangement $\sum a_{\sigma_n}$, and denote their partial sums by

$$S_n = \sum_{k=1}^{n} a_k \text{ and } \overline{S}_n = \sum_{k=1}^{n} a_{\sigma_k}.$$

Now, $\forall n \in \mathbb{N}$, $\sigma_1, \sigma_2, \cdots, \sigma_n \in \{1, 2, \cdots, m\}$, where $m = \max\{\sigma_k : k = 1, 2, \cdots, n\}$. Thus, since the terms are nonnegative,

$$\overline{S}_n \leq S_m \leq \sum_{k=1}^{\infty} a_k.$$

Thus, $\{\overline{S}_n\}$ is a monotone increasing sequence with an upper bound, so it must have a limit. And, since limits preserve inequalities,

$$\sum_{k=1}^{\infty} a_{\sigma_k} \leq \sum_{k=1}^{\infty} a_k.$$

We have just shown that every rearrangement of a convergent nonnegative series converges and has a sum less than or equal to that of the original series.

But $\sum a_n$ is also a rearrangement of $\sum a_{\sigma_n}$. Applying the result just proved, $\sum a_n \leq \sum a_{\sigma_n}$. Therefore, equality holds, and we have proved the desired result for nonnegative series.

(b) From Part (a) we can conclude that every rearrangement of an absolutely convergent series converges absolutely. It remains to prove that a rearrangement of an absolutely convergent series must have the same sum as the original series.

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4. A **permutation** of a set A is a 1-1 correspondence $\sigma : A \to A$. See Appendix B.2.

(c) Let $\sum a_n$ be an absolutely convergent series, and use the same notation used in Part (a) above. Let $\varepsilon > 0$. By the Cauchy criterion for series (8.1.11),

$$\exists\, n_0 \in \mathbb{N} \ni n > m \geq n_0 \Rightarrow \sum_{k=m+1}^{n} |a_k| < \frac{\varepsilon}{2}.$$

Since the function $\sigma : \mathbb{N} \to \mathbb{N}$ is 1-1 and onto, $\exists\, m \geq n_0 \ni$

$$\{1, 2, \cdots, n_0\} \subseteq \{\sigma_1, \sigma_2, \cdots, \sigma_m\}. \tag{12}$$

Suppose $n > m$. Then,

$$|S_n - \overline{S}_n| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} a_{\sigma_k} \right|. \tag{13}$$

By relation (12) we see that the terms $a_1, a_2, \cdots, a_{n_0}$ appear in both of the sums in (13), so they will cancel, leaving only terms with subscript $> n_0$. Thus,

$$n \geq m \Rightarrow |S_n - \overline{S}_n| = \left| \sum_{k=n_0+1}^{n} a_k - \sum_{\sigma_k \geq n_0+1}^{k \leq n} a_{\sigma_k} \right| \leq \left| \sum_{k=n_0+1}^{n} a_k \right| + \left| \sum_{\sigma_k \geq n_0+1}^{k \leq n} a_{\sigma_k} \right|$$

$$\leq \sum_{k=n_0+1}^{n} |a_k| + \sum_{k=n_0+1}^{M} |a_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ where } M = \max\{\sigma_k : k \leq n\}.$$

Therefore, $\lim_{n\to\infty} S_n - \lim_{n\to\infty} \overline{S}_n = \lim_{n\to\infty} (S_n - \overline{S}_n) = 0$. That is,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\sigma_k}. \qquad \blacksquare$$

With the help of the previous theorem, we can now see one of the deep differences between absolute and conditional convergence. The following theorem shows that this difference is quite dramatic.

Theorem 8.3.14 *Given any conditionally convergent series* $\sum a_n$*,*

(a) $\sum a_n$ *has a divergent rearrangement;*

(b) $\forall r \in \mathbb{R}$, $\sum a_n$ *has a rearrangement converging to* r*.*

Proof. Let $\sum a_n$ be a conditionally convergent series. By Theorem 8.3.10 (c), both $\sum a_n^+$ and $\sum a_n^-$ diverge. After eliminating the zeros from these two series we can say, in words, that the series of positive terms of $\sum a_n$ diverges to $+\infty$, and the series of negative terms of $\sum a_n$ diverges to $-\infty$.

(a) Form the desired divergent rearrangement as follows. Since the series of positive terms diverges, add the first consecutive positive terms until the sum is greater than 1; then add the first negative term. Then, add the next consecutive positive terms until the sum is greater than 2, after which add the

second negative term. Continue by induction. In general, after the n^{th} negative term has been added, add the next consecutive positive terms until the sum is greater than $n+1$. Since this process accounts for all the terms of $\sum a_n$, it yields a rearrangement of $\sum a_n$. This rearrangement diverges because its sequence of partial sums is unbounded above.

(b) Let $r \in \mathbb{R}$. We shall construct a rearrangement of $\sum a_n$ that converges to r. The following argument uses the fact that the series of positive terms of $\sum a_n$ diverges to $+\infty$ and the series of negative terms of $\sum a_n$ diverges to $-\infty$. It also uses the general term test (8.1.4), assuring us that $a_n \to 0$.

Add consecutive positive terms, but (only) enough positive terms so that the sum is greater than r. Then add consecutive negative terms, but (only) enough so that the cumulative sum is less than r. Continue, adding positive terms, but (only) enough positive terms so that the cumulative sum is greater than r. Then continue, adding consecutive negative terms, but (only) enough so that the cumulative sum is less than r. Continue this process by mathematical induction. The resulting series will use each term of $\sum a_n$ exactly once, and thus be a rearrangement $\sum a_{\sigma_k}$ of the series $\sum a_n$. As noted above, $a_n \to 0$, so $a_{\sigma_n} \to 0$. (See Exercise 2.2.23.)

Because of the way in which we have constructed the rearrangement, the partial sum $\overline{S}_n = \sum_{k=1}^{n} a_{\sigma_k}$ never deviates from r by more than the absolute value of the last term added. That is, $\forall n \in \mathbb{N}$,

$$\left|\overline{S}_{\sigma_n} - r\right| < |a_{\sigma_n}| \to 0.$$

Therefore, by the second squeeze principle, $\overline{S}_{\sigma_n} \to r$. ∎

So, what is conditional about a "conditionally" convergent series? One way of putting it is to say that the terms of an absolutely convergent series can be added in any order (i.e., unconditionally) without affecting the sum. In contrast, the terms of a conditionally convergent series can be added only on the "condition" that we add them in the right order. If we alter that order we may alter the sum or even lose convergence.

The next theorem shows another contrast between absolutely and conditionally convergent series. A conditionally convergent series always has at least two divergent subseries; namely, the series of its positive terms and the series of its negative terms. So, a subseries of a conditionally convergent series may diverge. However, we shall now show that an absolutely convergent series cannot have a divergent subseries.

Definition 8.3.15 A **subseries** of a series $\sum a_n$ is a series of the form $\sum a_{n_k}$, where $\{n_k\}$ is a subsequence of $\{n\}$.

Theorem 8.3.16 *A series converges absolutely if and only if each of its subseries converges (absolutely).*

Proof. Part 1 ($\Rightarrow$): Suppose $\sum a_n$ converges absolutely, and let $\sum a_{n_k}$ be a subseries. Then, $\forall m \in \mathbb{N}$, $m \leq n_m$, so

$$\sum_{k=1}^{m} |a_{n_k}| \leq \sum_{k=1}^{n_m} |a_k| \leq \sum_{k=1}^{\infty} |a_k|.$$

Thus, the sequence of partial sums of the subseries $\sum |a_{n_k}|$ is bounded above, so the subseries $\sum |a_{n_k}|$ converges absolutely.

Part 2 ($\Leftarrow$): Suppose every subseries of the series $\sum a_n$ converges. Then, both the series of positive terms and the series of negative terms of $\sum a_n$ converge, so both $\sum a_n^+$ and $\sum a_n^-$ converge. By Theorem 8.3.10, this implies that $\sum a_n$ converges absolutely. ∎

EXERCISE SET 8.3

1. Prove the assertions made in Example 8.3.6.

2. Prove Lemma 8.3.9.

3. Prove Theorem 8.3.10 (d).

4. Prove that the series $\frac{1}{2} - \frac{1}{2} + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} + \frac{1}{16} - \cdots$ converges conditionally, and find its sum.

5. Prove that the series $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{9} + \frac{1}{8} - \frac{1}{27} + \cdots + \frac{1}{2^k} - \frac{1}{3^k} + \cdots$ converges to $\frac{1}{2}$. Does it converge absolutely or conditionally?

6. Determine whether each of the following alternating series converges absolutely, converges conditionally, or diverges. In case of convergence, find an integer n_0 such that $n \geq n_0 \Rightarrow |S - S_n| < .01$.

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+4}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n+5}$ (d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{(n+2)!}$

   (e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n^2}$ (f) $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

7. **Sum of the Alternating Harmonic Series:** Let $\{S_n\}$ denote the sequence of partial sums of the alternating harmonic series, $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. First, show that $S_{2n} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k}$. Then use this, along with the properties of the sequence $\{\gamma_n\}$ obtained in Exercise 8.2.41, to show that $S_{2n} = \gamma_{2n} - \gamma_n + \ln 2$. Finally, show how this proves that $S_n \to \ln 2$.

8. Prove that $\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) \to \ln 2$ by relating these sums to the partial sums of the harmonic series and using insights gained from the previous exercise.

9. Use Theorem 8.3.10 to prove that the alternating series

$$1 - 1/2 + 1/2^2 - 1/3 + 1/3^2 - 1/4 + 1/4^2 - \cdots - 1/n + 1/n^2 - \cdots$$

diverges. Which conditions of the alternating series test do not hold?

10. Determine whether the given series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$ (b) $\sum_{n=1}^{\infty} \frac{\sin \frac{\pi n}{2}}{n^{3/2}}$

(c) $\sum_{n=1}^{\infty} \frac{n \sin \frac{\pi n}{2}}{3n+8}$ (d) $\sum_{n=1}^{\infty} \frac{\tan\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)}{\sqrt[3]{n}}$

(e) $\sum_{n=1}^{\infty} \left(\frac{\sin n}{n}\right)^n$ (f) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{\pi}{n}}{n}$ [See Example 6.4.7.]

11. Prove that for every real number x, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges absolutely.

12. Prove that if $\sum a_n$ converges absolutely and $\{b_n\}$ is a bounded sequence, then $\sum a_n b_n$ converges absolutely.

13. Prove that the following series converges conditionally:

$$\frac{1}{2} - \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots + (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n)}.$$

[See Exercises 8.2.43 and 8.2.49; for the sum, see Example 8.7.11.]

14. Form a rearrangement of the alternating harmonic series by adding the first two positive terms, then the first two negative terms, then the next two positive terms, then the next two negative terms, and so on. Prove that the resulting series converges, and find its sum.

15. Beginning with the harmonic series, form a new series by adding the first two terms, then subtracting the next two terms, then adding the next two

terms, subtracting the next two terms, and so on. How does this differ from the series in Exercise 14? Prove that the resulting series converges.

16. Beginning with the harmonic series, form a new series by adding the first two terms, then subtracting the next term, then adding the next two terms, subtracting the next term, and so on—always adding two terms and subtracting the next term. Prove that the resulting series diverges.

17. By Theorem 8.3.14 (a), the alternating harmonic series can be rearranged to a series that diverges to $+\infty$. Write out the first 24 terms of the rearranged series described in the proof of this theorem. [Use a calculator or computer to check the required inequalities.]

18. By Theorem 8.3.14 (b), the alternating harmonic series can be rearranged to a series that converges to 0. Write out the first 24 terms of the rearranged series described in the proof of this theorem. [Use a calculator or computer to check the required inequalities.]

19. Prove that although the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n+1}{n}$ diverges, the sequence obtained by grouping consecutive terms in pairs (using parentheses) converges absolutely. [See Theorem 8.1.10 (c).]

8.4 The Cauchy Product of Series

Given convergent series $\sum a_k$ and $\sum b_k$, it is often desirable to express their product as another convergent series:

$$\left(\sum a_k\right)\left(\sum b_k\right) = \sum c_k.$$

We seek an appropriate definition of such a product series $\sum c_k$, a formula for its terms c_k, and conditions under which the product series converges. To get a good start in that direction we take a look at a simpler, finite case:

$$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3 + b_4) = ?$$

Because multiplication of one sum by another must obey the distributive law, each term in the one sum must be multiplied by each term in the other sum, and then all these products must be added together. Thus, $(a_1 + a_2 + a_3)(b_1 + b_2 + b_3 + b_4)$ is the sum of all twelve entries in the following matrix:

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \end{bmatrix}.$$

If we collect terms along the northeast-to-southwest diagonals, we have $(a_1 + a_2 + a_3)(b_1 + b_2 + b_3 + b_4) =$
$a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + (a_1b_4 + a_2b_3 + a_3b_2) + (a_2b_4 + a_3b_3) + a_3b_4.$

Guided by this understanding, we want the series representing the product $(\sum a_k)(\sum b_k)$ to represent the sum of all the entries in the infinite matrix:

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & a_1b_5 & \cdots \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & a_2b_5 & \cdots \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & a_3b_5 & \cdots \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & a_4b_5 & \cdots \\ a_5b_1 & a_5b_2 & a_5b_3 & a_5b_4 & a_5b_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Any "product" of $\sum a_k$ and $\sum b_k$ must add all the entries in this infinite matrix. But how can we do that? Having studied conditionally convergent series, we know that the order of summing can affect the results. The idea behind the "Cauchy product" series is to add all the entries in this matrix by a procedure patterned after the one we used in the finite case above. That is, we add along the northeast-to-southwest diagonals and group these sums together as terms of a series. More specifically, we make the following definition.

Definition 8.4.1 The **Cauchy product** of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ is the series $\sum_{k=1}^{\infty} c_k$, where

$$c_k = \sum_{i=1}^{k} a_i b_{k+1-i} = a_1b_k + a_2b_{k-1} + \cdots + a_{k-1}b_2 + a_kb_1.$$

For series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$, the formula for c_k is

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} = a_0b_k + a_1b_{k-1} + \cdots + a_{k-1}b_1 + a_kb_0.$$

Two important questions are (1) under what conditions are we guaranteed that the Cauchy product series converges, and (2) when the Cauchy product of two series converges, does it converge to the product of their sums? Perhaps surprisingly, absolute convergence plays a key role in answering these questions. Before proving any theorems, we give an example showing that the Cauchy product of two convergent series does not necessarily converge.

Example 8.4.2 (Two Convergent Series Whose Cauchy Product Series Diverges) Let $\sum a_k$ and $\sum b_k$ both be the same series, with $a_k = b_k = \frac{(-1)^k}{\sqrt{k}}$. Then both $\sum a_k$ and $\sum b_k$ converge, but their Cauchy product series diverges.

Proof. The two series converge by the alternating series test. On the other hand, the k^{th} term of the Cauchy product series is

$$c_k = \sum_{i=1}^{k} \frac{(-1)^{i+1}}{\sqrt{i}} \frac{(-1)^{k+1-i}}{\sqrt{k+1-i}} = \sum_{i=1}^{k} \frac{(-1)^{k+2}}{\sqrt{i}\sqrt{k+1-i}} = (-1)^k \sum_{i=1}^{k} \frac{1}{\sqrt{i}\sqrt{k+1-i}}.$$

Thus, $|c_k| \geq \sum_{i=1}^{k} \frac{1}{\sqrt{k}\sqrt{k}} = \sum_{i=1}^{k} \frac{1}{k} = \frac{k}{k} = 1.$

So, the Cauchy product series diverges, since it fails the general term test. $\square$

Now we show the connection between absolute convergence and the behavior of Cauchy product series.

Theorem 8.4.3 *The Cauchy product of two absolutely convergent series is absolutely convergent, and its sum is the product of their sums.*

Proof. Suppose $\sum a_k$ and $\sum b_k$ converge absolutely, and denote their sums by A and B, respectively. Let $\sum c_k$ denote their Cauchy product. That is,

$$c_k = \sum_{i=1}^{k} a_i b_{k+1-i}.$$

Then the terms c_k represent the sums along the indicated diagonals of the infinite matrix

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 & a_1b_5 & \cdots \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 & a_2b_5 & \cdots \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 & a_3b_5 & \cdots \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 & a_4b_5 & \cdots \\ a_5b_1 & a_5b_2 & a_5b_3 & a_5b_4 & a_5b_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \tag{14}$$

Define the series $\sum d_k$ as the series obtained by removing the parentheses from the series $\sum c_k$. That is,
$d_k = a_1b_1 + a_1b_2 + a_2b_1 + a_1b_3 + a_2b_2 + a_3b_1 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_1b_5 + a_2b_4 + +a_3b_3 + a_4b_2 + a_5b_1 + \cdots$.

Denote the partial sums of these four series by

$$A_n = \sum_{k=1}^{n} a_k, \quad B_n = \sum_{k=1}^{n} b_k, \quad C_n = \sum_{k=1}^{n} c_k, \quad D_n = \sum_{k=1}^{n} d_k.$$

Note that

(a) Each c_k is obtained by grouping k successive terms of the series $\sum d_k$.

(b) Each C_n is a partial sum of the series $\sum d_k$, and is the sum of terms occupying a triangle in the upper left corner of the matrix (14) above.

(c) Each $A_n B_n$ is a sum (but not, strictly speaking, a "partial sum") of terms of $\sum d_k$, with terms occupying a square in the upper left corner of the matrix (14) above.

(d) Every D_n is the sum of terms contained within some square in the upper left corner of (14) above, and also contained within some triangle in the upper left corner of (14).

Case 1: Suppose $\sum a_k$ and $\sum b_k$ are nonnegative series. By note (d) above, every D_n is less than or equal to some $A_m B_m$. Since $\sum a_k$ and $\sum b_k$ are nonnegative series, we have for all n,

$$D_n \leq \lim_{m\to\infty} (A_m B_m) = \left(\lim_{m\to\infty} A_m\right)\left(\lim_{m\to\infty} B_m\right)$$
$$\text{i.e., } D_n \leq \left(\sum a_k\right)\left(\sum b_k\right).$$

Since limits preserve inequalities,

$$\sum d_k \leq \left(\sum a_k\right)\left(\sum b_k\right). \tag{15}$$

On the other hand, by note (c) above, we have for all n,

$$A_n B_n \leq \sum d_k, \text{ so}$$
$$\left(\lim_{n\to\infty} A_n\right)\left(\lim_{n\to\infty} B_n\right) \leq \sum d_k$$

$$\text{i.e., } \left(\sum a_k\right)\left(\sum b_k\right) \leq \sum d_k. \tag{16}$$

Therefore, putting (15) and (16) together, we have

$$\sum a_k \sum b_k = \sum d_k.$$

By note (b) above, $\{C_n\}$ is a subsequence of $\{D_n\}$, so

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} D_n$$

$$\text{i.e., } \sum c_k = \sum d_k.$$

Therefore, $\sum a_k \sum b_k = \sum c_k$.

Case 2: Suppose $\sum a_k$ and $\sum b_k$ are not both nonnegative series. By applying the argument of Case 1 to the series $\sum |a_k|$, $\sum |b_k|$, and $\sum |d_k|$, we conclude that $\sum d_k$ converges absolutely.

Now, by note (b), the sequence $\{C_n\}$ is a subsequence of $\{D_n\}$, so $\lim\limits_{n\to\infty} C_n = \lim\limits_{n\to\infty} D_n$. That is, $\sum c_k$ exists and equals $\sum d_k$. Also, by note (c), each $A_n B_n$

is a partial sum of a rearrangement of $\sum d_k$. By Theorem 8.3.13, every rearrangement of an absolutely convergent series converges to the same sum. Thus,

$$\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} D_n$$

$$\text{i.e., } \sum a_k \sum b_k = \sum d_k.$$

Therefore, $\sum a_k \sum b_k = \sum c_k$. ■

Actually, for the Cauchy product of two convergent series to converge, it is not necessary that both of the series converge absolutely. As the next theorem shows, it is sufficient that *one* of them converges absolutely.

***Theorem 8.4.4** (*Mertens' Theorem*) *The Cauchy product of an absolutely convergent series and a convergent series converges (but not necessarily absolutely), and its sum is the product of their sums.*

Proof. Suppose $\sum a_k$ converges absolutely to A and $\sum b_k$ converges to B. Using the same notation used in the proof of Theorem 8.4.3, let $\sum c_k$ denote their Cauchy product, and let A_n, B_n, and C_n denote the partial sums of $\sum a_k$, $\sum b_k$, and $\sum c_k$, respectively. We define a new sequence $\{\overline{B}_n\}$ by

$$\overline{B}_n = B_n - B.$$

$$\begin{aligned}
\text{Then,}\quad C_n = \sum_{k=1}^{n} c_k &= \sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_i b_{k+1-i}\right) \\
&= \sum_{k=1}^{n} a_k B_{n+1-k} \qquad (17) \\
&\quad \text{(show, in Exercise 13)} \\
&= \sum_{k=1}^{n} a_k (B + \overline{B}_{n+1-k}) \\
&= A_n B + \sum_{k=1}^{n} a_k \overline{B}_{n+1-k}.
\end{aligned}$$

Now, $\lim_{n\to\infty} A_n B = AB$. Thus, our proof will be complete if we can prove the following **claim**: $\lim_{n\to\infty} \sum_{k=1}^{n} a_k \overline{B}_{n+1-k} = 0.$

To prove this claim, let $\varepsilon > 0$. Since $\sum a_k$ converges absolutely, we may let $A' = \sum |a_k|$. Since $\overline{B}_n \to 0$, $\exists\ M > 0 \ni \forall n \in \mathbb{N}$, $|\overline{B}_n| \le M$ and $\exists\ n_0 \in \mathbb{N} \ni k \ge n_0 \Rightarrow |\overline{B}_k| \le \frac{\varepsilon}{2A'}$. (We can assume $A' > 0$, since if $A' = 0$ there is nothing to prove.)

$$
\begin{aligned}
\text{Then, } n \geq n_0 \Rightarrow \left|\sum_{k=1}^{n} a_k \overline{B}_{n+1-k}\right| &\leq \sum_{k=1}^{n} |a_{n+1-k}\overline{B}_k| \\
&= \sum_{k=1}^{n_0} |a_{n+1-k}\overline{B}_k| + \sum_{k=n_0+1}^{n} |a_{n+1-k}\overline{B}_k| \\
&\leq M\sum_{k=1}^{n_0} |a_{n+1-k}| + \frac{\varepsilon}{2A'} \sum_{k=n_0+1}^{n} |a_{n+1-k}| \\
&\leq M \sum_{k=n+1-n_0}^{n} |a_k| + \frac{\varepsilon}{2}. \qquad (18) \\
&\text{(show, in Exercise 13)}
\end{aligned}
$$

Since $\sum |a_k|$ converges, the Cauchy criterion guarantees that $\exists\ n_1 \in \mathbb{N} \ni$ $n > m > n_1 \Rightarrow \sum_{k=m+1}^{n} |a_k| < \frac{\varepsilon}{2M}$. Then

$$n > n_0 + n_1 \Rightarrow n > n_0,\ n > n_1, \text{ and } n+1-n_0 > n_1 \Rightarrow \sum_{k=n+1-n_0}^{n} |a_k| < \frac{\varepsilon}{2M}.$$

Therefore,

$$n \geq n_0 + n_1 \Rightarrow \left|\sum_{k=1}^{n} a_k \overline{B}_{n+1-k}\right| < \varepsilon,$$

which proves the desired claim. ■

In the statement of the previous theorem we were careful to indicate that the convergence of the Cauchy product of an absolutely convergent series and a convergent series need not be absolute. For an example illustrating this situation, see Exercise 9. For an example showing that a Cauchy product can converge even when both the series diverge, see Exercise 1. In 1826 the Norwegian mathematician Niels Henrik Abel proved that if the Cauchy product of two convergent series converges, its sum must be the product of their sums. Abel's theorem makes no use of absolute convergence. We shall be able to give an easy proof of this result after we study power series. (See Exercise 8.6.18.)

EXERCISE SET 8.4

1. Consider the series $\sum a_k = 1 + 2 + 2 + 2 + 2 + 2 + \cdots$ and $\sum b_k = 1 - 2 + 2 - 2 + 2 - 2 + \cdots$. Show that, although both of these series diverge, their Cauchy product converges. Find the sum of their Cauchy product.

2. Let $\sum c_k$ denote the Cauchy product series of $\sum_{k=0}^{\infty} r^k$ and $\sum_{k=0}^{\infty} (-1)^k r^k$. Find and simplify the expression for c_k and use it to prove that the product

series converges when $|r| < 1$, and find its sum. Verify the conclusion of Theorem 8.4.3 in this case.

3. Let $|r| < 1$. Although the series $\sum\limits_{k=0}^{\infty} kr^k$ is not a geometric series, we can find its sum by considering the Cauchy product of the geometric series $\sum\limits_{k=0}^{\infty} r^k$ with itself. Use the Cauchy product formula to write a power series for $\left(\sum\limits_{k=0}^{\infty} r^k\right)^2$, thus getting a power series for $\dfrac{1}{(1-r)^2}$. Massage this formula to show that the desired sum is $\dfrac{r}{(1-r)^2}$.

4. Use the result of Exercise 3 to find the sum of the given series.

   (a) $\sum\limits_{k=1}^{\infty} \dfrac{k}{5^k}$. (b) $1 + 1 + \dfrac{3}{2^2} + \dfrac{4}{2^3} + \dfrac{5}{2^4} + \cdots$.

5. Prove that if $|r| < 1$, $\sum\limits_{k=0}^{\infty} \dfrac{(k+1)(k+2)}{2} r^k = \dfrac{1}{(1-r)^3}$.

6. Define the function $E(x) = \sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!}$. This series converges absolutely, for all real numbers x.[5] Use the Cauchy product of series to prove that $\forall x, y \in \mathbb{R}$, $E(x)E(y) = E(x+y)$.

7. Let $\sum c_k$ denote the Cauchy product of the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{k}$ and $\sum\limits_{k=1}^{\infty} \dfrac{1}{3^k}$. Find and simplify the expression for c_k. Use one of the theorems of this section to prove that this Cauchy product series converges and find its sum.

8. Suppose $\{a_n\}$ and $\{b_n\}$ are nonnegative sequences. Prove that the Cauchy product of the alternating series $\sum(-1)^{k+1}a_k$ and $\sum(-1)^{k+1}b_k$ is also an alternating series, and can be obtained by inserting alternating signs in the Cauchy product of $\sum a_k$ and $\sum b_k$.

9. Consider[6] the series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^{3/2}}$ and $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^{1/2}}$. Show that one of these series converges absolutely and the other converges conditionally. Show that the k^{th} term of their Cauchy product series is

$$c_k = (-1)^{k+1} \sum_{i=1}^{k} \frac{1}{i^{3/2}(k+1-i)^{1/2}}.$$

5. See Exercise 8.3.11.

6. This example may be found in Burrill and Knudsen [25], page 143.

By Theorem 8.4.4, $\sum c_k$ must converge. Prove that this convergence is not absolute by showing that $\forall k$, $|c_k| \geq \frac{1}{k}$.

10. State and prove the commutative law for the Cauchy product of series.
11. State and prove the distributive law for sums and Cauchy products of series.
12. State and prove the associative law for the Cauchy product of series.
13. Prove the claims made in (17) and (18) of the proof of Theorem 8.4.4.

8.5 Series of Products

In the previous section we studied products of series; in this section we study series of products. In particular, we study series of the form $\sum c_k = \sum a_k b_k$ and investigate conditions on the sequences $\{a_k\}$ and $\{b_k\}$ that will guarantee convergence of such a series.

To determine whether $\sum a_k b_k$ converges, it is not enough to check whether both $\sum a_k$ and $\sum b_k$ converge. Indeed, it is easy to find convergent series $\sum a_k$ and $\sum b_k$ for which $\sum a_k b_k$ diverges (see Exercise 1). It is also easy to find divergent series $\sum a_k$ and $\sum b_k$ for which $\sum a_k b_k$ converges (Exercise 2). However, with the help of the following theorem, we can easily prove that if $\sum a_k$ and $\sum b_k$ are both absolutely convergent, then so is $\sum a_k b_k$ (Exercise 4).

Theorem 8.5.1 *If $\sum a_k$ converges absolutely and $\{b_n\}$ is a bounded sequence, then $\sum a_k b_k$ converges absolutely.*

Proof. Exercise 3. ■

We seek weaker conditions on the sequences $\{a_k\}$ and $\{b_k\}$ that will guarantee convergence of the series $\sum a_k b_k$. The following result will prove very useful in that investigation.

Theorem 8.5.2 (*Abel's Summation by Parts Formula*) *Let $\{a_k\}$ and $\{b_k\}$ be sequences, and define*

$$S_0 = 0, \text{ and } \forall n \geq 1 \quad S_n = \sum_{k=1}^{n} a_k.$$

Then, for all $1 \leq m < n$,

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} S_k(b_k - b_{k+1}) + S_n b_{n+1} - S_{m-1} b_m.$$

Proof. Let $\{a_k\}$, $\{b_k\}$, and $\{S_n\}$ satisfy the hypotheses. Then, for all $k \geq 0$,

$$\begin{aligned} a_k b_k &= (S_k - S_{k-1}) b_k \\ &= S_k(b_k - b_{k+1}) + S_k b_{k+1} - S_{k-1} b_k. \end{aligned}$$

Thus, $\forall\ 1 \le m < n$,

$$\begin{aligned}
\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n} S_k(b_k - b_{k+1}) + \sum_{k=m}^{n} (S_k b_{k+1} - S_{k-1} b_k) \\
&= \sum_{k=m}^{n} S_k(b_k - b_{k+1}) + (\cancel{S_m b_{m+1}} - S_{m-1} b_m) + (\cancel{S_{m+1} b_{m+2}} - \cancel{S_m b_{m+1}}) \\
&\quad + (\cancel{S_{m+2} b_{m+3}} - \cancel{S_{m+1} b_{m+2}}) + \cdots + (S_n b_{n+1} - \cancel{S_{n-1} b_n}) \\
&= \sum_{k=m}^{n} S_k(b_k - b_{k+1}) + S_n b_{n+1} - S_{m-1} b_m. \qquad \blacksquare
\end{aligned}$$

The formula in the conclusion of Theorem 8.5.2 is called "summation by parts" because of its similarity with the integration by parts formula. Admittedly, this similarity is not readily apparent in the form in which it appears here. To see the similarity, work Exercise 5.

Abel's summation by parts formula is important because it allows us to prove Dirichlet's convergence test, which opens the door to the study of trigonometric series and other applications.

Theorem 8.5.3 (*Dirichlet's Test*) *Suppose $\{a_k\}$ and $\{b_k\}$ are sequences such that*

(a) *the sequence $\{S_n\}$ of partial sums, $S_n = \sum_{k=1}^{n} a_k$, is bounded,*

(b) *the sequence $\{b_k\}$ is monotone decreasing and nonnegative, and*

(c) $b_k \to 0$.

Then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. Suppose $\{a_k\}$, $\{b_k\}$, and $\{S_n\}$ satisfy the hypotheses. Let $\varepsilon > 0$. By (a), $\exists M > 0 \ni \forall n \in \mathbb{N}$, $|S_n| \le M$. By (c), $\exists\ n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow 0 \le b_n < \dfrac{\varepsilon}{2M}$.

Then $n_0 \le m < n \Rightarrow$ (by Abel's summation by parts formula)

$$\begin{aligned}
\left| \sum_{k=m+1}^{n} a_k b_k \right| &= \left| \sum_{k=m+1}^{n} S_k(b_k - b_{k+1}) + S_n b_{n+1} - S_m b_{m+1} \right| \\
&\le \sum_{k=m+1}^{n} |S_k|\,(b_k - b_{k+1}) + |S_n|\, b_{n+1} + |S_m|\, b_{m+1} \\
&\le M \left(\sum_{k=m+1}^{n} (b_k - b_{k+1}) + b_{n+1} + b_{m+1} \right) \\
&= M\,(b_{m+1} - b_{n+1} + b_{n+1} + b_{m+1}) \\
&= 2M b_{m+1} < \varepsilon.
\end{aligned}$$

Therefore, by the Cauchy criterion for convergence of series (Theorem 8.1.11), $\sum\limits_{k=1}^{n} a_k b_k$ converges. ■

APPLICATION TO TRIGONOMETRIC SERIES

We shall apply Dirichlet's test to "trigonometric" series of the form

$$\sum_{k=1}^{\infty} a_k \sin kt \quad \text{and} \quad \sum_{k=1}^{\infty} b_k \cos kt.$$

We first want to prove that their partial sums $\sum\limits_{k=1}^{n} a_k \sin kt$ and $\sum\limits_{k=1}^{n} b_k \cos kt$ are bounded. Toward that end, we begin by proving two trigonometric identities.

Lemma 8.5.4 *For all real numbers t not an integral multiple of 2π, and $\forall n \in \mathbb{N}$,*

(a) $\displaystyle\sum_{k=1}^{n} \sin kt = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$, *and*

(b) $\displaystyle\sum_{k=1}^{n} \cos kt = \frac{\sin\left(n+\frac{1}{2}\right)t - \sin\frac{t}{2}}{2\sin\frac{t}{2}}$.

Proof. (a) Let t be a real number, not an integral multiple of 2π, and $\forall n \in \mathbb{N}$, let $S_n = \sum\limits_{k=1}^{n} \sin kt$. Recall the trigonometric identity

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

$$\begin{aligned} \text{Thus, } \left(\sin\tfrac{t}{2}\right) S_n &= \sum_{k=1}^{n} \sin\tfrac{t}{2}\sin kt \\ &= \tfrac{1}{2}\sum_{k=1}^{n}\left[\cos\left(\tfrac{t}{2}-kt\right) - \cos\left(\tfrac{t}{2}+kt\right)\right] \\ &= \tfrac{1}{2}\sum_{k=1}^{n}\left[\cos\left(kt-\tfrac{t}{2}\right) - \cos\left(kt+\tfrac{t}{2}\right)\right] \end{aligned}$$

$$= \tfrac{1}{2}\left[\cos\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{3t}{2}\right) + \cos\left(\tfrac{3t}{2}\right) - \cos\left(\tfrac{5t}{2}\right) + \cdots + \cos\left(nt-\tfrac{t}{2}\right) - \cos\left(nt+\tfrac{t}{2}\right)\right]$$

$$= \tfrac{1}{2}\left[\cos\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{(2n+1)t}{2}\right)\right].$$

Therefore, $S_n = \dfrac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$.

(b) Exercise 6. ■

Lemma 8.5.5 (a) *$\forall t \in \mathbb{R}$, the partial sums $\sum_{k=1}^{n} \sin kt$ are bounded.*

(b) *$\forall t \neq 2p\pi$, $(p \in \mathbb{Z})$, the partial sums $\sum_{k=1}^{n} \cos kt$ are bounded.*

Proof. (a) Let t be a real number and $\forall n \in \mathbb{N}$, let $S_n = \sum_{k=1}^{n} \sin kt$. If $t = 2p\pi$ for some $p \in \mathbb{Z}$, all terms of $\sum_{k=1}^{n} \sin kt$ are 0, hence this sum is bounded. So in the remainder of the proof we assume t is not an integral multiple of 2π. By Lemma 8.5.4, $S_n = \dfrac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$, so $\forall n \in \mathbb{N}$,

$$|S_n| \leq \frac{\left|\cos(\frac{t}{2})\right| + \left|\cos\left(n+\frac{1}{2}\right)t\right|}{\left|2\sin\frac{t}{2}\right|} \leq \frac{2}{\left|2\sin\frac{t}{2}\right|}.$$

Since t is a fixed real number, $|S_n|$ is bounded above.

(b) Exercise 7. ■

With the help of the two previous lemmas and Dirichlet's test, we can easily prove the following important result about trigonometric series.

Theorem 8.5.6 *If $\{a_k\}$ is a monotone decreasing sequence converging to 0, then*

(a) $\sum_{k=1}^{\infty} a_k \sin kt$ *converges $\forall t \in \mathbb{R}$, and*

(b) $\sum_{k=1}^{\infty} a_k \cos kt$ *converges $\forall t \neq 2p\pi$, $(p \in \mathbb{Z})$*

(and may also converge when $t = 2p\pi$).

Proof. Exercise 8. ■

Examples 8.5.7 (a) $\sum_{k=1}^{\infty} \frac{1}{k} \sin kt$ converges $\forall t \in \mathbb{R}$.

(b) $\sum_{k=1}^{\infty} \frac{1}{k} \cos kt$ converges $\forall t \neq 2p\pi$, and diverges $\forall t = 2p\pi$, $(p \in \mathbb{Z})$.

(c) Both $\sum_{k=1}^{\infty} \frac{1}{k^2} \sin kt$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos kt$ converge (absolutely) $\forall t \in \mathbb{R}$.

Another test of convergence of series of products, closely related to Dirichlet's test, is Abel's test.

Theorem 8.5.8 (*Abel's Test*) *If $\sum a_k$ converges and $\{b_k\}$ is a bounded, monotone sequence, then $\sum a_k b_k$ converges.*

Proof. Apply Dirichlet's test to the series $\sum(b - b_k)a_k$ or to the series $\sum(b_k - b)a_k$. (Work out the details in Exercise 11.) ■

Example 8.5.9 The series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k \tan^{-1} k}{k}$ converges by Abel's test, since $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{k}$ converges and $\{\tan^{-1} k\}$ is monotone increasing and bounded above by $\pi/2$.

DOT PRODUCT OF SEQUENCES

In a linear algebra course, a "finite sequence" of real numbers,

$$(x_1, x_2, x_3, \cdots, x_n)$$

is called an ***n*-tuple** or ***n*-vector**. We often use "vector" notation $\overrightarrow{x}$ to denote an n-tuple:

$$\overrightarrow{x} = (x_1, x_2, x_3, \cdots, x_n).$$

Recall that we **add** n-vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ according to the rule

$$\begin{aligned}\overrightarrow{x} + \overrightarrow{y} &= (x_1, x_2, x_3, \cdots, x_n) + (y_1, y_2, y_3, \cdots, y_n)\\ &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots, x_n + y_n),\end{aligned}$$

and we **multiply** an n-vector $\overrightarrow{x}$ by a real number r (called a "scalar" in this context) according to the rule

$$r\overrightarrow{x} = r(x_1, x_2, x_3, \cdots, x_n) = (rx_1, rx_2, rx_3, \cdots, rx_n).$$

Examples 8.5.10 (a) $(2, -4, 5, 0, -8) + (-1, 8, -9, 7, 0) = (1, 4, -4, 7, -8)$.

(b) $-3(5, -2, 1, 0) = (-15, 6, -3, 0)$.

(c) $2(3, 4, -5) - 5(-2, 1, 7) = (6, 8, -10) + (10, -5, -35) = (16, 3, -45)$.

The set of all n-tuples of real numbers, together with the two algebraic operations we have just defined, is called **Euclidean *n*-space** and is denoted $\mathbb{R}^n$. The basic algebraic properties of $\mathbb{R}^n$ are listed in the following theorem, which is one of the fundamental results of elementary linear algebra.

Theorem 8.5.11 $\mathbb{R}^n$, *together with the operations of addition and multiplication by scalars defined above, has the following properties*:

1. $\forall \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$, $\overrightarrow{x} + \overrightarrow{y} \in \mathbb{R}^n$.

2. $\forall \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in \mathbb{R}^n,\ \overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z}) = (\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z}$.
3. $\forall \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n,\ \overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$.
4. $\exists\ \overrightarrow{0} \in \mathbb{R}^n \ni \forall \overrightarrow{x} \in \mathbb{R}^n,\ \overrightarrow{x} + \overrightarrow{0} = \overrightarrow{0} + \overrightarrow{x} = \overrightarrow{x}$. $[\overrightarrow{0} = (0,0,0,\cdots,0)]$.
5. $\forall \overrightarrow{x} \in \mathbb{R}^n, \exists -\overrightarrow{x} \in \mathbb{R}^n \ni \overrightarrow{x} + -\overrightarrow{x} = \overrightarrow{0}$. $[-\overrightarrow{x} = (-x_1, -x_2, -x_3, \cdots, -x_n)]$.
6. $\forall \overrightarrow{x} \in \mathbb{R}^n, \forall r \in \mathbb{R},\ \ r\overrightarrow{x} \in \mathbb{R}^n$.
7. $\forall \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n, \forall r \in \mathbb{R},\ \ r(\overrightarrow{x} + \overrightarrow{y}) = r\overrightarrow{x} + r\overrightarrow{y}$.
8. $\forall \overrightarrow{x} \in \mathbb{R}^n, \forall r, s \in \mathbb{R},\ \ (r+s)\overrightarrow{x} = r\overrightarrow{x} + s\overrightarrow{x}$.
9. $\forall \overrightarrow{x} \in \mathbb{R}^n, \forall r, s \in \mathbb{R},\ \ (rs)\overrightarrow{x} = r(s\overrightarrow{x})$.
10. $\forall \overrightarrow{x} \in \mathbb{R}^n,\ 1\overrightarrow{x} = \overrightarrow{x}$.

Proof. Consult any standard textbook in elementary linear algebra. ■

Because $\mathbb{R}^n$ has these properties it is called a **vector space**. All of the algebraic properties of a general vector space are derivable from these properties. They are well known from your linear algebra course and are not repeated here.

In $\mathbb{R}^n$ there is a kind of product often called an "inner product." Specifically, we define the **dot product** of two n-vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ in $\mathbb{R}^n$ as the sum of the products of their components:

$$\begin{aligned} \overrightarrow{x} \cdot \overrightarrow{y} &= (x_1, x_2, x_3, \cdots, x_n) \cdot (y_1, y_2, y_3, \cdots, y_n) \\ &= x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n \\ &= \sum_{i=1}^{n} x_iy_i. \end{aligned}$$

Example 8.5.12 $(5,2,-4,1) \cdot (3,-7,0,-8) = 15 - 14 - 0 - 8 = -7$.

The following theorem lists the basic algebraic properties of the dot product in $\mathbb{R}^n$.

Theorem 8.5.13 $\forall \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in \mathbb{R}^n$, *and* $\forall r \in \mathbb{R}$,
1. $\overrightarrow{x} \cdot \overrightarrow{y} = \overrightarrow{y} \cdot \overrightarrow{x}$
2. $(\overrightarrow{x} + \overrightarrow{y}) \cdot \overrightarrow{z} = (\overrightarrow{x} \cdot \overrightarrow{z}) + (\overrightarrow{y} \cdot \overrightarrow{z})$
3. $(r\overrightarrow{x}) \cdot \overrightarrow{y} = r(\overrightarrow{x} \cdot \overrightarrow{y})$
4. $\overrightarrow{x} \cdot \overrightarrow{x} \geq 0$; *moreover,* $\overrightarrow{x} \cdot \overrightarrow{x} = 0 \Leftrightarrow \overrightarrow{x} = 0$.

Proof. Consult any standard textbook in elementary linear algebra. ■

All of the algebraic properties of the dot product are derivable from the four properties listed in Theorem 8.5.13. Since their proofs are basic in any

linear algebra course, we omit them here. Because of property 4, given any $\overrightarrow{x} \in \mathbb{R}^n$, the square root $\sqrt{\overrightarrow{x} \cdot \overrightarrow{x}}$ is a real number. In $\mathbb{R}^2$ and $\mathbb{R}^3$, this quantity represents the **length** of the vector $\overrightarrow{x}$. So we generalize to $\mathbb{R}^n$ and call $\sqrt{\overrightarrow{x} \cdot \overrightarrow{x}}$ the **length** of any given vector $\overrightarrow{x}$ in $\mathbb{R}^n$.

One property of the dot product is so important to us that we prove it here as our next theorem.

Theorem 8.5.14 (***Cauchy-Schwarz Inequality***) $\forall \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$,

$$|\overrightarrow{x} \cdot \overrightarrow{y}| \leq \sqrt{\overrightarrow{x} \cdot \overrightarrow{x}}\sqrt{\overrightarrow{y} \cdot \overrightarrow{y}}.$$

(The absolute value of the dot product of two n-vectors is less than or equal to the product of their lengths.)

Proof. Let $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$. The algebraic properties of the dot product in $\mathbb{R}^n$ assure us that $\forall r \in \mathbb{R}$,

$$(r\overrightarrow{x} + \overrightarrow{y}) \cdot (r\overrightarrow{x} + \overrightarrow{y}) \geq 0$$

$$(r\overrightarrow{x} \cdot r\overrightarrow{x}) + (\overrightarrow{y} \cdot r\overrightarrow{x}) + (r\overrightarrow{x} \cdot \overrightarrow{y}) + (\overrightarrow{y} \cdot \overrightarrow{y}) \geq 0$$

$$r^2(\overrightarrow{x} \cdot \overrightarrow{x}) + 2r(\overrightarrow{x} \cdot \overrightarrow{y}) + (\overrightarrow{y} \cdot \overrightarrow{y}) \geq 0.$$

The left side of this inequality can be regarded as a quadratic expression in the variable r. Since this quadratic is always ≥ 0, its discriminant must be ≤ 0.

That is,

$$[2(\overrightarrow{x} \cdot \overrightarrow{y})]^2 - 4(\overrightarrow{x} \cdot \overrightarrow{x})(\overrightarrow{y} \cdot \overrightarrow{y}) \leq 0$$

$$\text{i.e., } 4(\overrightarrow{x} \cdot \overrightarrow{y})^2 \leq 4(\overrightarrow{x} \cdot \overrightarrow{x})(\overrightarrow{y} \cdot \overrightarrow{y}).$$

Dividing out the 4's and taking the square root of both sides, we have the desired inequality. ■

The reader may wonder what relevance the dot product in $\mathbb{R}^n$ has to the study of the convergence of $\sum_{k=1}^{\infty} a_k b_k$. The relevance becomes clear when we think of this series of products as a kind of "dot product" of infinite sequences:

$$\sum_{k=1}^{\infty} a_k b_k = (a_1, a_2, a_3, \cdots, a_n, \cdots) \cdot (b_1, b_2, b_3, \cdots, b_n, \cdots). \tag{19}$$

We shall find that this approach is exactly what we need, especially as it allows us to use the power of the Cauchy-Schwarz inequality.

SQUARE SUMMABLE SEQUENCES

We are seeking conditions on the series $\sum a_k$ and $\sum b_k$, weaker than absolute convergence, that will guarantee that $\sum_{k=1}^{\infty} a_k b_k$ converges. That is, we seek conditions that guarantee the existence of the "dot product" of the sequences $\{a_k\}$ and $\{b_k\}$, as defined in Equation (19) above. With the help of the Cauchy-Schwarz inequality we shall show that "square summability," as defined below, is sufficient.

Definition 8.5.15 We shall call a sequence $\{x_k\}$ of real numbers **summable**[7] if $\sum_{k=1}^{\infty} x_k$ converges, **absolutely summable** if $\sum_{k=1}^{\infty} |x_k|$ converges, and **square summable** if $\sum_{k=1}^{\infty} x_k^2$ converges.

Absolute summability is stronger than both summability and square summability (see Exercise 13). However, summability and square summability are not comparable; that is, neither is stronger than the other (see Exercise 14).

The following theorem shows that square summability of both $\{a_k\}$ and $\{b_k\}$ is enough to guarantee the convergence of $\sum a_k b_k$. It is the condition we have been seeking.

Theorem 8.5.16 *If $\{a_k\}$ and $\{b_k\}$ are square summable sequences, then $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.*

Proof. Suppose $\{a_k\}$ and $\{b_k\}$ are square summable and denote the partial sums of their associated series by $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. Consider a fixed

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7. These are definitions for temporary convenience only, since the term "summable" has a more refined definition in higher analysis.

$n \in \mathbb{N}$. Define $\overrightarrow{a} = (|a_1|, |a_2|, \cdots, |a_n|)$ and $\overrightarrow{b} = (|b_1|, |b_2|, \cdots, |b_n|)$. Then, using the Cauchy-Schwarz inequality, the partial sums of $\sum_{k=1}^{\infty} |a_k b_k|$ satisfy

$$\begin{aligned}
\sum_{k=1}^{n} |a_k b_k| &= |a_1 b_1| + |a_2 b_2| + \cdots + |a_n b_n| \\
&= (|a_1|, |a_2|, \cdots, |a_n|) \cdot (|b_1|, |b_2|, \cdots, |b_n|) \\
&= \overrightarrow{a} \cdot \overrightarrow{b} \leq \sqrt{\overrightarrow{a} \cdot \overrightarrow{a}} \sqrt{\overrightarrow{b} \cdot \overrightarrow{b}} \quad \text{(Cauchy-Schwarz inequality)} \\
&\leq \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2} \sqrt{|b_1|^2 + |b_2|^2 + \cdots + |b_n|^2} \\
&= \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 \\
&= \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \\
&\leq \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} b_k^2.
\end{aligned}$$

(These series converge since $\{a_k\}$ and $\{b_k\}$ are square summable.)

Hence the sequence of partial sums $\left\{ \sum_{k=1}^{n} |a_k b_k| \right\}$ is bounded above. So, by Theorem 8.2.2, $\sum_{k=1}^{\infty} |a_k b_k|$ converges; i.e., $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely. ■

Square summable sequences are of great importance in the study of "sequence spaces," especially Hilbert Space.

EXERCISE SET 8.5

1. Find convergent series $\sum a_k$ and $\sum b_k$ such that $\sum a_k b_k$ diverges.

2. Find divergent series $\sum a_k$ and $\sum b_k$ such that $\sum a_k b_k$ converges.

3. Prove Theorem 8.5.1.[8]

4. Use Theorem 8.5.1 to prove that if $\sum a_k$ and $\sum b_k$ are both absolutely convergent, then so is $\sum a_k b_k$.

5. To see the similarity between Abel's "summation by parts" formula and the familiar integration by parts formula, work out the details of the following. Define $a_0 = 0$ and regard the sequence $\{a_n\}$ as the sequence of partial sums of a series $\sum x_k$ (see Exercise 8.1.6). Also, $\forall k \in \mathbb{N}$, define $\Delta a_k = a_{k+1} - a_k$ and $\Delta b_k = b_{k+1} - b_k$. With these definitions, using

8. This exercise is identical to Exercise 8.3.12.

$m = 1$, and using the sequence $\{x_k\}$ instead of $\{a_k\}$, show that the conclusion of Theorem 8.5.2 becomes: $\sum\limits_{k=1}^{n} b_k \Delta a_{k-1} = a_n b_{n+1} - \sum\limits_{k=1}^{n} a_k \Delta b_k$.

6. Prove Lemma 8.5.4 (b).

7. Prove Lemma 8.5.5 (b).

8. Prove Lemma 8.5.6.

9. Prove that the convergence in Examples 8.5.7 (a) and (b) is not absolute, except in (a) when t is an integral multiple of π.

10. Prove that $\sum\limits_{k=1}^{\infty} \dfrac{\sin(ks)\cos(kt)}{k}$ converges for all real numbers t.

11. Work out the details of the proof of Theorem 8.5.8.

12. Investigate the convergence of $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{\sin^2 k}{k}$, as follows:

    (a) Use a trigonometric identity for $\sin^2 k$ to express the nth partial sum of this series as a combination of $\displaystyle\sum_{k=1}^{n} \frac{(-1)^k}{2k}$ and $\displaystyle\sum_{k=1}^{n}(-1)^k \frac{\cos 2k}{2k}$.

    (b) Use a trigonometric identity to show that $(-1)^k \cos 2k = \cos(\pi + 2)k$.

    (c) Apply Lemma 8.5.4 to show that $\displaystyle\left|\sum_{k=1}^{n} \cos(\pi + 2)k\right| \leq \frac{1}{\cos 1}$.

    (d) Apply Dirichlet's test to the second of the two series in (a).

    (e) Apply all of the above to prove that the given series converges.

    (f) Show that the convergence is not absolute.

13. Prove that absolute summability is weaker than square summability by showing that

    (a) an absolutely summable sequence must be square summable.

    (b) $\exists$ a square summable sequence that is not absolutely summable.

14. Prove that square summability is neither weaker nor stronger than summability (that is, neither condition implies the other) by finding

    (a) a summable sequence that is not square summable.

    (b) a square summable sequence that is not summable.

15. Prove that if $\{a_k\}$ is square summable, then $\displaystyle\sum_{k=1}^{\infty} \frac{a_k}{k}$ converges absolutely.

16. Prove that if $\{a_k\}$ is square summable, then $\lim_{k\to\infty} a_k = 0$.

17. Prove that if $\sum a_k$ is a convergent nonnegative series then $\sum_{k=1}^{\infty} \frac{\sqrt{a_k}}{k}$ converges.

18. Prove that the sum of two square summable sequences is square summable.

8.6 Power Series

The study of power series has been an essential part of analysis for over 300 years, and remains so today. Since the early days of the development of calculus, power series have been an indispensable tool for calculating the values of many complicated algebraic and transcendental functions, and a powerful technique for solving a wide variety of problems. Many brilliant discoveries in pure and applied mathematics have been made using power series. Undergraduates are often skeptical of such claims and only begin to accept the importance of power series when they see them at work in other courses such as differential equations, statistics, complex variables, and physics. In this section we present only the essential core of this important subject. We begin by saying what we mean by a power series.

Definition 8.6.1 A **power series** is a series of the form

$$\sum_{k=0}^{\infty} a_k(x-c)^k,$$

where c is a fixed real number and all the "coefficients" a_k are real numbers. Such a series is sometimes called a power series "in $x - c$" or a power series "about c."

When $c = 0$, we have the power series

$$\sum_{k=0}^{\infty} a_k x^k,$$

which is said to be a power series "in x" or "about the origin."

If we regard power series as functions of x we can write

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \quad \text{or} \quad f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then, of course, it is natural to ask what is the domain of such a function: i.e., for what values of x does the given power series converge? Fortunately, the set of points where a power series converges is a well-behaved set. In fact,

it is always an interval centered at c. The next theorem is the first step in understanding this situation.

Theorem 8.6.2 *If a power series* $\sum_{k=0}^{\infty} a_k(x-c)^k$ *converges for some* $x_1 \neq c$, *then it converges absolutely whenever* $|x-c| < |x_1 - c|$; *that is, for all* x *in the interval* $(c-\varepsilon, c+\varepsilon)$, *where* $\varepsilon = |x_1 - c|$.

Proof. Suppose $\sum_{k=0}^{\infty} a_k(x_1-c)^k$ converges for some $x_1 \neq c$. Let $\varepsilon = |x_1 - c|$ and choose any $x \in (c-\varepsilon, c+\varepsilon)$. By the general term test, $a_k(x_1-c)^k \to 0$.

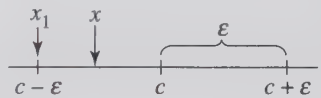

Figure 8.2

Hence $\{a_k(x_1-c)^k\}$ is a bounded sequence, so $\exists M > 0$ such that $\forall k$,

$$\begin{aligned} |a_k(x_1-c)^k| &< M \\ |a_k|\,|x_1-c|^k &< M. \end{aligned}$$

Now, $x \in (c-\varepsilon, c+\varepsilon)$, so $|x-c| < \varepsilon$.

Thus $\forall k$, $\quad |a_k(x-c)^k| = |a_k|\,|x-c|^k = |a_k|\,|x_1-c|^k \left|\dfrac{x-c}{x_1-c}\right|^k$

$$< M\left|\frac{x-c}{x_1-c}\right|^k = Mr^k, \text{ where } r = \left|\frac{x-c}{x_1-c}\right|.$$

Now $0 < r < 1$ since $r = \dfrac{|x-c|}{|x_1-c|} < 1$. Thus, $\sum Mr^k$ is a convergent (geometric) series. Therefore, by the comparison test, $\sum_{k=0}^{\infty} |a_k(x-c)^k|$ converges. ■

Corollary 8.6.3 *If a power series* $\sum_{k=0}^{\infty} a_k(x-c)^k$ *does not converge absolutely for some* $x_1 \neq c$, *then it diverges whenever* $|x-c| > |x_1-c|$; *that is, for all* x *outside the interval* $[c-\varepsilon, c+\varepsilon]$, *where* $\varepsilon = |x_1-c|$.

Proof. Exercise 1. ■

Corollary 8.6.4 *The set of points x for which a given power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges is a nonempty interval centered at c. That is, it must be one of the following: $\{c\}$, $\mathbb{R}$, or a bounded interval with endpoints $c-\rho$ and $c+\rho$, for some $\rho > 0$. The series converges absolutely in the interior of this interval, but may or may not converge at the endpoints.*

Proof. Let $A = \left\{ x : \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges} \right\}$, and suppose $A \neq \{c\}$ and $A \neq \mathbb{R}$. Then $\exists x_0 \neq c \ni$ the given series diverges when $x = x_0$. By Corollary 8.6.3, the series diverges for all x outside the interval $[c-\varepsilon, c+\varepsilon]$, where $\varepsilon = |x_0 - c|$. Hence, $A \subseteq [c-\varepsilon, c+\varepsilon]$. Thus, A is a bounded set. Hence, the set

$$B = \left\{ |x-c| : \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges} \right\} = \{|x-c| : x \in A\}$$

is also a bounded nonempty set. Thus, it has a least upper bound, say $\rho = \sup B$. We shall prove that $(c-\rho, c+\rho) \subseteq A \subseteq [c-\rho, c+\rho]$.

(a) Suppose $x \in (c-\rho, c+\rho)$. Then $|x-c| < \rho = \sup B$ so $\exists\ |x_1 - c| \in B \ni |x-c| < |x_1 - c|$. That is, $\exists x_1 \in A \ni |x-c| < |x_1-c|$. By Theorem 8.6.2, the given series converges (absolutely) at x. Therefore, $(c-\rho, c+\rho) \subseteq A$.

(b) Suppose $x \in A$. Then $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges, so $|x-c| \in B$. Thus, $|x-c| \leq \sup B = \rho$. That is,

$$c-\rho \leq x \leq c+\rho$$
$$\text{i.e., } x \in [c-\rho, c+\rho].$$

Therefore, $A \subseteq [c-\rho, c+\rho]$.

(c) Putting together (a) and (b) we have

$$(c-\rho, c+\rho) \subseteq A \subseteq [c-\rho, c+\rho].$$

As noted in (a) above, the series converges absolutely for all x in $(c-\rho, c+\rho)$. ■

Definition 8.6.5 We call the set of real numbers x for which a given power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges the **interval of convergence** of the series. If that interval is bounded, we call the number ρ of Theorem 8.6.4 the **radius of convergence** of the series. If the series converges only at $x = c$ we say that

the radius of convergence is 0, while if the series converges for all real numbers, we say that the radius of convergence is $+\infty$.

Example 8.6.6 Find the interval of convergence and the radius of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-3)^k}{k2^k}$.

Solution: As we may recall from elementary calculus, the ratio test (8.2.11) is useful here. We calculate the limit

$$L = \lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)2^{k+1}} \frac{k2^k}{(x-3)^k} \right| = \lim_{k\to\infty} \frac{k}{k+1} \left|\frac{x-3}{2}\right| = \left|\frac{x-3}{2}\right|.$$

The ratio test tells us that the series converges absolutely if $L < 1$ and diverges if $L > 1$. That is, the series converges absolutely if $|x-3| < 2$ and diverges if $|x-3| > 2$. Thus the series converges absolutely in the interval $(1,5)$ and diverges outside the interval $[1,5]$.

We test the endpoint $x = 1$:
$\displaystyle\sum_{k=1}^{\infty} \frac{(1-3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-2)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$, the alternating harmonic series, which converges.

We test the endpoint $x = 5$:
$\displaystyle\sum_{k=1}^{\infty} \frac{(5-3)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{(2)^k}{k2^k} = \sum_{k=1}^{\infty} \frac{1}{k}$, the harmonic series, which diverges.

Therefore, the interval of convergence of the given power series is $[1,5)$. The radius of convergence is 2. $\square$

Notice that in testing the endpoints of the interval of convergence in Example 8.6.6 we did not use the ratio test. In fact, we cannot use the ratio test at the endpoints, since they are precisely the points for which the ratio test is inconclusive.

Example 8.6.6 suggests the following formula for the radius of convergence of a power series.

Theorem 8.6.7 *The radius of convergence of the power series* $\displaystyle\sum_{k=0}^{\infty} a_k(x-c)^k$ *is* $\displaystyle\rho = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right|$ *if this limit exists, finite or infinite.*

Proof. Exercise 2. $\blacksquare$

The root test (8.2.15) may also be used in finding the interval of convergence of a power series. In fact, the upper limit form of the root test applies more generally because it does not require the existence of the limit.

Theorem 8.6.8 *The radius of convergence of the power series* $\sum_{k=0}^{\infty} a_k(x-c)^k$ *is* $\rho = \frac{1}{R}$, *where* $R = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}$. ($\rho = 0$ *if* $R = +\infty$, *and* $\rho = +\infty$ *if* $R = 0$.)

Proof. Exercise 3. ■

The reader has surely had experience finding the interval of convergence of power series in elementary calculus courses, so no additional examples will be worked here. More examples can be found in Exercise Set 8.6.

The next theorem will be no surprise. You probably already expect it.

Theorem 8.6.9 (*Algebra of Power Series*) *Suppose* $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ *and* $g(x) = \sum_{k=0}^{\infty} b_k(x-c)^k$, *with radii of convergence* ρ_f *and* ρ_g *respectively. Let* $\rho = \min\{\rho_f, \rho_g\}$. *Then, for all* x *in* $(c-\rho, c+\rho)$ *and all* $t \in \mathbb{R}$,

(a) $tf(x) = \sum_{k=0}^{\infty} ta_k(x-c)^k$,

(b) $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k)(x-c)^k$,

(c) $f(x)g(x) = \sum_{k=0}^{\infty} c_k(x-c)^k$, *where* $c_k = \sum_{k=0}^{\infty} a_j b_{k-j}$ *as in the Cauchy product of* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$.

Proof. Exercise 5. ■

POWER SERIES AS FUNCTIONS

It is clear that a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ may be regarded as a function of x. From Corollary 8.6.4 we know that the domain of this function is a nonempty interval centered at c, and may or may not include the endpoints of this interval if it has any. Naturally, we are concerned whether this function is differentiable on this interval. If so, it will also be continuous and integrable there.

Definition 8.6.10 Given a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$, the power series obtained from it by differentiating it term-by-term, $\sum_{k=0}^{\infty} ka_k(x-c)^{k-1}$, is called its **derived series**.

Theorem 8.6.11 *A power series and its derived series have the same radius of convergence.*

Proof. Consider a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$. Note that its derived series can be written as $\dfrac{1}{x-c}\sum_{k=0}^{\infty} ka_k(x-c)^k$. By Theorem 8.6.8, the radius of convergence of this series is

$$\overline{\lim_{k\to\infty}} \sqrt[k]{k\,|a_k|} = \overline{\lim_{k\to\infty}} \left(\sqrt[k]{k}\ \sqrt[k]{|a_k|}\right).$$

Recall from Example 2.3.8 that $\lim\limits_{k\to\infty} \sqrt[k]{k} = 1$. Thus, using Exercise 2.9.9,

$$\overline{\lim_{k\to\infty}} \sqrt[k]{k\,|a_k|} = \left(\lim_{k\to\infty} \sqrt[k]{k}\right)\left(\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}\right) = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|},$$

which establishes the desired result. ■

Corollary 8.6.12 *A power series* $\sum_{k=0}^{\infty} a_k(x-c)^k$ *and its term-by-term integrated series* $\sum_{k=0}^{\infty} \dfrac{a_k}{k+1}(x-c)^{k+1}$ *have the same radius of convergence.*

Proof. This is an immediate consequence of Theorem 8.6.11. ■

CAUTION: Theorem 8.6.11 does not say that the interval of convergence of a power series is the same as that of its derived series. The two series may behave differently at the endpoints of the interval. The following example should make that clear.

Example 8.6.13 The power series $\sum_{k=1}^{\infty} \dfrac{x^k}{k^2}$ and its derived series $\sum_{k=1}^{\infty} \dfrac{x^{k-1}}{k}$ have the same radius of convergence, $\rho = 1$. However, the interval of convergence of the given series is $[-1, 1]$ while the interval of convergence of its derived series is $[-1, 1)$.

Proof. Exercise 6. □

In the next group of results we show that a function representable as a power series with a positive radius of convergence must be quite well-behaved in its interval of convergence: it must be differentiable and, consequently, continuous and integrable there. We shall see that these results follow simply and elegantly from one fundamental result, Theorem 8.6.14. Unfortunately, the proof of this theorem is somewhat tedious. In Chapter 9 it will be seen as a straightforward consequence of "uniform convergence," to be defined in that chapter.

Theorem 8.6.14 *If a function f is representable as a power series with nonzero radius of convergence, then f is differentiable at every point in the interior of its interval of convergence; moreover, its derived series is its derivative. That is, if*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$$

with interval of convergence I, then at every point x in the interior of I,

$$f'(x) = \sum_{k=1}^{\infty} a_k k (x-c)^{k-1}.$$

Proof. Part 1: We first consider the case $c = 0$ to minimize the notational complexity. Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for every $x \in I^\circ$, the interior of the interval of convergence of this power series. Let x be a fixed member of I°. Then, $\forall y \in I^\circ$, $y \neq x$,

$$f(y) - f(x) = \sum_{k=0}^{\infty} a_k y^k - \sum_{k=0}^{\infty} a_k x^k = \sum_{k=1}^{\infty} a_k (y^k - x^k)$$

(When $k = 0$, the term is 0.)

So,

$$\frac{f(y)-f(x)}{y-x} - \sum_{k=1}^{\infty} a_k k x^{k-1} = \sum_{k=2}^{\infty} a_k \left\{ \frac{y^k - x^k}{y-x} - kx^{k-1} \right\}$$

(When $k = 1$, the term is 0.)

$$= \sum_{k=2}^{\infty} a_k \left\{ (y^{k-1} + y^{k-2}x + y^{k-3}x^2 + \cdots + yx^{k-2} + x^{k-1}) - kx^{k-1} \right\}$$

$$= \sum_{k=2}^{\infty} a_k \{ (y^{k-1} - x^{k-1}) + (y^{k-2}x - x^{k-1}) + (y^{k-3}x^2 - x^{k-1}) + \cdots$$
$$+ (y^2 x^{k-3} - x^{k-1}) + (yx^{k-2} - x^{k-1}) + (x^{k-1} - x^{k-1}) \}.$$

Writing the sum within braces in reverse order, this sum becomes

$$\sum_{k=2}^{\infty} a_k\{x^{k-2}(y-x)+x^{k-3}(y^2-x^2)+x^{k-4}(y^3-x^3)+\cdots+x^{k-k}(y^{k-1}-x^{k-1}).$$

Notice that each term in the sum within braces has $y-x$ as a factor. Factoring out $y-x$, the total sum above is

$$\begin{aligned}\sum_{k=2}^{\infty} a_k\, &\{x^{k-2}(y-x)+x^{k-3}(y-x)(y+x)+x^{k-4}(y-x)(y^2+yx+x^2)+\cdots \\ &+ x^0(y-x)(y^{k-2}+y^{k-3}x+\cdots+yx^{k-3}+x^{k-2})\} \\ = (y-x)&\sum_{k=2}^{\infty} a_k\,\{x^{k-2}+x^{k-3}(y+x)+x^{k-4}(y^2+yx+x^2)+\cdots \\ &+x^0(y^{k-2}+y^{k-3}x+\cdots+yx^{k-3}+x^{k-2})\}.\end{aligned}$$

Let $M=\max(|x|,|y|\}$. Then the absolute value of the above sum is

$$\begin{aligned}&\le |y-x|\sum_{k=2}^{\infty}|a_k|\,\{M^{k-2}+M^{k-3}(2M)+M^{k-4}(3M^2)+\cdots+(k-1)M^{k-2}\} \\ &= |y-x|\sum_{k=2}^{\infty}|a_k|M^{k-2}\{1+2+3+\cdots+(k-1)\} \\ &= |y-x|\sum_{k=2}^{\infty}|a_k|\frac{(k-1)k}{2}M^{k-2}. \qquad (20)\end{aligned}$$

Now $M\in I^\circ$ and the series in (20) above, without absolute values, is the term-by-term differentiation of the derived series of the given series at $x=M$. Hence, by Theorem 8.6.11, this series must converge, say to S. Then, $\forall y\in I^\circ$,

$$\left|\frac{f(y)-f(x)}{y-x}-\sum_{k=1}^{\infty}a_k k x^{k-1}\right|\le |y-x|S.$$

Therefore, by the squeeze principle (Theorem 4.2.20 (b)),

$$\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\sum_{k=1}^{\infty}a_k k x^{k-1}.$$

Part 2: Now consider the case $c\ne 0$. Apply the chain rule to the result of Part 1. (Exercise 7.) ■

Corollary 8.6.15 *If* $f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$ *has interval of convergence* I*, with nonempty interior, then*

(a) f *is continuous at every* x *in the interior of* I.

(b) *f has derivatives of all orders at every x in the interior of I.*

(c) *For any x in the interior of I, $\int f\,dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} + C$, where C is an arbitrary constant.*

(d) *$\forall k \in \mathbb{N}$, the k^{th} derivative of f at c is $f^{(k)}(c) = k!\, a_k$.*

Proof. Parts (a)–(c) follow from Theorem 8.6.14. To prove (d), note that

$$\begin{aligned}
f^{(0)}(x) &= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + a_5(x-c)^5 + \cdots \\
f^{(1)}(x) &= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + 5a_5(x-c)^4 + 6a_6(x-c)^5 + \cdots \\
f^{(2)}(x) &= 2a_2 + 2\cdot 3a_3(x-c) + 3\cdot 4a_4(x-c)^2 + 4\cdot 5a_5(x-c)^3 + 5\cdot 6a_5(x-c)^4 + \cdots \\
f^{(3)}(x) &= 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-c) + 3\cdot 4\cdot 5a_5(x-c)^2 + 4\cdot 5\cdot 6a_5(x-c)^3 + \cdots \\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

When we substitute $x = c$ into the above equations we get

$$\begin{aligned}
f^{(0)}(c) &= a_0 = 0!\, a_0 \\
f^{(1)}(c) &= a_1 = 1!\, a_1 \\
f^{(2)}(c) &= 2a_2 = 2!\, a_2 \\
f^{(3)}(x) &= 2\cdot 3a_3 = 3!\, a_3 \\
&\vdots \\
f^{(k)}(x) &= k!\, a_k. \qquad \blacksquare
\end{aligned}$$

Corollary 8.6.16 (***Uniqueness of Power Series Representation***) *If a function f is representable as a power series with interval of convergence I, then that power series is unique. In fact,*

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

for every x in the interior of I.

Proof. Immediate consequence of Corollary 8.6.15. ■

Definition 8.6.17 The series given in Corollary 8.6.16 is called the **Taylor series** of f about c. In case $c = 0$, the series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

is called the **Maclaurin series** of f.

Corollary 8.6.16 says that if a function has a power series representation valid in an interval centered at c, it must be the Taylor series of f about c; there are no other power series representations of f valid in this interval. We shall now explore some implications of these results.

Examples 8.6.18 Maclaurin Series for $\ln(1+x)$, $\tan^{-1} x$, $\dfrac{1}{(1+x)^2}$ and $\dfrac{x}{(1+x)^2}$.

From our knowledge of geometric series we know that

$$\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k = 1 - x + x^2 - x^3 + x^4 - \cdots \tag{21}$$

everywhere in the interval $(-1, 1)$. Thus, this series must be the Taylor series of the function $\dfrac{1}{1+x}$ about 0—i.e., the Maclaurin series of $\dfrac{1}{1+x}$. The radius of convergence of this series is 1, and the series diverges at both endpoints.

Formula (21) can be used to obtain the Maclaurin series of other functions.

(a) Recall that when $x > -1$, $\ln(1+x) = \displaystyle\int \frac{1}{1+x}\,dx + C$. Integrating the series (21) term-by-term, we have

$$\begin{aligned}\int \frac{1}{1+x}\,dx &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} + C\\ &= \left(x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \cdots\right) + C,\end{aligned}$$

which converges for all x in $(-1, 1)$. That is, for all $-1 < x < 1$,

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \cdots + C.$$

To find the constant C we let $x = 0$ in this equation, and find that $C = 0$.

Therefore, for all x in $(-1, 1)$,

$$\boxed{\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1},} \tag{22}$$

which must be the Maclaurin series for $\ln(1+x)$.

(b) Recall that $\tan^{-1} x = \displaystyle\int \frac{1}{1+x^2}\,dx + C$. From formula (21) above,

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k = 1 - x^2 + x^4 - x^6 + x^8 - \cdots. \tag{23}$$

By Corollary 8.6.16 and Definition 8.6.17, this must be the Maclaurin series for $\dfrac{1}{1+x^2}$. The radius of convergence is again 1. Integrating term-by-term,

$$\int \frac{1}{1+x^2}\,dx = \sum_{k=0}^{\infty} \int (-1)^k x^{2k}\,dx = \left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}\right) + C.$$

Thus, $\tan^{-1} x = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + C$. Letting $x = 0$, we find that $C = 0$. Thus, for all x in $(-1, 1)$,

$$\boxed{\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1},} \tag{24}$$

which must be the Maclaurin series for $\tan^{-1} x$. By Theorem 8.6.12, the radius of convergence of this series is 1.

(c) To derive a series representation of $\dfrac{1}{(1+x)^2}$ we first note that

$$\frac{1}{(1+x)^2} = -\frac{d}{dx}\frac{1}{1+x}.$$

Differentiating the series (21) term-by-term and multiplying by -1, we get

$$\boxed{\begin{aligned} \frac{1}{(1+x)^2} &= \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1} \\ &= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots. \end{aligned}} \tag{25}$$

(d) Finally, the Maclaurin series for $\dfrac{x}{(1+x)^2}$ is easily found from (25) using the algebra of limits:

$$\begin{aligned} \frac{x}{(1+x)^2} &= x \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k \\ &= x - 2x^2 + 3x^3 - 4x^4 + 5x^5 - \cdots. \qquad \square \end{aligned}$$

BEHAVIOR AT ENDPOINTS

When a function is representable as a power series everywhere in the interior of its interval of convergence, the behavior of the power series at the endpoints of this interval is unpredictable and must be investigated separately at each endpoint. The power series may or may not converge at these endpoints, and even when it converges, it may not converge to the value of the function at that endpoint. Notice that the interval of convergence of the series (23) above is $(-1, 1)$ but the interval of convergence of the term-by-term integrated series (24) above is $[-1, 1]$. By Corollary 8.6.15 we can be sure that when $-1 < x < 1$, the series (24) converges to $\tan^{-1} x$. But to what does the series (24) converge when $x = -1$ or $x = 1$? We hope, of course, that it converges to $\tan^{-1} x$, but we have not yet proved that.

The following theorem takes care of this problem. It assures us that if the function is continuous at an endpoint and its power series representation converges at that point, it will converge to the value of the function there. In Example 8.6.21 we show how to apply this theorem.

Theorem 8.6.19 (*Abel's Theorem*) *Suppose* $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ *for all* $|x| < 1$.

(a) *If* $\sum\limits_{k=0}^{\infty} a_k$ *converges, then* $\lim\limits_{x\to 1^-} f(x) = \sum\limits_{k=0}^{\infty} a_k$.

(b) *If* $\sum\limits_{k=0}^{\infty} (-1)^k a_k$ *converges, then* $\lim\limits_{x\to -1^+} f(x) = \sum\limits_{k=0}^{\infty} (-1)^k a_k$.

Proof. Suppose $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ for all $|x| < 1$.

(a) Suppose $\sum\limits_{k=0}^{\infty} a_k = S$, and let $\{S_n\}$ denote its sequence of partial sums. By Abel's summation by parts formula (8.5.2), for all $n \geq 1$ and all $|x| < 1$,

$$\begin{aligned}\sum_{k=0}^{n} a_k x^k &= \sum_{k=0}^{n} S_k(x^k - x^{k+1}) + S_n x^{n+1} \\ &= (1-x)\sum_{k=0}^{n} S_k x^k + S_n x^{n+1}.\end{aligned}$$

Now, $\{S_n\}$ converges and $x^{n+1} \to 0$, so taking the limit of both sides of this equation as $n \to \infty$, the algebra of limits assures us that $\forall\ |x| < 1$, $\sum\limits_{k=0}^{\infty} S_k x^k$ exists and

$$f(x) = (1-x)\sum_{k=0}^{\infty} S_k x^k.$$

Let $\varepsilon > 0$. Since $S_n \to S$, $\exists\ n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |S_n - S| < \frac{\varepsilon}{2}$. Recall that $\forall\ |x| < 1$, $\sum\limits_{k=0}^{\infty} x^k = \dfrac{1}{1-x}$, so $(1-x)\sum\limits_{k=0}^{\infty} x^k = 1$. Thus, $\forall\ 0 < x < 1$,

$$\begin{aligned}|f(x) - S| &= \left|(1-x)\sum_{k=0}^{\infty} S_k x^k - S(1-x)\sum_{k=0}^{\infty} x^k\right| = |1-x|\left|\sum_{k=0}^{\infty} S_k x^k - \sum_{k=0}^{\infty} S x^k\right| \\ &= |1-x|\left|\sum_{k=0}^{\infty}(S_k - S)x^k\right| \leq (1-x)\sum_{k=0}^{\infty}|S_k - S||x^k| \\ &< |1-x|\left[\sum_{k=0}^{n_0}|S_k - S| + \sum_{k=n_0+1}^{\infty}\frac{\varepsilon}{2}|x|^k\right] \\ &< |1-x|\left[M + \frac{\varepsilon}{2}\frac{1}{1-|x|}\right], \quad \text{where } M = \sum_{k=0}^{n_0}|S_k - S| \\ &= |1-x|M + \frac{\varepsilon}{2}\frac{|1-x|}{1-|x|} = |1-x|M + \frac{\varepsilon}{2}. \qquad (\text{since } 0 < x < 1)\end{aligned}$$

Choose $\delta < \min\left\{1, \dfrac{\varepsilon}{2M}\right\}$. Then,

$$\begin{aligned} 1-\delta < x < 1 &\Rightarrow 1-x<\delta \\ &\Rightarrow (1-x)M + \frac{\varepsilon}{2} < \frac{\varepsilon}{2M}M + \frac{\varepsilon}{2} = \varepsilon \\ &\Rightarrow |f(x)-S| < \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x\to 1^-} f(x) = S = \sum\limits_{k=0}^{\infty} a_k$.

(b) Suppose $\sum\limits_{k=0}^{\infty}(-1)^k a_k$ converges. Then, letting $y=-x$,

$$\begin{aligned} \lim_{x\to -1^+} f(x) = \lim_{y\to 1^-} f(-y) = \lim_{y\to 1^-}\sum_{k=0}^{\infty} a_k(-y)^k = \lim_{y\to 1^-}\sum_{k=0}^{\infty}(-1)^k a_k y^k \\ = \sum_{k=0}^{\infty}(-1)^k a_k \text{ by (a).} \quad ■ \end{aligned}$$

Corollary 8.6.20 *Suppose* $f(x) = \sum\limits_{k=0}^{\infty} a_k x^k$ *for all* $|x| < \rho$.

(a) *If* $\sum\limits_{k=0}^{\infty} a_k\rho^k$ *converges, then* $\lim\limits_{x\to\rho^-} f(x) = \sum\limits_{k=0}^{\infty} a_k\rho^k$.

(b) *If* $\sum\limits_{k=0}^{\infty}(-1)^k a_k\rho^k$ *converges, then* $\lim\limits_{x\to -\rho^+} f(x) = \sum\limits_{k=0}^{\infty}(-1)^k a_k\rho^k$.

Proof. Exercise 14. ■

The following example will show how Theorem 8.6.19 is used in practice.

Examples 8.6.21 (a) In Example 8.6.18 (a) we showed that $\ln(1+x) = \sum\limits_{k=0}^{\infty}\dfrac{(-1)^k x^{k+1}}{k+1}$ for all x in $(-1,1)$. This series diverges when $x=-1$ but converges when $x=1$ (verify). According to Abel's theorem (8.6.19),

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1} = \lim_{x\to 1^-}\ln(1+x) = \ln 2$$

since $\ln(1+x)$ is continuous at $x=1$. Thus, the Maclaurin series for $\ln(1+x)$ is a valid representation on the interval $(-1,1]$. As a bonus we have another derivation of the sum of the alternating harmonic series (see Exercise 8.3.7),

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2.$$

(b) Applying similar reasoning to Example 8.6.18 (b), we can show that the Maclaurin series for $\tan^{-1} x$ is a valid representation of $\tan^{-1} x$ on the interval $[-1, 1]$. As a consequence we have the interesting result,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

(See Exercise 15.) From this we get a series representation for π, although a very slowly converging one. □

EXERCISE SET 8.6

1. Prove Corollary 8.6.3.

2. Prove Theorem 8.6.7.

3. Prove Theorem 8.6.8.

4. Find the radius of convergence and the interval of convergence of each of the following power series.

(a) $\sum_{k=1}^{\infty} \ln k(x+1)^k$ (b) $\sum_{k=0}^{\infty} \frac{k!}{2^k} x^k$

(c) $\sum_{k=1}^{\infty} \frac{x^k}{k^k}$ (d) $\sum_{k=1}^{\infty} \frac{(x+5)^k}{k^3 3^k}$

(e) $\sum_{k=0}^{\infty} \frac{k^2}{2^k}(x-3)^k$ (f) $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^k (x+4)^k$

(g) $\sum_{k=0}^{\infty} \frac{(-3)^k}{k+2}(x+1)^k$ (h) $\sum_{k=1}^{\infty} \frac{1}{\ln 2k}(x-2)^k$

(i) $\sum_{k=0}^{\infty} \left(\cos \frac{k\pi}{6}\right)^k (x-3)^k$ (j) $\sum_{k=0}^{\infty} \left(1 + \sin \frac{k\pi}{4}\right)^k (x+2)^k$

(k) $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}(x+2)^k$ (l) $\sum_{k=1}^{\infty} \frac{k!}{k^k}(x+7)^k$

(m) $1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}x^4 - \cdots$ [See Exercise 8.3.13.]

(n) $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{2^3} + \frac{x^4}{3^4} + \frac{x^5}{2^5} + \cdots = \sum_{k=0}^{\infty} a_k x^k$,

where $a_k = \left\{ \begin{array}{ll} \left(\frac{1}{2}\right)^k & \text{if } k \text{ is odd} \\ \left(\frac{1}{3}\right)^k & \text{if } k \text{ is even} \end{array} \right\}$.

5. Prove Theorem 8.6.9.

6. Prove the claims made in Example 8.6.13.

7. Complete Part 2 of the proof of Theorem 8.6.14.

8. Find the Taylor series for $\ln x$ $(x > 0)$ about 1, and find its interval of convergence. [Write $\frac{1}{x} = \frac{1}{1-(1-x)}$ and apply the method of Example 8.6.18.]

9. Use known power series to find Maclaurin series representing each of the following functions, and find the interval of convergence in each case:

   (a) $\dfrac{1}{1+x^3}$ (b) $\dfrac{x^2}{1+x^3}$ (c) $x^2\ln(1+x)$

   (d) $x^3\tan^{-1}x$ (e) $\dfrac{x^2}{(1+x^3)^2}$ (f) $\dfrac{x^4}{(1+x^3)^3}$

10. Use the Maclaurin series for $\dfrac{1}{1-x}$ and its derivatives to find each of the following sums. Determine the interval of convergence where appropriate.

    (a) $\sum\limits_{k=1}^{\infty} kx^k$ (b) $\sum\limits_{k=1}^{\infty} k^2x^k$ (c) $\sum\limits_{k=1}^{\infty} \dfrac{k}{3^k}$ (d) $\sum\limits_{k=1}^{\infty} \dfrac{k^2}{3^k}$

11. Explain why $|x|$, $\ln x$, and $\sqrt{x}$ have no power series representations about 0.

12. Show that x^p has no power series representation about 0 if p is a real number other than a positive integer.

13. Prove that a Maclaurin series representing an even function has only even-powered terms, and a Maclaurin series representing an odd function has only odd-powered terms.

14. Prove Corollary 8.6.20. [Apply Theorem 8.6.19 to $f(u)$, where $u = \rho x$.]

15. Prove the assertions made in Example 8.6.21 (b).

16. Find a power series representation of $\int_0^x \tan^{-1} t\, dt$ and determine its interval of convergence (Abel's theorem may be helpful). Use this result to show that

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \cdots = \frac{\pi}{4} - \ln\sqrt{2}.$$

    (Partial fractions may also be helpful.)

17. For each of the following series, determine the values of x for which the series converges, and the values of x for which it converges absolutely.

    (a) $\sum\limits_{k=0}^{\infty} \dfrac{(-1)^k k^2}{(x-3)^k}$ (b) $\sum\limits_{k=0}^{\infty} \dfrac{\cos^k x}{k}$

18. Use Abel's theorem to prove Abel's claim that if the Cauchy product of two convergent series converges, its sum must be the product of their sums. (See notes after Theorem 8.4.4.)

8.7 Analytic Functions

In the concluding pages of Section 8.6, we found that if a function f is representable as a power series about a real number c then f must be infinitely differentiable at c, and the k^{th} coefficient in the power series is expressible in terms of the k^{th} derivative of f at c. We also saw that there can be only one such power series—the Taylor series of f about c, as specified in Definition 8.6.17. In this section we focus on the converse question: Given a function f that is infinitely differentiable at a real number c, how can we tell if f is representable as a power series in some neighborhood of c? We begin with a definition.

Definition 8.7.1 A function f is said to be **analytic at** c if there is a power series about c that represents f in some neighborhood of c. Equivalently, f is analytic at c if the Taylor series of f about c converges to $f(x)$ for all x in some neighborhood of c.

So, given a function f that is infinitely differentiable at c, we want to determine whether f is analytic at c. The fundamental tool used in answering this question is Taylor's theorem, which was proved and discussed in Section 6.5. Let us take another look.

Definition 8.7.2 Suppose f is infinitely differentiable at c. Then, $\forall n \in \mathbb{N}$, the n^{th} **Taylor polynomial** for f about c is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k, \tag{26}$$

and the n^{th} Taylor **remainder** for f about c is

$$R_n(x) = f(x) - T_n(x). \tag{27}$$

Thus, $\forall x \in I$,

$$f(x) = T_n(x) + R_n(x). \tag{28}$$

To say that f has a power series representation in a neighborhood of c is to say that for all x in this neighborhood, $\lim_{n\to\infty} T_n(x) = f(x)$. Equivalently, $\lim_{n\to\infty} R_n(x) = 0$. Taylor's theorem is the principal tool we use in showing that this limit is 0; it enables us to get a handle on $R_n(x)$.

Theorem 8.7.3 (***Taylor's Theorem, with Various Forms of the Remainder***) *Suppose f is n times differentiable on an open interval containing*

c and x, where $x \neq c$, and $f^{(n+1)}(t)$ exists for all t in the open interval I between c and x. With $T_n(x)$ and $R_n(x)$ as defined above,

(a) $\exists z \in I \ni$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}. \tag{29}$$

*[Formula (29) is called the "**Lagrange form**" of the remainder.]*

(b) *if $f^{(n+1)}$ exists and is integrable on $[c, x]$ if $c < x$, or $[x, c]$ if $x < c$,*

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)dt. \tag{30}$$

*[Formula (30) is called the "**integral form**" of the remainder.]*

(c) *if $f^{(n+1)}$ is continuous on $I = [c, x]$ if $c < x$, or $I = [x, c]$ if $x < c$, then $\exists z \in I \ni$*

$$R_n(x) = \frac{f^{(n+1)}(z)}{n!}(x-z)^n(x-c). \tag{31}$$

*[Formula (31) is called "**Cauchy's form**" of the remainder.]*

Proof. Part (a) was proved as Theorem 6.5.11 and Part (b) as Theorem 7.6.16. We can derive (c) from (b) by applying the first mean value theorem for integrals (Theorem 7.6.17) to (b). See Exercise 1. ■

We shall now show by means of examples how Taylor's theorem, with its various forms of the remainder, can be used to show that some familiar functions are analytic at a given real number c.

Examples 8.7.4 (Some Analytic Functions[9] and Their Maclaurin Series)

(a) $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ (valid for all $x \in \mathbb{R}$).

(b) $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ (valid for all $x \in \mathbb{R}$).

(c) $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ (valid for all $x \in \mathbb{R}$).

9. For a discussion of the problem of defining these functions, see Section 7.7.

Proof. (a) See Examples 6.5.5 and 6.5.13.
(b) One can easily verify that the Maclaurin polynomials are (Exercise 4)

$$T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$
$$T_0(x) = 0, \text{ and } T_{2n}(x) = T_{2n-1}(x).$$

We use the ratio test to determine convergence of $\sum a_k$ with $a_k = \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$.

$$\lim_{k\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+3}}{(2k+3)!}\cdot\frac{(2k+1)!}{x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{x^2}{(2k+3)(2k+2)}\right| = 0.$$

Thus, this series converges for all $x \in \mathbb{R}$; the interval of convergence is $(-\infty, +\infty)$.

To prove that the series converges to $\sin x$, we must prove that $\forall\, x \in \mathbb{R}$, $R_n(x) \to 0$. Let $x \in \mathbb{R}$. Using Taylor's theorem with the Lagrange form of the remainder, we have $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = \dfrac{\pm\sin z \text{ or } \pm\cos z}{(n+1)!}x^{n+1}$ for some z between x and 0. Thus,

$$|R_n(x)| \le \frac{1}{(n+1)!}|x|^{n+1}.$$

Now x is fixed in this inequality, so by Corollary 2.3.11, $R_n(x) \to 0$.

(c) Exercise 5. □

Our proof of Example 8.7.4 (b) illustrates a common method often useful in proving that $R_n(x) \to 0$. It is summarized in the following theorem.

Theorem 8.7.5 *Suppose f and all its derivatives exist and are bounded by a single constant on an open interval I containing c. If the Taylor series of f about c converges on I, then it converges to $f(x)$ for all $x \in I$;*

i.e., if $\exists\, M > 0 \ni \forall x \in I$ and $\forall k \in \mathbb{N}$, $\left|f^{(k)}(x)\right| \le M$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k,\ \forall x \in I.$$

Proof. See Theorem 6.5.14. ■

Example 8.7.6 Taylor series for e^x about c.

Using the Maclaurin series for e^x and the algebra of power series (8.6.9), we find that

$$e^x = e^c e^{x-c} = e^c \sum_{k=0}^{\infty} \frac{(x-c)^k}{k!} = \sum_{k=0}^{\infty} \frac{e^c(x-c)^k}{k!},$$

which is valid for all $x \in \mathbb{R}$. $\square$

Example 8.7.7 Taylor series for $\sin x$ about c.

Using the Maclaurin series for $\sin x$ and $\cos x$, and the algebra of power series Theorem (8.6.9), we find that for all $x \in \mathbb{R}$,

$$\begin{aligned}
\sin x &= \sin[c + (x-c)] = \sin c \cos(x-c) + \cos c \sin(x-c) \\
&= \sin c \sum_{k=0}^{\infty} \frac{(-1)^k (x-c)^{2k+1}}{(2k+1)!} + \cos c \sum_{k=0}^{\infty} \frac{(-1)^k (x-c)^{2k}}{(2k)!} \\
&= \sin c \left[1 - \frac{(x-c)^2}{2!} + \frac{(x-c)^4}{4!} - \cdots\right] + \cos c \left[(x-c) - \frac{(x-c)^3}{3!} + \frac{(x-c)^5}{5!} - \cdots\right] \\
&= \sum_{k=0}^{\infty} \frac{a_k (x-c)^k}{k!}, \text{ where } a_k = \left\{ \begin{array}{l} (-1)^{k/2} \sin c \text{ if } k \text{ is even} \\ (-1)^{(k-1)/2} \cos c \text{ if } k \text{ is odd} \end{array} \right\}. \quad \square
\end{aligned}$$

THE BINOMIAL SERIES

Recall the binomial theorem,[10] which says that $\forall n \in \mathbb{N}$, and $\forall x, y \in \mathbb{R}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k,$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. If we write $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$, then this formula makes sense even when $k > n$. In fact, for natural numbers $k > n$, $\binom{n}{k} = 0$. Because of this we have the infinite series representation

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k} y^k.$$

As a young man, Newton developed a power series expansion of $(1+x)^\alpha$ when α is not a positive integer. He considered his derivation and analysis of this series among his finest achievements. We now call this series the binomial series, but we use more modern methods in its analysis.

Definition 8.7.8 Given an arbitrary real number α, the series

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k,$$

10. See Exercise 1.3.24.

where $\binom{\alpha}{0} = 1$ and $\binom{\alpha}{k} = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$ when $k \geq 1$, is called the **binomial series**.

We shall show that this series converges to $(1+x)^\alpha$ for all $|x| < 1$. But first we prove a lemma that we will need.

Lemma 8.7.9 $\forall |x| < 1$, *and* $\forall \alpha \in \mathbb{R}$, $\lim_{n\to\infty} n\binom{\alpha}{n}|x|^n = 0$.

Proof. Suppose $|x| < 1$. Apply the ratio test to the series $\sum_{k=0}^{n} k\binom{\alpha}{k}x^k$:

$$\lim_{k\to\infty}\left|\frac{(k+1)\binom{\alpha}{k+1}x^{k+1}}{k\binom{\alpha}{k}x^k}\right| = |x|\lim_{k\to\infty}\frac{(k+1)}{k}\lim_{k\to\infty}\left|\frac{\binom{\alpha}{k+1}}{\binom{\alpha}{k}}\right|$$

$$= |x|\lim_{k\to\infty}\left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k)}{(k+1)!}\cdot\frac{k!}{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}\right|$$

$$= |x|\lim_{k\to\infty}\left|\frac{\alpha-k}{k+1}\right| = |x|\lim_{k\to\infty}\left|\frac{\frac{\alpha}{k}-1}{1+\frac{1}{k}}\right| = |x| < 1.$$

So by the ratio test, this series converges. Therefore, its general term must have limit 0. ■

Theorem 8.7.10 *Given an arbitrary real number* α, $\forall\, x \in (-1,1)$,

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k.$$

(*Convergence at the endpoints* -1 *and* 1 *depends on the value of* α.)[11]

Proof. Let $f(x) = (1+x)^\alpha$, where α is an arbitrary but fixed real number. We find the Maclaurin series for f.

$$\begin{aligned}
f(x) &= (1+x)^\alpha & f(0) &= 1\\
f'(x) &= \alpha(1+x)^{\alpha-1} & f'(0) &= \alpha\\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & f''(0) &= \alpha(\alpha-1)\\
&\ \vdots & &\ \vdots\\
f^{(k)}(x) &= \alpha(\alpha-1)\cdots(\alpha-k+1)(1+x)^{\alpha-k} & &\\
&= k!\binom{\alpha}{k}(1+x)^{\alpha-k} & f^{(k)}(0) &= k!\binom{\alpha}{k}.
\end{aligned}$$

11. For more complete details, see pages 567–572 of [32].

Then, $\forall k \in \mathbb{N}$, the k^{th} Maclaurin coefficient is

$$\frac{f^{(k)}(0)}{k!} = \binom{\alpha}{k}.$$

Therefore, the Maclaurin series for f is $\sum\limits_{k=0}^{\infty} \binom{\alpha}{k} x^k$. The radius of convergence of this series is $\lim\limits_{k\to\infty} \left| \frac{\binom{\alpha}{k+1}}{\binom{\alpha}{k}} \right| = 1$, as shown in the proof of Lemma 8.7.9.

Now that we know that the series $\sum\limits_{k=0}^{\infty} \binom{\alpha}{k} x^k$ converges everywhere in $(-1,1)$, it remains to prove that it converges to $(1+x)^\alpha$ for every x in this interval. We use the integral form of the remainder to show that $R_n(x) \to 0$. By Theorem 8.7.3 (b),

$$\begin{aligned}
R_n(x) &= \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)dt \\
&= \frac{1}{n!} \int_0^x (x-t)^n (n+1)! \binom{\alpha}{n+1} (1+t)^{\alpha-(n+1)} dt \\
&= (n+1)\binom{\alpha}{n+1} \int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{\alpha-1} dt.
\end{aligned}$$

$$\text{So, } |R_n(x)| = (n+1)\left|\binom{\alpha}{n+1}\right| \left| \int_0^x \left|\frac{x-t}{1+t}\right|^n |1+t|^{\alpha-1}\, dt \right|. \tag{32}$$

Case 1 ($\alpha > 1$): To eliminate the absolute value bars around the integral in (32) we must decide whether we integrate from 0 to x or from x to 0.

If $0 < x < 1$, then we integrate over the interval $0 \le t \le x$, and

$$0 \le \frac{x-t}{1+t} \le x \quad \text{and} \quad 0 \le 1+t \le 2.$$

If $-1 < x < 0$, then we integrate over the interval $x \le t \le 0$, and

$$x \le \frac{x-t}{1+t} \le 0 \text{ (Why?)} \quad \text{and} \quad 0 \le 1+t \le 1 < 2.$$

Plugging this information into (32) we have

$$\begin{aligned}
|R_n(x)| &\le (n+1)\left|\binom{\alpha}{n+1}\right| \left| \int_0^x |x|^n\, 2^{\alpha-1} dt \right| \\
&= (n+1)\left|\binom{\alpha}{n+1}\right| |x|^n\, 2^{\alpha-1} \left| \int_0^x dt \right| = (n+1)\left|\binom{\alpha}{n+1}\right| |x|^{n+1}\, 2^{\alpha-1}.
\end{aligned}$$

Applying Lemma 8.7.9, we see that this expression converges to 0 as $n \to \infty$. Therefore, $\lim\limits_{n\to\infty} R_n(x) = 0$.

Case 2 ($\alpha < 1$): Let $M = \max\{1, |1+x|^{\alpha-1}\}$. Arguing as in Case 1 above (Exercise 10) we can show that

$$|R_n(x)| \le (n+1)\left|\binom{\alpha}{n+1}\right|\left|\int_0^x |x|^n\, M\, dt\right| = (n+1)\left|\binom{\alpha}{n+1}\right| |x|^{n+1}\, M \to 0,$$

which completes the proof. ■

Example 8.7.11 Maclaurin series for $\dfrac{1}{\sqrt{1+x}}$:

$$\boxed{\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots \quad \text{on } (-1,1].}$$

Proof. This is a binomial $(1+x)^{-1/2}$. By Theorem 8.7.10 its Maclaurin series is

$$(1+x)^{-1/2} = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} x^k.$$

Now, when $k \ge 1$, $\dbinom{-\frac{1}{2}}{k} = \dfrac{(-1)^k 1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)}{2^k k!}$. (Show details.)

Thus, the Maclaurin series is $1 + \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k 1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)}{2^k k!} x^k$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2 2!}x^2 - \frac{1\cdot 3\cdot 5}{2^3 3!}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^5 5!}x^5 + \cdots.$$

By Theorem 8.7.10 we know the radius of convergence is 1. The series diverges when $x = -1$ but converges when $x = 1$. (See Exercises 8.2.43 and 8.3.13.) Since $(1+x)^{-1/2}$ is continuous at $x = 1$, Abel's theorem 8.6.19 guarantees that it converges to $(1+x)^{-1/2}$ everywhere on the interval $(-1,1]$. □

Example 8.7.12 Maclaurin series for $\sin^{-1} x$:

$$\boxed{\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\frac{x^9}{9} + \cdots \quad \textit{on } [-1,1].}$$

Proof. Recall that $\sin^{-1} x = \displaystyle\int \frac{dx}{\sqrt{1-x^2}}$. From Example 8.7.11, we have the Maclaurin series

$$(1-x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-x^2)^k$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2^k k!} (-1)^k x^{2k}.$$

For $|x| < 1$, we integrate term-by-term and find that

$\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = C + x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{(2k+1)2^k k!} x^{2k+1}$. Since $\sin^{-1} 0 = 0$, $C = 0$.

Therefore, when $|x| < 1$,

$$\sin^{-1} x = x + \frac{1}{3 \cdot 2 \cdot 1!} x^3 + \frac{1 \cdot 3}{5 \cdot 2^2 \cdot 2!} x^5 + \frac{1 \cdot 3 \cdot 5}{7 \cdot 2^3 \cdot 3!} x^7 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{9 \cdot 2^4 \cdot 4!} x^9 + \cdots. \quad (33)$$

We now test the endpoints of the interval $(-1, 1)$ for convergence of this series.

Test the endpoint $x = 1$. We apply Raabe's test (8.2.21):

$$R = \lim_{k \to \infty} k \left(1 - \frac{a_{k+1}}{a_k}\right)$$

$$= \lim_{k \to \infty} k \left(1 - \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdots (2k)(2k+2)(2k+3)} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2k)(2k+1)}{1 \cdot 3 \cdot 5 \cdots (2k-1)}\right)$$

$$= \lim_{k \to \infty} k \left(1 - \frac{(2k-1)^2}{(2k+2)(2k+3)}\right) = \lim_{k \to \infty} \frac{6k^2 + 5k}{4k^2 + 10k + 6} = \frac{3}{2}.$$ Since $R > 1$, Raabe's test tells us that the series converges.

Test the endpoint $x = -1$. Since the series (33) contains only odd-degree terms, the partial sums for $-x$ are the negatives of the partial sums for x. Therefore, the series (33) converges for $x = -1$.

Therefore, the interval of convergence of the series (33) is $[-1, 1]$. Since $\sin^{-1} x$ is continuous from the right at -1 and continuous from the left at 1, Abel's theorem 8.6.19 guarantees that this power series converges to $\sin^{-1} x$ everywhere on $[-1, 1]$. □

FURTHER THEORETICAL CONSIDERATIONS

Any function that is infinitely differentiable at c "has" a Taylor series in the sense that all its Taylor coefficients exist. However, it would be wishful thinking to conclude that such a function is necessarily analytic at c. For a

function f to be analytic at c its Taylor series must converge to $f(x)$ in some neighborhood of c. Thus, a function f that is infinitely differentiable at c can fail to be analytic at c in either of the following circumstances:

1. Its Taylor series converges in some neighborhood of c, but not to $f(x)$ in any neighborhood of c. [See Exercise 13.]

2. Its Taylor series diverges everywhere except at c; that is, its radius of convergence is 0. [See Exercise 14.]

For an example of a function whose Taylor series converges on an interval that is strictly smaller than the interval on which it is infinitely differentiable, see Exercise 15. For an example of a function whose Maclaurin series converges everywhere, but to $f(x)$ only on $(-\infty, 0]$, see Exercise 6.6.15.

In contrast to these negative examples, Boas [16] proves a very interesting theorem of S. Bernstein: "If f and all its derivatives are nonnegative in an interval, then f is analytic in that interval." Further examples and interesting discussions of these concerns can be found in [16] pages 179–183, [49] pages 68–70, and [61] page 318.

Finally, it is important to point out that analyticity does not occur at isolated points; if a function is analytic at c, it is analytic on an entire neighborhood of c. The following theorem makes this explicit.

***Theorem 8.7.13** *Suppose f has a power series representation $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, with radius of convergence $\rho > 0$. Then, $\forall$ d in the interior of the interval of convergence, f has a power series representation about d with radius of convergence at least $\rho - |c-d|$. In fact,*

$$f(x) = \sum_{k=0}^{\infty} b_k(x-d)^k, \text{ where } b_k = \sum_{j=0}^{\infty} \binom{k+j}{j} a_{k+j}(d-c)^j.$$

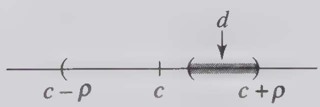

Figure 8.3

Proof. Suppose $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$, with radius of convergence $\rho > 0$, and let $d \in (c-\rho, c+\rho)$. Then

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} a_k[(d-c)+(x-d)]^k \text{ and, using the binomial theorem,} \\
&= \sum_{k=0}^{\infty} a_k \left[\sum_{j=0}^{k} \binom{k}{j} (d-c)^{k-j}(x-d)^j \right] \\
&= \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} a_k \binom{k}{j} (d-c)^{k-j}(x-d)^j \right]. \qquad (34)
\end{aligned}
$$

We first prove that this series converges absolutely in an open interval centered at d. Choose any x such that $|x-d| < \rho - |c-d|$, and let $t = |x-d| + |c-d|$. Then $0 < t < \rho$, and

$$
\begin{aligned}
\sum_{k=0}^{\infty} |a_k| \left| \sum_{j=0}^{k} \binom{k}{j} (d-c)^{k-j}(x-d)^j \right| &\leq \sum_{k=0}^{\infty} |a_k| \left(\sum_{j=0}^{k} \binom{k}{j} |d-c|^{k-j}|x-d|^j \right) \\
&= \sum_{k=0}^{\infty} |a_k| \left[|d-c| + |x-d|\right]^k \text{ (binomial theorem)} \\
&= \sum_{k=0}^{\infty} |a_k| t^k, \text{ which converges since } 0 < t < \rho.
\end{aligned}
$$

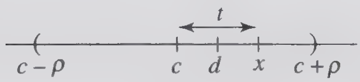

Figure 8.4

Thus the series (34) converges absolutely whenever $|x-d| < \rho - |c-d|$, and represents the sum of all terms in the "infinite matrix" below, adding first across the rows and then adding the row sums:

$$
\begin{bmatrix}
a_0\binom{0}{0} & 0 & 0 & 0 & \cdots \\
a_1\binom{1}{0}(d-c) & a_1\binom{1}{1}(x-d) & 0 & 0 & \cdots \\
a_2\binom{2}{0}(d-c)^2 & a_2\binom{2}{1}(d-c)(x-d) & a_2\binom{2}{2}(x-d)^2 & 0 & \cdots \\
a_3\binom{3}{0}(d-c)^3 & a_3\binom{3}{1}(d-c)^2(x-d) & a_3\binom{3}{2}(d-c)(x-d)^2 & a_3\binom{3}{3}(x-d)^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

In Theorem 8.7.16 below,[12] we prove that we get the same sum by adding down the columns and then adding the column sums. Now the sum of column

12. We place Theorem 8.7.16 below to avoid interupting the flow of ideas here.

k $(k = 0, 1, 2, 3, \cdots)$ is

$$\left[\sum_{j=0}^{\infty} \binom{k+j}{k} a_{k+j}(d-c)^j\right](x-d)^k.$$

(We know that the column sums exist because they are subseries of an absolutely convergent series.)

By Theorem 8.7.16, the sum of these column sums must be $f(x)$. Therefore,

$$f(x) = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{\infty} \binom{k+j}{j} a_{k+j}(d-c)^j\right](x-d)^k.$$

Since this is true whenever $|x-d| < \rho - |c-d|$, the radius of convergence is at least $\rho - |c-d|$. ∎

Theorem 8.7.13 is of little practical computational value. For example, try applying it to the Maclaurin series for $\ln(1+x)$ to calculate the Taylor coefficients b_k in the Taylor series for $\ln(1+x)$ about $c = 1/2$.

*DOUBLE SERIES

Definition 8.7.14 The notation $\sum\limits_{i,j=1}^{\infty} a_{ij}$ is used to represent the "sum" of all the entries in the infinite matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{35}$$

We shall not concern ourselves with all the various ways one can define such a sum, but shall focus on only two. We define the **row sums** $R_1, R_2, \cdots, R_i, \cdots$ and **column sums** $C_1, C_2, \cdots, C_j, \cdots$ of (35) by

$$R_i = \sum_{j=1}^{\infty} a_{ij} \quad \text{and} \quad C_j = \sum_{i=1}^{\infty} a_{ij}.$$

If all the row sums (series) R_i converge, then the series

$$\sum_{i=1}^{\infty} R_i = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right)$$

is called the **sum by rows** of $\sum\limits_{i,j=1}^{\infty} a_{i\,j}$. If all the column sums (series) C_j converge, then the series

$$\sum_{j=1}^{\infty} C_j = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{i\,j}\right)$$

is called the **sum by columns** of $\sum\limits_{i,j=1}^{\infty} a_{i\,j}$.

Example 8.7.15 To see that the sum by rows and the sum by columns can be quite different, consider the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \cdots \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \cdots \\ 0 & 0 & 0 & -\frac{1}{8} & -\frac{1}{8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

Here, the sum by columns is 2 but the sum by rows diverges. (Exercise 17.)

***Lemma 8.7.16** *Suppose all the entries of* (35) *are nonnegative. If every row sum R_i converges and the "sum by rows" $\sum\limits_{i=1}^{\infty} R$ converges, then*

(a) *every column sum C_j converges, and*

(b) *the sum by columns converges and equals the sum by rows; i.e.,*

$$\sum_{i=1}^{\infty} R_i = \sum_{j=1}^{\infty} C_j, \quad or \quad \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{i\,j}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{i\,j}\right).$$

Proof. Suppose all the entries of (35) are nonnegative, every row sum R_i converges, and the "sum by rows" $\sum\limits_{i=1}^{\infty} R_i$ converges.

(a) Let $i, j \in \mathbb{N}$. Then since the terms are all nonnegative, $a_{i\,j} \leq \sum\limits_{j=1}^{\infty} a_{i\,j}$, which converges (to R_i), so $a_{i\,j} \leq R_i$. Therefore, $\forall j \in \mathbb{N}$, $\sum\limits_{i=1}^{\infty} a_{i\,j}$ converges by comparison with the series $\sum\limits_{i=1}^{\infty} R_i$. That is, every column sum converges.

(b) $\forall n \in \mathbb{N}$, $\sum_{j=1}^{n} C_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{\infty} a_{i\,j} \right)$

$$= \sum_{i=1}^{\infty} \left(\sum_{j=1}^{n} a_{i\,j} \right) \quad \text{(by the linearity of series, Theorem 8.1.12)}$$

$$\leq \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{i\,j} \right) = \sum_{i=1}^{\infty} R_i.$$

Therefore, $\sum_{j=1}^{\infty} C_j$ converges and $\sum_{j=1}^{n} C_j \leq \sum_{i=1}^{\infty} R_i$.

Similarly, $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} a_{i\,j} \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{n} a_{i\,j} \right) \leq \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a_{i\,j} \right) = \sum_{j=1}^{\infty} C_j$.

■

Theorem 8.7.17** (Absolute Convergence of Double Series***) *Suppose the sum by rows of the double series $\sum_{i,j=1}^{\infty} |a_{ij}|$ converges. Then both the sum by rows and the sum by columns of the double series $\sum_{i,j=1}^{\infty} a_{ij}$ converge and are equal. In fact, $\sum_{i=1}^{n} R_i$ and $\sum_{j=1}^{\infty} C_j$ converge absolutely.*

Proof. Suppose the sum by rows of the double series $\sum_{i,j=1}^{\infty} |a_{ij}|$ converges. Using the notation of Definition 8.3.8, let $a_{i\,j}^{+} = \max\{a_{i\,j}, 0\}$ and $a_{i\,j}^{-} = \max\{-a_{i\,j}, 0\}$. By Lemma 8.3.9 and Theorem 8.3.10,

$$R_i = \sum_{j=1}^{\infty} a_{i\,j} = \sum_{j=1}^{\infty} (a_{i\,j}^{+} - a_{i\,j}^{-}) = \sum_{j=1}^{\infty} a_{i\,j}^{+} - \sum_{j=1}^{\infty} a_{i\,j}^{-}.$$

(Both of these nonnegative series converge by comparison with $\sum_{i,j=1}^{\infty} |a_{ij}|$.)

Consider the two matrices

$$\begin{bmatrix} a_{11}^{+} & a_{12}^{+} & a_{13}^{+} & \cdots & a_{1j}^{+} & \cdots \\ a_{21}^{+} & a_{22}+ & a_{23}^{+} & \cdots & a_{2\,j}^{+} & \cdots \\ a_{31}^{+} & a_{32}^{+} & a_{33}^{+} & \cdots & a_{3\,j}^{+} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}^{+} & a_{i2}^{+} & a_{i3}^{+} & \cdots & a_{i\,j}^{+} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11}^{-} & a_{12}^{-} & a_{13}^{-} & \cdots & a_{1j}^{-} & \cdots \\ a_{21}^{-} & a_{22}^{-} & a_{23}^{-} & \cdots & a_{2\,j}^{-} & \cdots \\ a_{31}^{-} & a_{32}^{-} & a_{33}^{-} & \cdots & a_{3\,j}^{-} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}^{-} & a_{i2}^{-} & a_{i3}^{-} & \cdots & a_{i\,j}^{-} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

(36) (37)

We have just seen that all the row sums of both of these nonnegative matrices (double series) converge. Further, $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{+}\right) \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} |a_{ij}|\right) \leq \text{sum by rows of } \sum_{i,j=1}^{\infty} |a_{ij}|, \text{ and}$$

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{-}\right) \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} |a_{ij}|\right) \leq \text{sum by rows of } \sum_{i,j=1}^{\infty} |a_{ij}|.$$

Thus, since their partial sums are bounded, the "sum by rows" of each of the double series (36) and (37) converge. Therefore, by Lemma 8.7.16, all their column sums converge and their sums by columns equal their sums by rows. That is,

$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{+}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{+}\right) \quad \text{and} \quad \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{-}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{-}\right).$$

Therefore $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{+}\right) - \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{-}\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{+} - \sum_{j=1}^{\infty} a_{ij}^{-}\right)$$

$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{+}\right) - \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty} a_{ij}^{-}\right).$$ Taking the limit as $n \to \infty$, we have

$$\sum_{i=1}^{\infty} R_i = \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{+}\right) - \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}^{-}\right)$$

$$= \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{+}\right) - \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{-}\right). \tag{38}$$

On the other hand, the sum by columns of $\sum_{i,j=1}^{\infty} a_{ij}$ is

$$\sum_{j=1}^{n} C_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty}(a_{ij}^{+} - a_{ij}^{-})\right)$$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_{ij}^{+}\right) - \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_{ij}^{-}\right).$$ Taking the limit as $n \to \infty$, we have

$$\sum_{j=1}^{\infty} C_j = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{+}\right) - \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}^{-}\right). \tag{39}$$

Putting together (38) and (39), we have $\sum_{i=1}^{\infty} R_i = \sum_{j=1}^{\infty} C_j$.

To see that $\sum_{i=1}^{\infty} R_i$ and $\sum_{j=1}^{\infty} C_j$ converge absolutely, observe that $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} |R_i| = \sum_{i=1}^{n} \left|\sum_{j=1}^{\infty} a_{i\,j}\right| \le \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} |a_{i\,j}|\right) \le \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a_{i\,j}|\right) \quad \text{and}$$

$$\sum_{j=1}^{n} |C_j| = \sum_{j=1}^{n} \left|\sum_{i=1}^{\infty} a_{i\,j}\right| \le \sum_{j=1}^{n} \left(\sum_{i=1}^{\infty} |a_{i\,j}|\right) \le \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a_{i\,j}|\right).$$

Since their partial sums are bounded, these nonnegative series converge. ■

EXERCISE SET 8.7

1. Complete the proof of Theorem 8.7.3 by deriving (c) from (b).

2. Consider the polynomial function $p(x) = 3x^4 - 5x^3 + x^2 - 8$.

   (a) Prove that p is analytic at 0; find its Maclaurin series and the radius of convergence.

   (b) Prove that p is analytic at 2; find its Taylor series about 2 and the radius of convergence. Simplify the result to show that it equals $p(x)$.

   (c) Is p analytic everywhere? Justify your answer.

3. On the basis of Exercise 2, state a theorem about polynomials, their analyticity, their Maclaurin series, and their Taylor series about $c \neq 0$.

4. Verify that the Maclaurin polynomials for $\sin x$ are as given in Example 8.7.4 (b).

5. Prove the claim made in Example 8.7.4 (c).

6. Use the results of Example 8.7.4 and the algebra of power series (8.6.9) to find Maclaurin series for the "hyperbolic" functions, $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Find their intervals of convergence and prove that these series converge to these functions everywhere in these intervals. Compare these series with the series for $\sin x$ and $\cos x$.

7. Use known power series and the methods of this section to derive Maclaurin series representations for each of the following functions. In each case find the interval of convergence.

   (a) $x^2 e^x$ (b) $x^3 \sin x$
   (c) $\sin x + \cos x$ (d) $x \ln(1 + x)$
   (e) $\cos^2 x$ [Use trig. identity.] (f) $\sin^2 x$

8. Prove the following slightly stronger version of Theorem 8.7.5: If f is infinitely differentiable on an open interval I containing c and $\exists\ M > 0 \ni \forall x \in I, \forall n \in \mathbb{N}, \left|f^{(n)}(x)\right| \leq M^n$, then $\forall x \in I$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

9. Use the method of Example 8.7.7 to find the Taylor series of $\cos x$ about c.

10. Complete the proof of Case 2 of Theorem 8.7.10.

11. Find each of the following integrals using power series. Can you find these integrals without using power series?

    (a) $\int e^{x^2}\,dx$ (b) $\int \frac{e^{-x}}{x}\,dx$

12. The Maclaurin series for e^x, $\sin x$, and $\cos x$ converge to these functions for all real numbers. Assume that the same is true for all "complex" numbers (real numbers in combination with the "imaginary" number $i = \sqrt{-1}$). Derive the identity $e^{ix} = \cos x + i \sin x$ and from it deduce Euler's famous identity, $e^{i\pi} = -1$. (If you have never seen this amazing identity before, you may need a little time to let it sink in.)

13. Consider the function[13] $f(x) = \left\{\begin{matrix} e^{-1/x^2} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{matrix}\right\}$. If you have not worked out Exercise 6.6.16, do so now. Use the result to show that the Maclaurin series of this function converges everywhere, but does not converge to $f(x)$ for any nonzero x.

14. Consider the function[14] $f(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k x)}{k!}$. Assuming that the successive derivatives of f can be found by term-by-term differentiation,[15] show that $\forall$ odd $k \in \mathbb{N}$, $f^{(k)}(0) = 0$ and that $\forall$ even $k \in \mathbb{N}$, $\left|f^{(k)}(0)\right| = e^{2^k} - 1$. Show that this yields an infinitely differentiable function whose Maclaurin series diverges everywhere except at 0.

15. Show that the function[16] $f(x) = \frac{1}{1+x^2}$ is infinitely differentiable everywhere, but that its Maclaurin series converges only for $|x| < 1$.

16. Prove that the sum by columns of the matrix given in Example 8.7.15 is 2, but the sum by rows diverges.

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13. This example appears in almost every textbook on this subject.

14. This example may be found in [61], page 256.

15. This will be shown in Chapter 9.

16. This function is suggested in [16], page 179.

17. For the double series $\sum\limits_{i,j=1}^{\infty} a_{ij}$ defined by the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

find the sum by rows and the sum by columns.

*18. Write the infinite matrix representing the "double geometric series" $\sum\limits_{i,j=0}^{\infty} r^i s^j$. Assuming $0 < |i|, |j| < 1$, find the sum by rows (or columns).

*19. We could say that a double series $\sum\limits_{i,j=1}^{\infty} a_{ij}$ converges to $S \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow \left| \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_{ij} - S \right| < \varepsilon$. In this definition, $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_{ij}$ plays the role of "partial sum." Explain why this method of summing a double series could be called "summing by upper left rectangles." Prove that if the sum by rows (or the sum by columns) of $\sum\limits_{i,j=1}^{\infty} |a_{ij}|$ converges, then $\sum\limits_{i,j=1}^{\infty} a_{ij}$ converges by this definition. Show that the double series of Exercise 17 does not converge by this definition.

*20. Given series $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$, find an infinite matrix whose sum by columns is their Cauchy product.

8.8 *Elementary Transcendental Functions (Project)

This section is designed as a project to be completed by students. It has no exercise set at the end. Instead, students are asked to furnish proofs for claims left unproved in the text.

Recall that a real number x is called an **algebraic number** if it is a solution of some polynomial equation

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$$

with integer coefficients $a_0, a_1, \cdots, a_n$. A **transcendental number** is a real number that is not algebraic. For example, 97, $\sqrt{13}$, and $\sqrt[5]{7 - \sqrt[3]{11}}$ are algebraic, but π, e, e^{100}, and $\sin 10$ are transcendental.

Similarly, a function $y = y(x)$ is an **algebraic function** if it is a solution of some polynomial identity (true for all values of x)

$$a_0(x) + a_1(x)y + a_2(x)y^2 + \cdots + a_n(x)y^n = 0,$$

where the coefficients $a_i(x)$ are polynomials with integer coefficients. A **transcendental function** is a function that is not algebraic. At one time mathematicians thought that polynomial equations could be solved by explicit formula using only "algebraic" methods (addition, subtraction, multiplication, division, and extraction of n^{th} roots for integral n). The work of Abel and Galois in the early nineteenth century proved this impossible in general. Nevertheless, algebraic functions are still regarded as simpler than transcendental functions since transcendental functions cannot be defined using polynomial equations.

Elementary functions are rational functions, e^x, $\sin x$, $\cos x$, and all functions that can be expressed as finite combinations of these and their inverses, using algebraic operations, composition, and inverses. These include polynomials, fractional powers, logarithms, all trigonometric functions and hyperbolic functions, and their inverses. Virtually all functions encountered in beginning calculus courses are elementary functions. Calling them elementary is just a convention of no great importance. Nonelementary functions abound in mathematics. Indeed, applied mathematicians may use nonelementary functions so routinely in their line of work that they find them as familiar as elementary functions.

Of concern in this section is how elementary transcendental functions can be defined and their values calculated. It is naive to believe, for example, that $2^{3.71}$, $\log_{10} 56$, $\sin 5.68$, or e^π can be calculated from the "definitions" given in elementary courses. We have devoted some space to this problem in earlier chapters of this book. In Section 5.6 we gave rigorous definitions of exponentials, powers, and logarithms using only continuity arguments. In Section 7.7 we gave definitions of $\ln x$, e^x, and the trigonometric functions using the Riemann integral. While these definitions are rigorous, they are not very useful in computation. We shall now show that power series provide the most direct and computationally useful definitions of the elementary transcendental functions.

We begin by asking you to imagine that you have never seen a definition of exponential or logarithm functions or any of the trigonometric functions. In what follows, you may use all the results of real analysis we have developed so far except those related to these functions. We are going to define these

functions and derive their familiar properties as if we had never encountered them before. We will use power series as our foundation.

EXPONENTIAL AND LOGARITHM FUNCTIONS

Definition 8.8.1 We define the function $E : \mathbb{R} \to \mathbb{R}$ by $E(x) = \sum\limits_{k=0}^{\infty} \frac{x^k}{k!}$.

(This series converges absolutely $\forall x \in \mathbb{R}$.)

Theorem 8.8.2 *The function $E : \mathbb{R} \to \mathbb{R}$ has the following properties:*

(a) $E(0) = 1$.

(b) $\forall x \in \mathbb{R}$, $E(x) > 0$.

(c) $\forall x \in \mathbb{R}$, *E is differentiable at x, and $E'(x) = E(x)$.*

(d) *E is strictly increasing on $\mathbb{R}$.*

Definition 8.8.3 $e = E(1)$. Note that $2 < e < 3$.

Remark 8.8.4 The series for e converges rapidly; in fact, $\left| e - \sum\limits_{k=0}^{n} \frac{1}{k!} \right| < \frac{1}{n!n}$.

$\left(\text{To prove this, show that } \sum\limits_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{n!} \sum\limits_{k=0}^{\infty} \frac{1}{(n+1)^k}. \right)$

Corollary 8.8.5 *e is irrational.*

Proof. For contradiction, suppose e is rational; say $e = m/n$ where $m, n \in \mathbb{N}$. Let $S_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$. Then, applying Remark 8.8.4,

$$0 < n!(e - S_n) < \tfrac{1}{n} < 1.$$

On the other hand, $n!(e - S_n) = n! \left(\frac{m}{n} - 1 - \frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \cdots - \frac{1}{n!} \right)$, which must be a positive integer. In that case we would have a positive integer between 0 and 1, a contradiction. Therefore, e is irrational. ■

Theorem 8.8.6 *The function $E : \mathbb{R} \to \mathbb{R}$ has the following properties:*

(a) $\forall x, y \in \mathbb{R}$, $E(x)E(y) = E(x + y)$. *[Use the Cauchy product formula.]*

(b) $\forall x, y \in \mathbb{R}$, $E(x)/E(y) = E(x - y)$ *and* $E(-x) = 1/E(x)$.

(c) $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$, $E(nx) = [E(x)]^n$.

(d) $\forall n \in \mathbb{N}$, $E(n) = e^n$, *and* $\forall m \in \mathbb{Z}$, $E(m) = e^m$.

(e) $\forall n \in \mathbb{N}$, $E(1/n) = e^{1/n}$, *and* $\forall r \in \mathbb{Q}$, $E(r) = e^r$.

(f) $\lim_{x\to\infty} E(x) = +\infty$; $\lim_{x\to-\infty} E(x) = 0$;

(g) *The range of* E *is* $(0, +\infty)$.

Theorem 8.8.7 *The function* $E(x)$ *is identical with the function* e^x *defined in Sections 5.6 and 7.7. That is,* $\forall x \in \mathbb{R}$, $E(x) = e^x$.

Proof. Apply Exercise 5.1.29. ■

Theorem 8.8.8 *The function* e^x *has an inverse.*

Definition 8.8.9 The inverse of the function e^x is (temporarily) called $L(x)$.

Theorem 8.8.10 *The function* $L(x)$ *is identical with the function* $\ln x$ *defined in Sections 5.6 and 7.7.*

TRIGONOMETRIC FUNCTIONS

Definition 8.8.11 We define the functions $S : \mathbb{R} \to \mathbb{R}$ and $C : \mathbb{R} \to \mathbb{R}$ by

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

(These series converge absolutely $\forall x \in \mathbb{R}$.)

Theorem 8.8.12 *The functions* $S : \mathbb{R} \to \mathbb{R}$ *and* $C : \mathbb{R} \to \mathbb{R}$ *have the following properties:*

(a) $\forall x \in \mathbb{R}$, $S(-x) = -S(x)$ *and* $C(-x) = C(x)$.

(b) $S(0) = 0$ *and* $C(0) = 1$.

(c) $S(x)$ *and* $C(x)$ *are differentiable everywhere, and* $\forall x \in \mathbb{R}$,

  (1) $S'(x) = C(x)$ *and* $C'(x) = -S(x)$;

  (2) $S''(x) = -S(x)$ *and* $C''(x) = -C(x)$.

Theorem 8.8.13 (a) $\forall x \in \mathbb{R}$, $S^2(x) + C^2(x) = 1$.

(b) $\forall x \in \mathbb{R}$, $|S(x)| \le 1$ *and* $|C(x)| \le 1$.

(c) $\forall x \in \mathbb{R}$, $S(x+y) = S(x)C(y) + C(x)S(y)$;

(d) $\forall x \in \mathbb{R}$, $C(x+y) = C(x)C(y) - S(x)S(y)$;

(e) $\forall x \in \mathbb{R}$, $S(x-y) = S(x)C(y) - C(x)S(y)$;

(f) $\forall x \in \mathbb{R}$, $C(x-y) = C(x)C(y) + S(x)S(y)$.

Proof. To prove (a), let $f(x) = S^2(x) + C^2(x)$, and prove that $f'(x) = 0$ for all x. This means that f must be a constant function. Find that constant and you will be done.

To prove (c) and (d) we can use a similar approach, but with more complicated functions. Let y be a fixed real number, and define F and G by

$$\begin{aligned} F(x) &= S(x+y) - S(x)C(y) - C(x)S(y), \\ G(x) &= C(x+y) - C(x)C(y) + S(x)S(y). \end{aligned}$$

Then prove that $\forall x \in \mathbb{R}$, $F^2(x) + G^2(x) = 0$ by the method used to prove (a).

Parts (e) and (f) follow easily from previously proved identities.

For a more challenging alternative approach, one could try to prove (c) and (d) by using Cauchy's product series formula on both terms of the right side of the equation and adding the results. ■

Lemma 8.8.14 *There exists a positive real number t such that $C(t) < 0$.*

Proof. Suppose that $\forall t > 0$, $C(t) \geq 0$. Then S is monotone increasing on $(0, \infty)$ since $S'(x) = C(x)$. Now,

$$S(1) = 1 - \tfrac{1}{3!} + \tfrac{1}{5!} - \tfrac{1}{7!} + \cdots > 0.$$

Let $x > 1$. By the mean value theorem, $\exists\, u \in (1, x) \ni$

$$\begin{aligned} \frac{C(x) - C(1)}{x-1} &= -S(u) \\ &\leq -S(1) \text{ since } S \text{ is monotone increasing.} \end{aligned}$$

Thus, $C(x) - C(1) \leq -(x-1)S(1)$, so

$$C(x) \leq C(1) - (x-1)S(1).$$

Now, when x is sufficiently large, $C(1) - (x-1)S(1) < 0$. Thus, $\exists\, x > 0 \ni C(x) < 0$. Contradiction. Therefore, $\exists\, t > 0 \ni C(t) < 0$. ■

Lemma 8.8.15 *There is a smallest positive real number t such that $C(t) = 0$.*

Proof. Let $A = \{t \geq 0 : C(t) = 0\}$. To see that A is nonempty, use Lemma 8.8.14 and the intermediate value theorem. Show that $\exists\; u = \inf A$, and then use Exercise 5.1.21, Theorem 3.2.8, and Exercise 3.2.7 to show that $u \in A$. ■

Definition 8.8.16 $\pi = 2u$, where $u = \min\{t > 0 : C(t) = 0\}$.
(That is, $\frac{\pi}{2}$ is the smallest positive real number x such that $C(x) = 0$.)

Theorem 8.8.17 (a) $C(\frac{\pi}{2}) = 0$, $S(\frac{\pi}{2}) = 1$.

(b) $S(\pi) = 0$; $C(\pi) = -1$; $S(2\pi) = 0$; $C(2\pi) = 1$.

(c) $\forall x \in \mathbb{R}$, $S(\frac{\pi}{2} - x) = C(x)$, *and* $C(\frac{\pi}{2} - x) = S(x)$.

(d) $S(x)$ *is increasing on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *and* $C(x)$ *is decreasing on* $[0, \frac{\pi}{2}]$.

Theorem 8.8.18 *$S(x)$ and $C(x)$ are periodic with period 2π. That is, 2π is the smallest real number k such that $\forall x \in \mathbb{R}$, $S(x+k) = S(x)$, and $C(x+k) = C(x)$.*

Proof. First show that when $k = 2\pi$, $\forall x \in \mathbb{R}$, $S(x + k) = S(x)$ and $C(x + k) = C(x)$. Then, for contradiction, suppose $\exists k \ni 0 < k < 2\pi$ satisfying these equations. Show that $S(k) = 0$ and $C(k) = 1$, and then using Theorem 8.8.13, show that $C(\frac{k}{2}) = 1$. This would contradict (8.8.15) and (8.8.16). Finally, show that $\forall x \in \mathbb{R}$, $S(x + k) = S(x) \Leftrightarrow \forall x \in \mathbb{R}$, $C(x + k) = C(x)$. ■

Theorem 8.8.19 (a) *The graph of $S(x)$ is symmetric relative to the line $x = \frac{\pi}{2}$; that is, $\forall x \in \mathbb{R}$, $S(\frac{\pi}{2} - x) = S(\frac{\pi}{2} + x)$;*

(b) $S(x)$ *is decreasing on* $[\frac{\pi}{2}, \frac{3\pi}{2}]$ *and increasing on* $[\frac{3\pi}{2}, 2\pi]$.

(c) $C(x)$ *is decreasing on* $[0, \pi]$ *and increasing on* $[\pi, 2\pi]$.

Theorem 8.8.20 *The functions $S(x)$ and $C(x)$ are identical to the functions $\sin x$ and $\cos x$ defined in Definitions 7.7.22 and 7.7.29.*

Proof. See Theorems 7.7.34 and 7.7.35. ■

Having defined the functions $S(x) = \sin x$ and $C(x) = \cos x$, we define the remaining trigonometric functions in the usual way, as given in Table 6.1 in Section 6.2.

Chapter 9

Sequences and Series of Functions

The chief concern of this chapter is uniform convergence and its consequences. By using the notion of distance between functions we set the stage for the study of function spaces in more advanced courses. In Section 9.4 we bring the course to a culmination in two famous theorems of Weierstrass: on the existence of continuous, nowhere differentiable functions, and on polynomial approximation of continuous functions.

In many areas of advanced analysis, significant power is gained by shifting our attention from sequences, series, and sets of *numbers* to sequences, series and "spaces" of *functions*. We begin this shift of attention here, by considering families of functions in Section 9.1. The meaning of convergence of a sequence of functions will be considered in Section 9.1, but a more satisfactory type of convergence will be defined in Section 9.2.

9.1 Families of Functions and Pointwise Convergence

Definition 9.1.1 Let $\mathcal{S}$ denote an arbitrary set. Any function $f : \mathcal{S} \to \mathbb{R}$ is called a **real-valued function** on $\mathcal{S}$. We shall consider the set of all such functions,

$$\mathcal{F}(\mathcal{S}, \mathbb{R}) = \{\text{all functions } f : \mathcal{S} \to \mathbb{R}\}.$$

On this set $\mathcal{F}(\mathcal{S}, \mathbb{R})$ we define algebraic operations. For every pair of functions $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, and $\forall r \in \mathbb{R}$, we define

(a) **Addition** of f and g by specifying that

$$\forall x \in \mathcal{S},\ (f+g)(x) = f(x) + g(x).$$

(b) **Multiplication of f by a "scalar"** r by specifying that

$$\forall x \in \mathcal{S},\ (rf)(x) = r \cdot f(x).$$

(c) **Multiplication of** f and g by specifying that

$$\forall x \in \mathcal{S},\ (fg)(x) = f(x) \cdot g(x).$$

(d) **Division of** f by g by specifying that

$$\forall x \in \mathcal{S},\ \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

(Notice that $\frac{f}{g} \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ only if $\forall x \in \mathcal{S},\ g(x) \neq 0$.)

(e) **The absolute value of** f: $\forall x \in \mathcal{S},\ |f|(x) = |f(x)|$.

(f) **The maximum of** f and g:

$$\forall x \in \mathcal{S},\ \max\{f, g\}(x) = \max\{f(x), g(x)\}.$$

(g) **The minimum of** f and g:

$$\forall x \in \mathcal{S},\ \min\{f, g\}(x) = \min\{f(x), g(x)\}.$$

Theorem 9.1.2 (***Algebra of Functions***) *If $\mathcal{S}$ is an arbitrary nonempty set then $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the operations (a) and (b) specified in Definition 9.1.1 above, satisfies the following ten properties:*

(1) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(2) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + (g + h) = (f + g) + h$;

(3) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + g = g + f$;

(4) $\exists\ 0 \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni \forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + 0 = 0 + f = f$;

(5) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \exists -f \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni f + (-f) = 0$;

(6) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ r \in \mathbb{R},\ rf \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(7) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\ r \in \mathbb{R},\ r(f + g) = rf + rg$;

(8) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\ r, s \in \mathbb{R},\ (r + s)(f) = rf + sf$;

(9) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $\forall\, r, s \in \mathbb{R}$, $r(sf) = (rs)f = s(rf)$;

(10) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $1f = f$;

Taking into account operation (c) *of Definition 9.1.1, $\mathcal{F}(\mathcal{S}, \mathbb{R})$ also satisfies the following five properties:*

(11) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $fg \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(12) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f(gh) = (fg)h$;

(13) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $fg = gf$;

(14) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, $f(g + h) = fg + fh$;

(15) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, *and* $\forall r \in \mathbb{R}$, $r(fg) = (rf)g = f(rg)$;

Proof. See Theorem B.3.3 in Appendix B. ■

Definition 9.1.3 Because $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the operations of addition and multiplication by scalars, satisfies the first ten properties of Theorem 9.1.2, it is called a **vector space**[1] of functions, or simply a **function space**. Any subset $\mathcal{G} \subseteq \mathcal{F}(\mathcal{S}, \mathbb{R})$ that also satisfies these ten properties relative to these two operations is called a **subspace** of $\mathcal{F}(\mathcal{S}, \mathbb{R})$.

Because $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the operations of addition, multiplication by scalars, and multiplication satisfies all fifteen properties of Theorem 9.1.2, it is called an **algebra** of functions. Any subset $\mathcal{G} \subseteq \mathcal{F}(\mathcal{S}, \mathbb{R})$ that also satisfies these fifteen properties relative to these three operations is called a **subalgebra** of $\mathcal{F}(\mathcal{S}, \mathbb{R})$.

In the remainder of this chapter we shall be concerned with vector spaces of functions and their subspaces, but we shall not pursue algebras of functions and their subalgebras.

Lemma 9.1.4 *Any subset of $\mathcal{F}(\mathcal{S}, \mathbb{R})$ that satisfies properties* (1) *and* (6) *of Theorem 9.1.2 is a subspace of $\mathcal{F}(\mathcal{S}, \mathbb{R})$.*

Proof. Consult any elementary linear algebra textbook. ■

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1. See also Theorem 8.5.11.

Examples 9.1.5 The following are among the many subspaces of $\mathcal{F}(\mathcal{S}, \mathbb{R})$:

(a) If $\mathcal{S}$ is any set, $B(\mathcal{S})$ = the set of all bounded[2] real-valued functions on S.

(b) If $\mathcal{S}$ is any set of real numbers, we define

(i) $C(\mathcal{S})$ = the set of all real-valued functions that are continuous on $\mathcal{S}$.

(ii) $D(\mathcal{S})$ = the set of all real-valued functions that are differentiable on $\mathcal{S}$.

(iii) $C^k(\mathcal{S})$ = the set of all real-valued functions that have continuous k^{th} derivative on $\mathcal{S}$.

(iv) $C^\infty(\mathcal{S})$ = the set of all real-valued functions that are infinitely differentiable on $\mathcal{S}$.

(c) If $\mathcal{S} = [a, b]$ is any compact interval, we define $R[a, b]$ to be the set of all real-valued functions that are Riemann integrable on $[a, b]$.

(d) If $\mathcal{F}_1$ is any subspace of $\mathcal{F}(\mathcal{S}, \mathbb{R})$ and x_0 is any member of $\mathcal{S}$, then the set of all $f \in \mathcal{F}_1$ such that $f(x_0) = 0$ is a subspace of $\mathcal{F}_1$.

Note that all subspaces listed in (b) above are subspaces of $B(\mathcal{S})$, defined in (a).

POINTWISE CONVERGENCE

Our first concern in this chapter is the notion of convergence of a sequence $\{f_n\}$ of functions. We must clarify what we mean by the limit statement

$$\lim_{n\to\infty} f_n = f$$

when $\{f_n\}$ is a sequence of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$ and $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. The simplest notion of convergence is called "pointwise" convergence, which we now define. Other types of convergence will be defined later.

Definition 9.1.6 (Pointwise Convergence) For a given sequence $\{f_n\}$ of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$ and a function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we say that f_n **converges pointwise** to f if $\forall x \in \mathcal{S}$, $f_n(x) \to f(x)$. We sometimes indicate this by writing

$$f_n \to f \text{ (pointwise).}$$

We say that a given **series** $\sum f_n$ of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$ **converges pointwise** to a function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ if its sequence of partial sums converges pointwise to f. Thus, when we say that a power series "represents" a function f on an interval we are saying that it converges pointwise to f on that interval.

While this type of convergence has a simple definition, it is not the most useful concept of convergence, as we shall see when we explore some examples.

2. See Definition 4.2.6.

Examples 9.1.7 We give here some examples of sequences $\{f_n\}$ and their pointwise limits. Exercise 2 asks you to prove the claims made here.

(a) Let $\mathcal{S} = [0,1]$ and $f_n(x) = x^n$. The graphs of f_n for $n = 1,2,3,4$ are shown in Figure 9.1. It is clear from Theorem 2.3.7 that the pointwise limit of the sequence $\{f_n\}$ is the function

$$f(x) = \begin{cases} 0 \text{ if } & 0 \le x < 1; \\ 1 \text{ if } & x = 1. \end{cases}$$

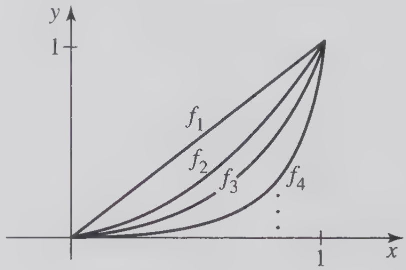

Figure 9.1

(See Figure 9.1.) Thus we say $f_n \to f$ (pointwise) on $[0,1]$.

(b) Let $\mathcal{S} = [0,1]$ and define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} 2n - 2n^2 x & \text{if } 0 \le x \le \frac{1}{n}; \\ 0 & \text{if } \frac{1}{n} < x \le 1. \end{cases}$

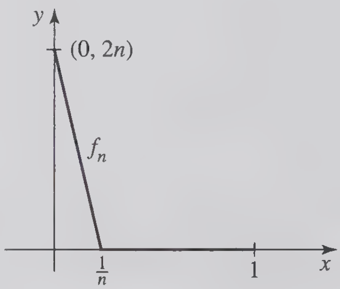

Figure 9.2

The graph of a typical f_n is shown in Figure 9.2. It is clear that the pointwise limit of $\{f_n\}$ on $(0,1]$ is $f(x) = 0$, but the pointwise limit does not exist on $[0,1]$ because $\lim\limits_{n\to\infty} f_n(0)$ does not exist.

(c) Let $\mathcal{S} = [-1,1]$ and define $f_n : [-1,1] \to \mathbb{R}$ by $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } |x| \le \frac{1}{n}; \\ |x| & \text{if } \frac{1}{n} < |x| \le 1. \end{cases}$

The graph of a typical f_n is shown in Figure 9.3.
It is clear that the pointwise limit of $\{f_n\}$ on $[-1,1]$ is $f(x) = |x|$.

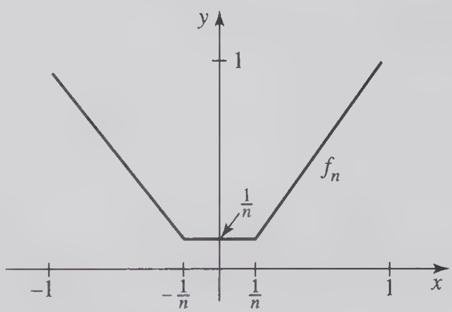

Figure 9.3

(d) Let $\mathcal{S} = [0,1]$ and define $f_n : [0,1] \to \mathbb{R}$

$$\text{by } f_n(x) = \begin{cases} 1 & \text{if } x = 0, \text{ or } x = \frac{m}{k} \text{ for some} \\ & \text{relatively prime } m, k \in \mathbb{N}, \\ & \text{where } k \le n; \\ 0 & \text{otherwise.} \end{cases}$$

The graph of a typical f_n is shown in Figure 9.4. It is clear that the pointwise limit of $\{f_n\}$ on $[0, 1]$ is the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \quad \square$$

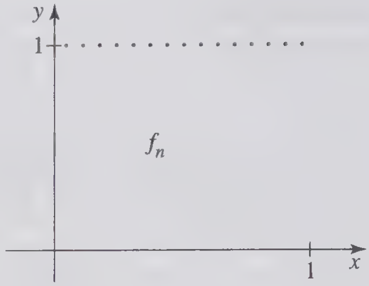

Figure 9.4

The examples we have just seen, along with additional examples provided in the exercises below, suggest that functions that are pointwise limits of sequences of functions do not necessarily share important features of the functions in the sequence. In Example 9.1.7(a) each function f_n is continuous on $[0, 1]$ but the pointwise limit function f is not continuous at 1. In Example 9.1.7(c) each function f_n is differentiable at 0 but the pointwise limit function f is not. In Example 9.1.7(d) each function f_n is integrable on $[0, 1]$ but the pointwise limit function f is not. (See Example 7.2.10.)

In Exercises 6 and 7 below, we shall see sequences $\{f_n\}$ of integrable functions that converge pointwise to an integrable function f for which $\lim_{n\to\infty} \int_a^b f_n \neq \int_a^b \lim_{n\to\infty} f_n$. In Exercise 8 we shall see a sequence $\{f_n\}$ of differentiable functions that converges pointwise to a differentiable function f for which $f_n' \not\to f'$.

Thus, while pointwise convergence is essential in establishing the limit function, it alone does not guarantee that the limit function inherits "nice" properties such as continuity, differentiability, and integrability from the functions in the sequence. In Section 9.2 we shall discuss a stronger type of convergence, known as "uniform" convergence, which does guarantee that limit functions inherit certain of these features from the functions in the sequence.

EXERCISE SET 9.1

1. Let $\mathcal{S} = [a, b]$ for some $a < b$. Which of the following are subspaces of $B(\mathcal{S})$? Verify your claim in each case.
   (a) The set of all $f \in B(\mathcal{S})$ that are piecewise-continuous on $\mathcal{S}$;
   (b) The set of all $f \in B(\mathcal{S})$ that are nonnegative on $\mathcal{S}$;
   (c) The set of all $f \in B(\mathcal{S})$ that are analytic on $\mathcal{S}$;
   (d) The set of all $f \in B(\mathcal{S})$ that are monotone increasing on $\mathcal{S}$;
   (e) The set of all $f \in B(\mathcal{S})$ that are monotone on $\mathcal{S}$;
   (f) The set of all $f \in B(\mathcal{S})$ such that $f(x_0) = 1$ for some $x_0 \in \mathcal{S}$.
2. Verify the claims made in Example 9.1.7.

3. For each of the following sequences of functions $\{f_n\}$, find the largest set S on which the sequence $\{f_n\}$ converges pointwise, and find the limit function f on that set. Sketch graphs of these functions where practical.

   (a) $f_n(x) = \dfrac{x}{n}$ (b) $f_n(x) = \tan^{-1}(nx)$

   (c) $f_n(x) = nxe^{-nx}$ (d) $f_n(x) = (1 - |x|)^n$

   (e) $f_n(x) = \sin^{2n} x$ (f) $f_n(x) = \cos^n(x)$

   (g) $f_n(x) = \dfrac{nx}{1+nx}$ (h) $f_n(x) = \dfrac{nx}{1+n^2x^2}$

   (i) $f_n(x) = \dfrac{x^n}{1+x^n}$ (j) $f_n(x) = \dfrac{x^2+nx}{3n+2}$

   (k) $f_n(x) = \dfrac{ne^x - n^2e^{-nx}}{n^2+1}$ (l) $f_n(x) = \left(1 + \dfrac{x}{n}\right)^n$

4. Despite the negative tone of Examples 9.1.7, some properties of functions are preserved by pointwise limits of sequences. For example, suppose $\{f_n\}$ is a sequence of functions defined on a set S, converging pointwise to a limit function f on S. Prove that

   (a) if each function f_n is bounded above (or below) by the same constant M on S, then so is f.

   (b) if each function f_n is monotone increasing (or decreasing) on S, then so is f.

5. Can you find a sequence $\{f_n\}$ of functions, each of which has a local maximum at some point x_0 but the limit function does not?

6. In a slight modification of Example 9.1.7 (b), define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2n - 2n^2x & \text{if } 0 < x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x \leq 1, \end{cases}$$

   and let f be the pointwise limit of f on $[0,1]$. Show that f_n and f are integrable on $[0,1]$ but $\lim\limits_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

7. Define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 3n+1 & \text{if } 0 < x \leq \frac{1}{n}, \\ 0 & \text{if } x = 0 \text{ or } x > \frac{1}{n}, \end{cases}$$

   and let f be the pointwise limit of f on $[0,1]$. Show that f_n and f are integrable on $[0,1]$ but $\lim\limits_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

8. $\forall n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$, define $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$. Prove that

    (a) $\{f_n\}$ converges to the 0 function on $(-\infty, +\infty)$.

    (b) every f_n is differentiable on $(-\infty, +\infty)$, and the limit function f has derivative 0 everywhere on $(-\infty, +\infty)$.

    (c) $\{f_n'\}$ does not converge pointwise to f' on $(-\infty, +\infty)$.

9. Prove that $\displaystyle\lim_{n\to\infty} \sqrt[2n-1]{x} = \left\{\begin{array}{rl} 1 & \text{if } 0 < x \le 1, \\ -1 & \text{if } -1 \le x < 0 \end{array}\right\}$. (See Example 2.3.9.) Use this to prove that the sequence $\left\{x^{\frac{2n}{2n-1}}\right\}$ converges pointwise to $|x|$ on $[-1, 1]$.

9.2 Uniform Convergence

The notion of "pointwise" convergence ignores one of the essential features of our definition of limit of a sequence. In Chapter 2, we agreed[3] that the statement "$\lim\limits_{n\to\infty} x_n = x$" means that x_n can be made arbitrarily "close to" x by making n sufficiently large. We would like our definition of the limit statement

$$\lim_{n\to\infty} f_n = f$$

to incorporate this notion of "closeness." We want it to mean that we can make the function f_n be arbitrarily "close to" the function f by making n sufficiently large. But what does it mean to say that two *functions* are "close to" each other? How do we measure the distance between two functions?

In defining closeness of real numbers we used the absolute value; thus, $|x-y|$ represents the distance between x and y, and $|x|$ represents the distance between x and 0. To represent the "distance" between functions f and g we can take a similar approach, as in the following definition.

Definition 9.2.1 Given a function $f \in B(\mathcal{S})$, we define

$$\|f\| = \sup\{|f(x)| : x \in \mathcal{S}\}.$$

The real number $\|f\|$ is called the **sup norm** of f on $\mathcal{S}$; it is guaranteed to exist by the completeness property. (See Figure 9.5.)

Given functions $f, g \in B(\mathcal{S})$, the **distance between** f and g on $\mathcal{S}$ is

$$d(f, g) = \|f - g\|.$$

3. See Definition 2.1.4 and the verbal paraphrase that follows it.

(See Figure 9.6.) Thus, $\|f\|$ is the distance between f and the 0 function.

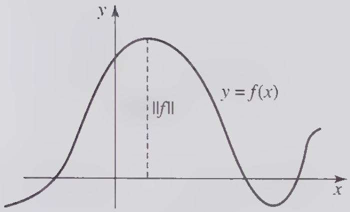

Figure 9.5

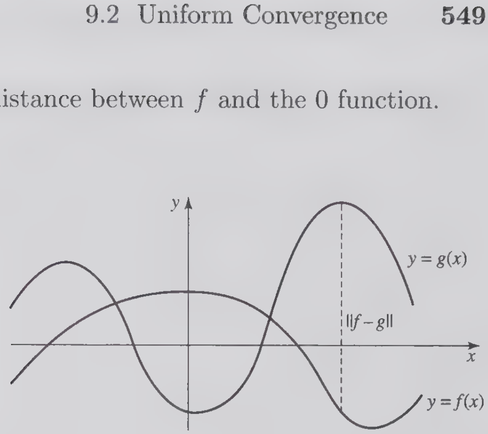

Figure 9.6

Thus, for a given $\varepsilon > 0$, a function $g \in B(\mathcal{S})$ is within a distance ε of a function $f \in B(\mathcal{S})$ if its graph lies within a "band" ε units above and below the graph of f. See Figure 9.7.

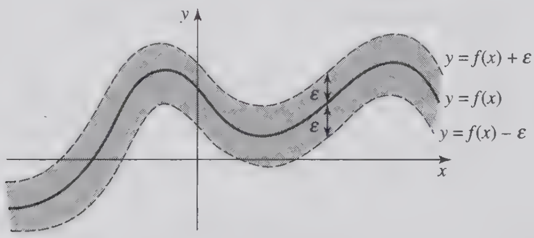

Figure 9.7

The "sup norm" has several important algebraic properties which make it resemble the absolute value, and that form the basis of its usefulness.

Theorem 9.2.2 *Given any $f, g \in B(\mathcal{S})$, and any real number r,*

(a) $\|f\| \geq 0$, *and* $\|f\| = 0$ *if and only if* $f = 0$;

(b) $\|f + g\| \leq \|f\| + \|g\|$ (*triangle inequality*);

(c) $\|rf\| = |r|\ \|f\|$;

(d) $\|fg\| \leq \|f\|\ \|g\|$.

Proof. Exercise 1. ■ (See also Exercise 2.)

We are finally ready to state the definition of "uniform convergence" of a sequence of functions. Although this definition seems complicated, we shall find that by using the notion of "norm" we can make it more understandable.

Definition 9.2.3 (Uniform Convergence) We say that a sequence $\{f_n\}$ of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$ converges uniformly to a function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ if

$$\boxed{\forall \varepsilon > 0, \exists\ n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \forall x \in \mathcal{S}, |f_n(x) - f(x)| < \varepsilon.}$$

(Compare this with pointwise convergence in Exercise 3. See also Exercise 4.)

Equivalently,

$$\forall \varepsilon > 0, \exists\ n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow f_n - f \text{ is bounded on } \mathcal{S} \text{ and } \|f_n - f\| < \varepsilon.$$

That is,

$$\boxed{\{f_n - f\} \text{ is eventually in } B(\mathcal{S}) \text{ and from that point on,}^4\ \|f_n - f\| \to 0.}$$

We often indicate this by writing $f_n \to f$ (uniformly).

For bounded functions, the definition of uniform convergence can be stated more simply.

Theorem 9.2.4 *For a given sequence $\{f_n\}$ of functions in $B(\mathcal{S})$ and a function $f \in B(\mathcal{S})$, $\{f_n\}$* ***converges uniformly*** *to f if and only if $\|f_n - f\| \to 0$.*

That is,

$$\boxed{\forall \varepsilon > 0, \exists\ n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \|f_n - f\| < \varepsilon.}$$

Notice how closely this parallels the definition of $\lim\limits_{n \to \infty} x_n = x$.

Theorem 9.2.5 *If $f_n(x) \to f$ (uniformly) on $\mathcal{S}$, then $f_n(x) \to f$ (pointwise) on $\mathcal{S}$.*

Proof. Exercise 6. ■

Examples 9.2.6 (a) As seen in Example 9.1.7 (a), the sequence $\{f_n\}$ of functions $f_n(x) = x^n$ converges pointwise on $[0, 1]$ to the limit function

$$f(x) = \begin{cases} 0 \text{ if } & 0 \leq x < 1; \\ 1 \text{ if } & x = 1. \end{cases}$$

However, the convergence is not uniform, since

$$\forall n \in \mathbb{N}, \|f_n - f\| = \sup\{|x^n| : 0 \leq x \leq 1\} = 1, \text{ so } \|f_n - f\| \not\to 0.$$

(b) Consider the same sequence $\{f_n\}$ of functions as well as the same f defined in (a), but consider convergence over the interval $[0, c]$ where $0 < c < 1$. This time, $\|f_n - f\| = \sup\{|x^n| : 0 \leq x \leq c\} = c^n \to 0$. Thus, $f_n(x) \to f$ uniformly on $[0, c]$.

4. That is, ignoring the (finite number of) terms before which $f_n - f$ is bounded on $\mathcal{S}$.

(c) In Example 9.1.7 (c), the sequence $\{f_n\}$ of functions
$$f_n(x) = \begin{cases} \frac{1}{n} \text{ if } |x| \le \frac{1}{n}; \\ |x| \text{ if } \frac{1}{n} < |x| \le 1 \end{cases}$$
converges pointwise on $[0, 1]$ to the limit function $f(x) = |x|$.

Since $\forall n \in \mathbb{N}$, $\|f_n - f\| = \frac{1}{n}$, $\|f_n - f\| \to 0$, and so $f_n(x) \to f$ uniformly on $[0, 1]$.

(d) On $\mathcal{S} = (-\infty, \infty)$, define $f(x) = x$ and
$$f_n(x) = \begin{cases} x + \frac{1}{n} & \text{if } x \ge 0 \text{ or } n \ge 11; \\ \frac{1}{n} & \text{if } x < 0 \text{ and } 1 < n \le 10. \end{cases}$$

(See Figure 9.8.) Notice that $f_n - f$ is not bounded when $1 < n \le 10$. For $n \ge 11$, $f_n - f$ is bounded and $\|f_n - f\| = \frac{1}{n} \to 0$. Thus, by Definition 9.2.3, $f_n(x) \to f$ uniformly.

Note that none of the functions in this example is bounded. Uniform convergence does not require that the functions themselves be bounded. $\square$

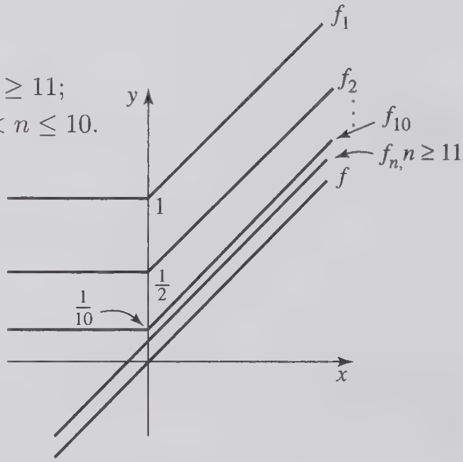

Figure 9.8

Theorem 9.2.7 (*Uniform Cauchy Criterion*) *Suppose $\{f_n\}$ is a sequence of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$. Then $\{f_n\}$ converges uniformly to some function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ if and only if*

$$\boxed{\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni m, n \ge n_0 \Rightarrow f_n - f_m \textit{ is bounded and } \|f_n - f_m\| < \varepsilon.}$$

Proof. Suppose $\{f_n\}$ is a sequence of functions in $\mathcal{F}(\mathcal{S}, \mathbb{R})$.

Part 1 ($\Rightarrow$): Exercise 10.

Part 2 ($\Leftarrow$): Suppose that $\forall \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N} \ni m, n \ge n_0 \Rightarrow f_n - f_m$ is bounded and $\|f_n - f_m\| < \varepsilon$. Then for each $x \in \mathcal{S}$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers, so it converges to some number, call it $f(x)$. In this way we get a function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. We shall show that $f_n \to f$ uniformly.

Let $\varepsilon > 0$. By hypothesis, $\exists\, n_0 \in \mathbb{N} \ni m, n \ge n_0 \Rightarrow f_n - f_m$ is bounded and $\|f_n - f_m\| < \varepsilon/2$. So,
$$m, n \ge n_0 \Rightarrow \sup\{|f_n(x) - f_m(x)| : x \in \mathcal{S}\} < \varepsilon/2.$$

Fix any $n \ge n_0$. Then,
$$\forall x \in \mathcal{S},\ |f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \le \varepsilon/2,$$

using the squeeze theorem. In summary,

$$\exists\, n_0 \in \mathbb{N} \ni m, n \ \geq n_0 \Rightarrow \forall x \in \mathcal{S},\ |f_n(x) - f(x)| < \varepsilon.$$

That is, $f_n \to f$ uniformly. ■

Theorem 9.2.8 (***Uniform Convergence Preserves Boundedness***) *If $\{f_n\}$ is a sequence of functions in $B(\mathcal{S})$ converging uniformly on $\mathcal{S}$ to a real-valued function f, then $f \in B(\mathcal{S})$.*

Proof. Suppose $\{f_n\}$ is a sequence of functions in $B(\mathcal{S})$ converging uniformly on $\mathcal{S}$ to a real-valued function f. By Definition 9.2.3, $\exists\, n_0 \in \mathbb{M} \ni \forall x \in \mathcal{S}$, $n \geq n_0 \Rightarrow |f_n(x) - f(x)| < 1$. Since f_{n_0} is a bounded function, $\exists\, M > 0 \ni \forall x \in \mathcal{S}$, $|f_{n_0}(x)| < M$. So, $\forall x \in \mathcal{S}$,

$$|f(x)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x)| < 1 + M.$$

That is, f is bounded on $\mathcal{S}$. ■

Definition 9.2.9 A sequence (or family) of functions is said to be **uniformly bounded** on a set $\mathcal{S}$ if there exists a positive real number M such that for all functions f in the sequence (or family), $\|f\| \leq M$; that is, $\forall x \in \mathcal{S}$, $|f(x)| \leq M$.

Theorem 9.2.10 *Every uniformly convergent sequence of functions in $B(\mathcal{S})$ is uniformly bounded.*

Proof. Suppose $\{f_n\}$ is a sequence in $B(\mathcal{S})$ and $f_n \to f$ uniformly. By Theorem 9.2.8, $f \in B(\mathcal{S})$. So, $\exists\ M > 0 \ni \forall x \in \mathcal{S}$, $|f(x)| \leq M$. Since $f_n \to f$ uniformly, $\exists\, n_0 \in \mathbb{N} \ni n \ \geq n_0 \Rightarrow \forall x \in \mathcal{S}$, $|f_n(x) - f(x)| \leq 1$. Then $\forall x \in \mathcal{S}$,

$$n \ \geq n_0 \Rightarrow |f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + M.$$

Since each function $f_1, f_2, \cdots, f_{n_0}$ is bounded on $\mathcal{S}$, $\exists\ M_1, M_2, \cdots, M_{n_0} > 0 \ni \|f_1\| \leq M_1$, $\|f_2\| \leq M_2$, $\cdots$, $\|f_{n_0}\| \leq M_{n_0}$. Then,

$$\forall n \in \mathbb{N},\ \forall x \in \mathcal{S},\ |f_n(x)| \leq 1 + \max\{M, M_1, M_2, \cdots, M_{n_0}\}.$$

That is, $\{f_n\}$ is uniformly bounded. ■

UNIFORM CONVERGENCE OF SERIES

We say that a series $\sum\limits_{k=0}^{\infty} f_k$ of functions $f_k \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ **converges uniformly** to a function $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ if its sequence of partial sums converges uniformly to f. As in Chapter 8, the Cauchy criterion applies to series as well as sequences.

Theorem 9.2.11 (***Uniform Cauchy Criterion for Series***) *A series* $\sum_{k=0}^{\infty} f_k$ *of functions in* $\mathcal{F}(\mathcal{S}, \mathbb{R})$ *converges uniformly on* $\mathcal{S}$ *if and only if* $\forall \varepsilon > 0$, $\exists$ $n_0 \in \mathbb{N} \ni n > m \geq n_0 \Rightarrow \sum_{k=m+1}^{n} f_k$ *is bounded and* $\left\| \sum_{k=m+1}^{n} f_k \right\| < \varepsilon$.

Proof. Exercise 13. ■

Corollary 9.2.12 *If* $\sum_{k=0}^{\infty} f_k = f$ *uniformly on a set* $\mathcal{S}$*, then* $\|f_k\| \to 0$.

Proof. Exercise 14. ■

The following simple test is often useful in proving that a series of functions is uniformly convergent.

Theorem 9.2.13 (***Weierstrass M-Test***) *Let* $\{f_n\}$ *be a sequence of functions defined and bounded on a set* S *of real numbers. If there is a sequence of positive real numbers* $\{M_k\}$ *such that* $\sum_{k=0}^{\infty} M_k < \infty$ *and* $\forall k \in \mathbb{N}$, $\|f_k\| \leq M_k$ *on* S*, then the series* $\sum_{k=0}^{\infty} f_k$ *converges (absolutely and) uniformly on* S.

Proof. Suppose $\{f_n\}$, S, and $\{M_k\}$ are as described in the hypotheses. Since $\sum_{k=0}^{\infty} M_k$ converges, it satisfies the Cauchy criterion for series (8.1.11), so

$$\exists\, n_0 \in \mathbb{N} \ni n > m \geq n_0 \Rightarrow \sum_{k=m+1}^{n} M_k < \varepsilon. \text{ Thus,}$$

$$n > m \geq n_0 \Rightarrow \left\| \sum_{k=m+1}^{n} f_k \right\| \leq \sum_{k=m+1}^{n} \|f_k\| \leq \sum_{k=n+1}^{m} M_k < \varepsilon.$$

By the uniform Cauchy criterion for series, $\sum_{k=0}^{\infty} f_k$ converges uniformly on S. ■

Corollary 9.2.14 *If* $\sum_{k=0}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=0}^{\infty} a_k \sin b_k x$ *and* $\sum_{k=0}^{\infty} a_k \cos b_k x$ *converge uniformly on* $(-\infty, \infty)$ *for any sequence* $\{b_k\}$.

Proof. Apply the Weierstrass M-test. ■

Corollary 9.2.15 *A power series* $\sum_{k=0}^{\infty} a_k(x-c)^k$ *converges uniformly (and absolutely) in any compact interval* $[c-r, c+r]$*, where* $0 < r < \rho$ *and* ρ *is the radius of convergence of the series.*

Proof. Exercise 15. ■

Thus, power series converge both absolutely and uniformly in any interval in the *interior* of the interval of convergence. But for series of functions in general, absolute and uniform convergence do not necessarily go together. For an example of a series of functions that converges absolutely but not uniformly, see Exercise 16. For a power series that converges uniformly but not absolutely on an interval, see Exercise 17.

Many of the facts we have proved about series of real numbers carry over to uniformly convergent series of functions. One such example is the following.

Theorem 9.2.16 (*Dirichlet's Test for Uniform Convergence of Series*) *Suppose* $\{f_k\}$ *and* $\{g_k\}$ *are sequences of functions in* $\mathcal{F}(\mathcal{S}, \mathbb{R})$ *such that*

(a) *the sequence* $\{S_n\}$ *of partial sums* $S_n = \sum_{k=1}^{n} f_k$ *is uniformly bounded;*

(b) $\forall x \in \mathcal{S}$*, the sequence* $\{g_k(x)\}$ *is monotone decreasing and nonnegative;*

(c) $g_k \to 0$ *uniformly on* $\mathcal{S}$.

Then $\sum_{k=1}^{\infty} f_k g_k$ *converges uniformly on* $\mathcal{S}$.

Proof. Exercise 18. ■

Example 9.2.17 The series $\sum_{k=1}^{\infty} \dfrac{\sin kx}{k}$

(a) converges uniformly on $[\delta, \pi-\delta]$ for any $0 < \delta < \dfrac{\pi}{2}$;

(b) converges pointwise, but not uniformly, on $[0, \pi]$.

Proof. (a) Let $0 < \delta < \dfrac{\pi}{2}$. We shall apply Dirichlet's test (9.2.16) with $f_k(x) = \sin kx$ and $g_k(x) = \frac{1}{k}$. Let $x \in [\delta, \pi-\delta]$. As shown in Lemma 8.5.5,

$$\left|\sum_{k=1}^{n} \sin kx\right| \le \frac{1}{\left|\sin \frac{x}{2}\right|} \le \frac{1}{\sin \frac{\delta}{2}}.$$

Hence the partial sums $\sum_{k=1}^{n} \sin kx$ are uniformly bounded on $[\delta, \pi-\delta]$. The sequence $\{g_k\} = \left\{\frac{1}{k}\right\}$ is monotone decreasing and nonnegative, and

$\|g_k\| = \frac{1}{k} \to 0$, so $g_k \to 0$ uniformly on $[\delta, \pi - \delta]$. Therefore, by Dirichlet's test for uniform convergence, $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ converges uniformly on $[\delta, \pi - \delta]$.

(b) The series $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ converges pointwise everywhere, as we showed in Example 8.5.7. To investigate uniform convergence we use the uniform Cauchy criterion. Consider the partial sums $S_n = \sum_{k=1}^{n} \frac{\sin kx}{k}$. Let $m \in \mathbb{N}$. Note that for $m \le k \le 2m$, $\frac{1}{2} \le \frac{k}{2m} \le 1$, so $\sin \frac{k}{2m} \ge \sin \frac{1}{2}$. Now, let $x = \frac{1}{2m}$. Then,

$$\sum_{k=m}^{2m} \frac{\sin kx}{k} = \sum_{k=m}^{2m} \frac{\sin \frac{k}{2m}}{k} \ge \sum_{k=m}^{2m} \frac{\sin \frac{1}{2}}{2m} = \frac{\sin \frac{1}{2}}{2m} \sum_{k=m}^{2m} 1 = \frac{\sin \frac{1}{2}}{2}.$$

Therefore, $\|S_{2m} - S_m\| \ge \frac{\sin \frac{1}{2}}{2}$. So, by the uniform Cauchy criterion, $\sum_{k=1}^{\infty} \frac{\sin kx}{k}$ cannot converge uniformly on $[0, \pi]$. □

EXERCISE SET 9.2

1. Prove Theorem 9.2.2.

2. Prove that if $f_1, f_2, \cdots, f_n \in B(\mathcal{S})$, then $f_1 + f_2 + \cdots + f_n \in B(\mathcal{S})$, and $\|f_1 + f_2 + \cdots + f_n\| \le \|f_1\| + \|f_2\| + \cdots + \|f_n\|$.

3. Write out the ε-n_0 definition of the statement that $f_n \to f$ pointwise on a set $\mathcal{S}$. Explain how this differs from the definition of uniform convergence of f_n to f on $\mathcal{S}$ given in Definition 9.2.3.

4. Which of the following are equivalent? Explain the relevance to Definition 9.2.3.

   (a) $\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni \forall n \ge n_0, x \in \mathcal{S} \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

   (b) $\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni \forall n \ge n_0$ and $\forall x \in \mathcal{S}, |f_n(x) - f(x)| < \varepsilon$.

   (c) $\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni \forall x \in \mathcal{S}$ and $\forall n \ge n_0, |f_n(x) - f(x)| < \varepsilon$.

   (d) $\forall \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni \forall x \in \mathcal{S}, n \ge n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

5. For each of the sequences in Examples 9.1.7 (b) and (d), calculate $\|f_n - f\|$ and use it to show that $\{f_n\}$ does not converge uniformly to f.

6. Prove Theorem 9.2.5.

7. Prove that $\left\{\sin\left(x + \frac{1}{n}\right)\right\}$ converges uniformly to $\sin x$ on $\mathbb{R}$. [Hint: use the mean value theorem.]

8. Determine whether the following sequences $\{f_n\}$ converge uniformly on $[0, +\infty)$:

   (a) $f_n(x) = x^n e^{-nx}$ (b) $f_n(x) = \dfrac{x}{n} e^{-x/n}$

   (c) $f_n(x) = \dfrac{1}{nx^2 + 1}$ (d) $f_n(x) = \dfrac{1}{n(x^2 + 1)}$

9. For each of the sequences $\{f_n\}$ in Exercise 9.1.3, determine whether $\{f_n\}$ converges uniformly on its set $\mathcal{S}$ of pointwise convergence. If it doesn't, find a subset of $\mathcal{S}$ on which $\{f_n\}$ does converge uniformly, if possible.

10. Prove Part 1 of Theorem 9.2.7.

11. Prove that if $\{f_n\}$ is a sequence in $B(\mathcal{S})$ converging uniformly on $\mathcal{S}$ to f, then $\|f_n\| \to \|f\|$. [First prove that $|\, \|f_n\| - \|f\| \,| \leq \|f_n - f\|$.]

12. Suppose $\{f_n\}$, $\{g_n\}$ converge uniformly on $\mathcal{S}$, and let $r \in \mathbb{R}$. Prove that

    (a) $\{f_n + g_n\}$ converges uniformly on $\mathcal{S}$.

    (b) $\{rf_n\}$ converges uniformly on $\mathcal{S}$.

    (c) if $\{f_n\}$ and $\{g_n\}$ are in $B(\mathcal{S})$, then $\{f_n g_n\}$ converges uniformly on $\mathcal{S}$.

13. Prove Theorem 9.2.11.

14. Prove Corollary 9.2.12

15. Prove Corollary 9.2.15.

16. Consider the functions $f_k(x) = \left\{ \begin{array}{l} 1 \text{ if } \frac{1}{k+1} < x \leq \frac{1}{k} \\ 0 \text{ otherwise} \end{array} \right\}$. Prove that on $[0, 1]$ the series $\sum\limits_{k=0}^{\infty} f_k$ converges pointwise (absolutely) but not uniformly.

17. Prove that the given series converges uniformly but not absolutely on the given interval.

    (a) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$ on $[0, 1]$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{x^2 + k}$ on $(-\infty, \infty)$

18. Modify the proof of Theorem 8.5.3 to obtain a proof of Theorem 9.2.16.

19. Determine whether the following series converge uniformly on the indicated set:

    (a) $\sum\limits_{k=1}^{\infty} \dfrac{\sin kx}{k^2}$ on $\mathbb{R}$ (b) $\sum\limits_{k=1}^{\infty} \dfrac{kx^k}{k+1}$ on $(0, 1)$ (c) $\sum\limits_{k=0}^{\infty} e^{-kx}$ on $(0, \infty)$

    (d) $\sum\limits_{k=0}^{\infty} e^{-kx}$ on $[1, \infty)$ (e) $\sum\limits_{k=1}^{\infty} x^k e^{-kx}$ on $[0, \infty)$ (f) $\sum\limits_{k=0}^{\infty} \dfrac{1}{1 + k^2 x^2}$ on $(0, 1]$

9.3 Implications of Uniform Convergence in Calculus

It is natural to inquire about the interaction between limits of sequences of functions and the fundamental operations of calculus such as limits, continuity, derivatives, and integrals. To bring this concern into sharper focus, we pose the following questions.

FOUR QUESTIONS:

Suppose $f_n \to f$ pointwise on a set $S \subseteq \mathbb{R}$ and let $x_0 \in S$.

Q#1 If each function f_n has a limit at x_0, does f have a limit at x_0? If so, is it true that $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to x_0} f_n(x)\right) = \lim\limits_{x\to x_0} f(x)$? That is, can we "interchange the limits," so that $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to x_0} f_n(x)\right) = \lim\limits_{x\to x_0}\left(\lim\limits_{n\to\infty} f_n(x)\right)$?

Q#2 If each function f_n is continuous on S, must f also be continuous on S?

Q#3 If each function f_n is integrable on $[a,b] \subseteq S$, must f also be integrable on $[a,b]$? If it is, must it be true that $\lim\limits_{n\to\infty}\int_a^b f_n = \int_a^b f$?

Q #4 If each function f_n is differentiable at x_0, must f also be differentiable at x_0? If it is, must it be true that $\lim\limits_{n\to\infty} f_n'(x_0) = f'(x_0)$?

The following examples will show that the answer to each of these questions is "no." However, when "pointwise convergence" is replaced by "uniform convergence," the answer to most of these questions changes to "yes." The remainder of this section will be devoted to proving these claims.

Examples 9.3.1

(a) Define $f_n : [-1,1] \to \mathbb{R}$

by $f_n(x) = \begin{Bmatrix} -1 \text{ if } -1 \le x \le -\frac{1}{n} \\ 0 \text{ if } -\frac{1}{n} < x < \frac{1}{n} \\ 1 \text{ if } \frac{1}{n} < x \le 1 \end{Bmatrix}$.

(See Figure 9.9.) The limit function is the signum function defined in Definition 5.1.5. Each function f_n has a limit at 0, but f does not. Thus, the answer to the first part of Q#1 is "no."

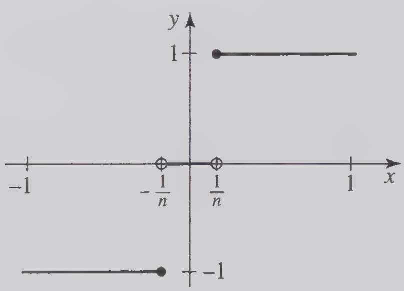

Figure 9.9

(b) Let $S = [0,1]$ and $f_n(x) = x^n$. As we saw in Example 9.1.7 (a), $\{f_n\}$ converges pointwise on $[0,1]$ to the function $f(x) = \left\{ \begin{array}{l} 0 \text{ if } 0 \leq x < 1 \\ 1, \text{ if } x = 1 \end{array} \right\}$. Here, $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to 0} f_n(x)\right) = \lim\limits_{n\to\infty}\left(\lim\limits_{x\to 0} x^n\right) = 1$, whereas $\lim\limits_{x\to 0}\left(\lim\limits_{n\to\infty} f_n(x)\right) = \lim\limits_{x\to 0} f(x) = 0$. Thus, the answer to the second part of Q#1 is "no." □

Example 9.3.2

In Example 9.3.1 (a) above, each function f_n is continuous on $[0,1]$ but the limit function f is not. Hence, the answer to Q#2 is "no."

Examples 9.3.3

(a) In Example 9.1.7 (d), each function f_n is integrable over $[0,1]$, since it differs from the constant function 0 at only finitely many points (See Theorem 7.4.9). But the limit function f is not integrable over $[0,1]$, as shown in Example 7.2.10. Hence, the answer to the first part of Q#3 is "no."

(b) Let $S = [0,1]$ and define $f_n : [0,1] \to \mathbb{R}$

by $f_n(x) = \left\{ \begin{array}{cl} 0 & \text{if } x = 0; \\ 2n - 2n^2 x & \text{if } 0 < x \leq \frac{1}{n}; \\ 0 & \text{if } \frac{1}{n} < x \leq 1 \end{array} \right\}$.

The graph of a typical f_n is shown in Figure 9.10. It is clear that the pointwise limit of $\{f_n\}$ on $[0,1]$ is $f(x) = 0$. In this case, $\int_0^1 f_n = 1$, but $\int_0^1 f = 0$; thus, $\int_{0f}^1 f_n \not\to \int_0^1 f$. Therefore, the answer to the second part of Q#3 is "no." □

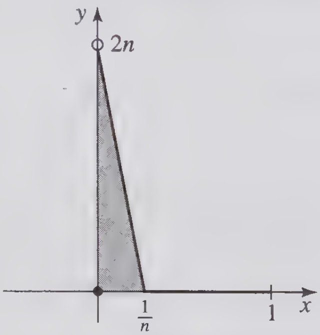

Figure 9.10

Examples 9.3.4

(a) In Example 9.1.7 (c), each function f_n is differentiable at 0 but the limit function is not. Hence, the answer to the first part of Q#4 is "no." But notice that in this case the convergence is uniform, so that even uniform convergence will not guarantee the differentiability of the limit function.

(b) Let $S = [0,1]$ and define $f_n : [0,1] \to \mathbb{R}$

by $f_n(x) = \left\{ \begin{array}{cl} x + \frac{1}{n} & \text{if } -1 \leq x \leq -\frac{1}{n}; \\ 0 & \text{if } -\frac{1}{n} < x < \frac{1}{n}; \\ x - \frac{1}{n} & \text{if } \frac{1}{n} \leq x \leq 1 \end{array} \right\}$.

The graph of a typical f_n is shown in Figure 9.11. The pointwise limit of $\{f_n\}$ on $[0,1]$ is $f(x) = x$. Each f_n is differentiable at 0, and $f_n'(0) = 0$, so $f_n'(0) \to 0$. However, $f'(0) = 1$. Therefore, the answer to the second part of Q#4 is "no." $\square$

We now show how uniform convergence affects the concerns raised in questions Q#1–Q#4. We address these concerns in the same order.

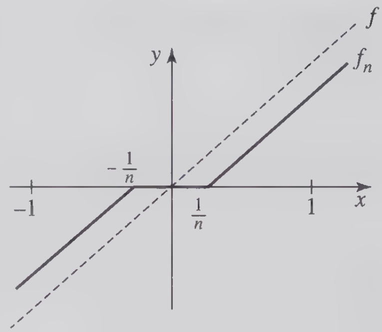

Figure 9.11

Theorem 9.3.5 (***Uniform Convergence Permits Interchange of Limits***[5]**):**
Suppose $\{f_n\}$ converges uniformly to f on a set $S - \{x_0\}$ for some $x_0 \in S$. If each f_n has a (finite) limit as $x \to x_0$ then so does f, and we can interchange the limits. More precisely, if $\forall n \in \mathbb{N}$, $\lim\limits_{x\to x_0} f_n(x)$ exists, then $\lim\limits_{x\to x_0} f(x)$ exists and equals $\lim\limits_{n\to\infty}\left(\lim\limits_{x\to x_0} f_n(x)\right)$. That is, $\lim\limits_{x\to x_0}\left(\lim\limits_{n\to\infty} f_n(x)\right) = \lim\limits_{n\to\infty}\left(\lim\limits_{x\to x_0} f_n(x)\right)$.

Proof. Suppose $f_n \to f$ uniformly on $S - \{x_0\}$ for some $x_0 \in S$, and suppose that each of the functions f_n in the sequence has a limit as $x \to x_0$,

$$\lim_{x\to x_0} f_n(x) = L_n \in \mathbb{R}. \tag{1}$$

Let $\varepsilon > 0$. By the uniform Cauchy criterion,

$$\exists n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow \|f_n - f_m\| < \tfrac{\varepsilon}{3}.$$

Now, $\forall x \in S - \{x_0\}$,

$$|L_n - L_m| \leq |L_n - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - L_m|. \tag{2}$$

By Equation (1), $\forall k \in \mathbb{N}$, $\exists \delta_k > 0 \ni 0 < |x - x_0| < \delta_k \Rightarrow |f_k(x) - L_k| < \frac{\varepsilon}{3}$.

Fix any $m, n \geq n_0$ and choose any x such that $0 < |x - x_0| < \min\{\delta_m, \delta_n\}$. For this x we have

$$|L_n - f_n(x)| < \tfrac{\varepsilon}{3} \text{ and } |f_m(x) - L_m| < \tfrac{\varepsilon}{3}, \text{ as well as } |f_n(x) - f_m(x)| < \tfrac{\varepsilon}{3}.$$

Plugging these into (2), we have

$$|L_n - L_m| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

---

5. Including one-sided limits.

Since this is true $\forall m, n \geq n_0$, $\{L_n\}$ is a Cauchy sequence, and so it must converge. Let

$$L = \lim_{n\to\infty} L_n.$$

The proof will be complete when we prove that $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} L_n \ (= L)$.

Let $\varepsilon > 0$. $\forall x \in S - \{x_0\}$, and $\forall n \in \mathbb{N}$,

$$|f(x) - L| \leq |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L|.$$

Since $f_n \to f$ uniformly on $S - \{x_0\}$,

$$\begin{aligned}\exists n_1 \in \mathbb{N} \ni n \geq n_1 &\Rightarrow \|f_n - f\| < \frac{\varepsilon}{3} \\ &\Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{3} \ \forall x \in S - \{x_0\}.\end{aligned}$$

Since $L_n \to L$, $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow |L_n - L| < \frac{\varepsilon}{3}$.

Choose any $n_3 \geq \max\{n_1, n_2\}$ and hold this n_3 fixed throughout the remainder of the proof. Since $\lim_{x\to x_0} f_{n_3}(x) = L_{n_3}$, we can choose $\delta > 0$ such that

$$\begin{aligned}0 < |x - x_0| < \delta &\Rightarrow |f_{n_3}(x) - L_{n_3}| < \tfrac{\varepsilon}{3}. \\ &\quad \left(\text{as well as } |f_{n_3}(x) - f(x)| < \tfrac{\varepsilon}{3} \text{ and } |L_{n_3} - L| < \tfrac{\varepsilon}{3}\right) \\ &\Rightarrow |f(x) - L| \leq |f_{n_3}(x) - L_{n_3}| + |f_{n_3}(x) - f(x)| + |L_{n_3} - L| \\ &\qquad < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Therefore, $\lim_{x\to x_0} f(x) = L$. ■

Corollary 9.3.6 (***Uniform Convergence Preserves Continuity***) *If $f_n \to f$ uniformly on a set $S \subseteq \mathbb{R}$, and if each f_n is continuous on S, then f is continuous on S.*

Proof. Suppose $f_n \to f$ uniformly on S and each f_n is continuous on S. Then $\forall x_0 \in S$, $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \left[\lim_{n\to\infty} f_n(x)\right]$ since $f_n \to f$ (uniformly) on S;

$$\begin{aligned}&= \lim_{n\to\infty} \left[\lim_{x\to x_0} f_n(x)\right] \text{ by Theorem 9.3.5;} \\ &= \lim_{n\to\infty} [f_n(x_0)] \text{ since } f_n \text{ is continuous at } x_0; \\ &= f(x_0) \text{ since } f_n \to f \text{ on } S.\end{aligned}$$

Therefore, f is continuous at x_0. ■

Corollary 9.3.7 *If $\sum_{k=0}^{\infty} f_k = f$ uniformly on a set S of real numbers, and each function f_k is continuous on S, then the limit function f is continuous on S.*

Theorem 9.3.8 (***Uniform Convergence Preserves Integrals***) *If $f_n \to f$ uniformly on a closed interval $[a,b]$, and if each f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$ and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Proof. Suppose $f_n \to f$ uniformly on $[a,b]$, and each f_n is integrable on $[a,b]$. Note that $\forall x \in [a,b]$, $|f_n(x) - f(x)| \le \|f_n - f\|$, so

$$f_n(x) - \|f_n - f\| \le f(x) \le f_n(x) + \|f_n - f\|.$$

Then $\int_a^b (f_n - \|f_n - f\|) = \underline{\int_a^b} (f_n - \|f_n - f\|) \le \underline{\int_a^b} f \le \overline{\int_a^b} f$

$$\le \overline{\int_a^b} (f_n + \|f_n - f\|) = \int_a^b (f_n + \|f_n - f\|). \tag{3}$$

So,
$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le \int_a^b (f_n + \|f_n - f\|) - \int_a^b (f_n - \|f_n - f\|)$$

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 2\int_a^b \|f_n - f\| = 2\|f_n - f\|(b-a).$$

Since $f_n \to f$ uniformly on $[a,b]$, $\|f_n - f\| \to 0$. So, when we take the limit as $n \to \infty$, we have

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 0; \text{ i.e., } \overline{\int_a^b} f = \underline{\int_a^b} f,$$

which means that f is integrable on $[a,b]$.

From inequality (3) we have

$$\left(\int_a^b f_n\right) - \|f_n - f\|(b-a) \le \int_a^b f \le \left(\int_a^b f_n\right) + \|f_n - f\|(b-a)$$

$$-\|f_n - f\|(b-a) \le \int_a^b f - \int_a^b f_n \le \|f_n - f\|(b-a)$$

$$\left|\int_a^b f - \int_a^b f_n\right| \le \|f_n - f\|(b-a) \to 0.$$

Therefore, by the second squeeze principle (2.3.2), $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. ■

Corollary 9.3.9 *If a series $\sum f_k$ of integrable functions converges uniformly to f on $[a,b]$, then f is integrable on $[a,b]$, and $\int_a^b f = \sum \int_a^b f_k$.*

That is, if $\sum f_k$ converges uniformly, then $\int_a^b \sum f_k = \sum \int_a^b f_k$.

Example 9.3.10 Evaluate $\displaystyle\lim_{n\to\infty} \int_a^\pi \frac{\sin nx}{nx}\,dx$, where $0 < a < \pi$.

Solution: Let $0 < a < \pi$. On the interval $[a, \pi]$, $\left\| \dfrac{\sin nx}{nx} \right\| \le \dfrac{1}{na} \to 0$. Thus, $\dfrac{\sin nx}{nx} \to 0$ uniformly on $[a, \pi]$. So, by Theorem 9.3.8,

$$\lim_{n\to\infty} \int_a^\pi \frac{\sin nx}{nx}\, dx = \int_a^\pi \left(\lim_{n\to\infty} \frac{\sin nx}{nx} \right) dx = \int_a^\pi 0 = 0. \quad \square$$

Uniform convergence does not necessarily preserve (improper) integrals over infinite intervals. See Exercise 7 for an example.

The relationship between uniform convergence and derivatives is not so straightforward. As we saw in Example 9.3.4(a), uniform convergence does not preserve differentiability. Even when a sequence $\{f_n\}$ of differentiable functions converges uniformly to a differentiable limit function f, the sequence $\{f_n'(x_0)\}$ might not converge to $f'(x_0)$. (See Exercise 13.) The following theorem shows that for sequences of differentiable functions, the focus shifts from uniform convergence of $\{f_n\}$ to that of $\{f_n'\}$.

Theorem 9.3.11 (***Uniform Convergence and Differentiation***) *Suppose $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that $\{f_n'\}$ converges uniformly on $[a, b]$, and suppose that $\{f_n(x_0)\}$ converges for at least one $x_0 \in [a, b]$. Then $\{f_n\}$ converges uniformly on $[a, b]$ to a differentiable function f, and $\forall x \in [a, b]$, $f'(x) = \lim_{n\to\infty} f_n'(x)$.*

Proof. Suppose $\{f_n\}$ and $\{f_n'\}$ satisfy the hypotheses. Then $a < b$.

Part 1: We first show that $\{f_n\}$ converges uniformly on $[a, b]$. Let $\varepsilon > 0$.

Since $\{f_n'\}$ converges uniformly on $[a, b]$, the uniform Cauchy criterion tells us that $\exists n_1 \in \mathbb{N} \ni$

$$m, n \ge n_1 \Rightarrow \|f_n' - f_m'\| < \frac{\varepsilon}{2(b-a)}.$$

Let x_0 be any point of $[a, b]$ for which $\{f_n(x_0)\}$ converges. By the Cauchy criterion for sequences of real numbers, $\exists n_2 \in \mathbb{N} \ni$

$$m, n \ge n_2 \Rightarrow |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}.$$

Suppose $m, n \ge \max\{n_1, n_2\}$. Consider any $x \ne x_0$ in $[a, b]$. By the mean value theorem applied to the function $f_n - f_m$ on the closed interval between x and x_0, $\exists t$ between x and x_0 such that

$$[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)] = (x - x_0)\,[f_n'(t) - f_m'(t)], \tag{4}$$

$$f_n(x) - f_m(x) = f_n(x_0) - f_m(x_0) + (x - x_0)\,[f_n'(t) - f_m'(t)].$$

Then,

$$|f_n(x) - f_m(x)| \le |f_n(x_0) - f_m(x_0)| + |x - x_0|\,|f_n'(t) - f_m'(t)|$$
$$< \frac{\varepsilon}{2} + (b-a)\frac{\varepsilon}{2(b-a)} = \varepsilon.$$

Thus, $m, n \ge \max\{n_1, n_2\} \Rightarrow \|f_n - f_m\| < \varepsilon$. Therefore, $\{f_n\}$ converges uniformly on $[a, b]$. Define

$$f = \lim_{n\to\infty} f_n. \tag{5}$$

Part 2: Remarks on Part 1.

(1) x_0 may be any point of $[a, b]$ for which $\{f_n(x_0)\}$ converges. Since $f_n \to f$ uniformly on $[a, b]$, Equation (4) holds for all $x_0 \in [a, b]$.

(2) Each f_n is differentiable on $[a, b]$, so it is continuous there. Thus, by Corollary 9.3.6, f is continuous on $[a, b]$.

Part 3: Let $x_0 \in [a, b]$. Define functions g_n and g on $[a, b]$ by

$$g_n(x) = \left\{\begin{array}{ll} \dfrac{f_n(x) - f_n(x_0)}{x - x_0} & \text{if } x \ne x_0 \\ f_n'(x_0) & \text{if } x = x_0 \end{array}\right\} \text{ and}$$

$$g(x) = \left\{\begin{array}{ll} \dfrac{f(x) - f(x_0)}{x - x_0} & \text{if } x \ne x_0 \\ f'(x_0) & \text{if } x = x_0 \end{array}\right\}.$$

Since each f is differentiable at x_0,

$$\lim_{x\to x_0} g_n(x) = f_n'(x_0) = g_n(x_0).$$

Thus, each g_n is continuous at x_0. Also, $\forall x \in [a, b]$,

$$g_n(x) - g_m(x) = \left\{\begin{array}{ll} \dfrac{[f_n(x) - f_n(x_0)] - [f_m(x) - f_m(x_0)]}{x - x_0} & \text{if } x \ne x_0 \\ f_n'(x_0) - f'(x_0) & \text{if } x = x_0 \end{array}\right\}$$

$$= \left\{\begin{array}{ll} \dfrac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0} & \text{if } x \ne x_0 \\ f_n'(x_0) - f'(x_0) & \text{if } x = x_0 \end{array}\right\}. \tag{6}$$

Let $\varepsilon > 0$. Since $\{f_n'\}$ converges uniformly on $[a, b]$, $\exists n_3 \in \mathbb{N} \ni m, n \ge n_3 \Rightarrow \|f_n' - f_m'\| < \varepsilon$.

Let $x \neq x_0$ in $[a, b]$. By Equation (4), $\exists t$ between x and x_0 such that

$$\left|\frac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0}\right| = |f_n'(t) - f_m'(t)| \leq \|f_n' - f_m'\|.$$

Thus, from (6), $\forall x \in [a, b] - \{x_0\}$,

$$m, n \geq n_3 \Rightarrow |g_n(x) - g_m(x)| \leq \|f_n' - f_m'\| < \varepsilon.$$

Hence, $m, n \geq n_3 \Rightarrow \|g_n - g_m\| < \varepsilon$ on $[a, b] - \{x_0\}$. Therefore, $\{g_n\}$ converges uniformly on $[a, b] - \{x_0\}$, by Theorem 9.2.7.

Part 4: We finish by showing that f is differentiable on $[a, b]$ and $\forall x_0 \in [a, b]$, $f'(x_0) = \lim\limits_{n\to\infty} f_n'(x_0)$. We first note that from Equation (5), $\forall x \neq x_0$ in $[a, b]$,

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{f_n(x) - f_n(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} = g(x).$$

Finally, by Theorem 9.3.5, uniform convergence of $\{g_n\}$ on $[a, b] - \{x_0\}$ allows us to interchange limits, as follows:

$$\begin{aligned}
f'(x_0) &= \lim_{x\to x_0} g(x), \quad \text{by definition of } g;\\
&= \lim_{x\to x_0} \left[\lim_{n\to\infty} g_n(x)\right], \quad \text{as shown above;}\\
&= \lim_{n\to\infty} \left[\lim_{x\to x_0} g_n(x)\right], \quad \text{by Theorem 9.3.5;}\\
&= \lim_{n\to\infty} [f_n'(x_0)], \quad \text{by definition of } g_n.
\end{aligned}$$

Therefore, f is differentiable on $[a, b]$ and $\forall x \in [a, b]$, $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$. ■

Theorem 9.3.11 provides an easy, short proof of the term-by-term differentiability of functions represented by power series. Compare the proof of the following corollary with that of Theorem 8.6.14.

Corollary 9.3.12 *Suppose* $f(x) = \sum\limits_{k=0}^{\infty} a_k(x - c)^k$, *with radius of convergence* $\rho > 0$. *Then* $\forall x \in (c - \rho, c + \rho)$, f *is differentiable at* x *and* $f'(x) = \sum\limits_{k=1}^{\infty} a_k k(x - c)^{k-1}$.

Moreover, the convergence of both series is uniform in $[c - R, c + R]$, *where* $0 < R < \rho$.

Proof. Exercise 8. ■

Example 9.3.13 $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k}$ converges uniformly on any compact interval to a function whose derivative is $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos \frac{x}{k}$.

Proof. Let $a < b$. We shall use Theorem 9.3.11 to prove that the given series converges uniformly as claimed on $[a, b]$. First, observe that $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{x}{k}$ converges when $x = 0$. If $0 \notin [a, b]$, choose $a' < b'$ so that $0 \in [a', b']$ and $[a, b] \subseteq [a', b']$. Next observe that the series of derivatives is $\sum_{k=1}^{\infty} \frac{1}{k^2} \cos \frac{x}{k}$. By the Weierstrass M-test, this series converges uniformly on $[a', b']$. Therefore, by Theorem 9.3.11, our proof is complete. □

There is one more theorem of this type that we include here. It is a partial converse of Corollary 9.3.6, which said that the uniform limit of a sequence of continuous functions must be continuous. The theorem says that, under certain circumstances, if the pointwise limit of a sequence of continuous functions is continuous then the convergence must be uniform.

***Theorem 9.3.14** (*Dini's Uniform Convergence Theorem*) *If $\{f_n\}$ is a sequence of continuous functions converging pointwise to a continuous function f on a compact set S and if, $\forall x \in S$, $\{f_n(x)\}$ is monotone decreasing, then $f_n \to f$ uniformly on S.*

Proof. Suppose $\{f_n\}$, f, and S satisfy the hypotheses. For contradiction, suppose the convergence $f_n \to f$ is not uniform on S. Then $\|f_n - f\| \not\to 0$, so $\exists c > 0 \ni$

$$\|f_n - f\| > c, \text{ for infinitely many } n. \tag{7}$$

Since S is compact, and since each $\{f_n(x)\}$ is monotone decreasing, Inequality (7) becomes

$$\max\{f_n(x) - f(x) : x \in S\} > c, \text{ for infinitely many } n.$$

Thus, there exists a sequence $\{x_n\}$ of points of S such that $\forall n \in \mathbb{N}$,

$$f_n(x_n) - f(x_n) > c. \tag{8}$$

By the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, say $x_{n_k} \to L$. Since S is compact, $L \in S$.

From Inequality (8), $\forall k \in \mathbb{N}$,

$$f_{n_k}(x_{n_k}) - f(x_{n_k}) > c. \tag{9}$$

Fix an $n_0 \in \mathbb{N}$. $\forall x \in S$, $\{f_n(x)\}$ is monotone decreasing, so whenever $n_k \geq n_0$, $f_{n_0}(x_{n_k}) \geq f_{n_k}(x_{n_k})$. Thus, from Equation (9),

$$n_k \geq n_0 \Rightarrow f_{n_0}(x_{n_k}) - f(x_{n_k}) > c.$$

Taking the limit as $k \to \infty$,

$$\lim_{k \to \infty} [f_{n_0}(x_{n_k}) - f(x_{n_k})] \geq c. \tag{10}$$

Since $f_n - f$ is continuous on S, Inequality (10) becomes

$$f_{n_0}(L) - f(L) \geq c > 0. \tag{11}$$

We have shown that (11) is true for all $n_0 \in \mathbb{N}$. Therefore, $f_n(L) \not\to f(L)$. But $\{f_n\}$ converges pointwise to f on S, and $L \in S$. Contradiction. Therefore, $f_n \to f$ uniformly on S. ■

EXERCISE SET 9.3

1. The series $\sum_{k=0}^{\infty} x^k$ converges uniformly on every closed interval $[0, a]$ for $0 < a < 1$ (Why?). Use Theorem 9.3.5 to prove that this series is not uniformly convergent on $[0, 1)$.

2. In each of the following, use Corollary 9.3.6 to prove that the given sequence does not converge uniformly on the given set.

   (a) $\{1 - x^n\}$ on $[0, 1]$ (b) $\{\sin^n x\}$ on $[0, \pi]$ (c) e^{-nx} on $[0, \infty)$

3. Find a sequence of functions that are discontinuous at every real number that converges uniformly to a function that is not only continuous, but differentiable at every real number.[6]

4. Let $f_n(x) = \dfrac{nx}{1 + nx}$. Show that even though $\{f_n\}$ does not converge uniformly[7] on $[0, 1]$, it is still true that $\lim_{n \to \infty} \int_0^1 f_n = \int_0^1 \lim_{n \to \infty} f_n$. Does this violate Theorem 9.3.8?

5. $\forall n \in \mathbb{N}$, define f_n on $[0, 2]$ by $f(x) = e^{-nx^2}$. Find the pointwise limit function f on $[0, 2]$ and show that the convergence is not uniform. Nevertheless, show that $\int_0^2 f_n \to \int_0^2 f$. [Hint: Theorem 9.3.8 applies to $[\varepsilon, 2]$ when $0 < \varepsilon < 2$.]

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6. It is often said that uniform convergence is good at preserving good properties but bad at preserving bad properties.

7. See Exercises 9.1.3(g) and 9.2.9(g).

6. Evaluate each of the following integrals and justify your answers:

(a) $\displaystyle\int_0^\pi \sum_{k=1}^\infty \frac{\cos kx}{k^2}\,dx$ (b) $\displaystyle\int_0^\pi \sum_{k=1}^\infty \frac{\sin kx}{k^2}\,dx$ (c) $\displaystyle\int_1^2 \sum_{k=1}^\infty ke^{-kx}\,dx$

7. Use the functions $f_n(x) = \left\{\begin{matrix} \frac{1}{n} - \frac{x}{n^2} & \text{if } 0 \le x \le n \\ 0 & \text{if } x > n \end{matrix}\right\}$ over the interval $(0, +\infty)$ to show that Theorem 9.3.8 does not apply over infinite intervals. That is, uniform convergence of $\{f_n\}$ does not always preserve (improper) integrability over infinite intervals.

8. Prove Corollary 9.3.12.

9. Use Corollary 9.3.12 to show that $\forall\ |x| < 1$, $\dfrac{d}{dx}\left(\sum\limits_{k=1}^\infty \dfrac{x^k}{k(k+1)}\right) = \sum\limits_{k=1}^\infty \dfrac{x^{k-1}}{k+1}$.

10. Use Corollary 9.3.12 to prove that the function $y(x) = \sum\limits_{k=0}^\infty \dfrac{(-1)^k}{k!}x^{3k}$ is a solution of the differential equation $y' + 3x^2y = 0$ on $(-\infty, \infty)$.

11. Use results of this section to find the sums of the following series, for $x \in (-1, 1)$:

(a) $1 + 2x + 3x^2 + 4x^3 + \cdots$. [Think geometric series.]

(b) $\dfrac{x^3}{1\cdot 3} + \dfrac{x^4}{2\cdot 4} + \dfrac{x^5}{3\cdot 5} + \dfrac{x^6}{4\cdot 6} + \cdots$. [See Equation (22) of Example 8.6.18.]

12. Show that $\left\{\sqrt{x^2 + \frac{1}{n}}\right\}$ converges uniformly to $|x|$ on $\mathbb{R}$. Verify that each function in the sequence is differentiable everywhere, but the limit function is not.

13. For all $n \in \mathbb{N}$, define $f_n(x) = \dfrac{\cos nx}{n}$ on $(-\infty, \infty)$. Prove that $\{f_n\}$ is a sequence of differentiable functions on $(-\infty, \infty)$ converging uniformly to 0 on $(-\infty, \infty)$, but $\{f_n'(x)\}$ diverges except when x is an integral multiple of π. [See Exercise 2.6.20.] Doesn't this contradict Theorem 9.3.11? Explain.

14. Explain what happens if you try to redo Example 9.3.13 using the following functions instead of the one given there?

(a) $\sum\limits_{k=1}^\infty \frac{1}{k}\cos\frac{x}{k}$ (b) $\sum\limits_{k=1}^\infty \sin\frac{x}{k}$ (c) $\sum\limits_{k=1}^\infty \cos\frac{x}{k}$

15. **Riemann's Zeta function**[8] is defined $\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$, for all $x > 1$. Prove that this series converges uniformly on the interval $[a, \infty)$ for any $a > 1$, and show that $\forall x > 1$, $\zeta(x)$ is differentiable and $\zeta'(x) = -\sum_{k=1}^{\infty} \frac{\ln k}{k^x}$.

9.4 *Two Results of Weierstrass

> This section is designed as another project to be completed by students. It has no exercise set at the end. Instead, students are asked to complete the proofs following guidelines provided in the text.

Karl Theodor Wilhelm Weierstrass (1815–1897), often called "the father of modern analysis," was a late bloomer. At his father's insistence he enrolled at the University of Bonn in 1834 to study law, finance, and economics instead of mathematics, the subject of his real interest. He left there after four years of intensive fencing and drinking, without having studied the required courses and without taking the final examinations. Continuing to study mathematics privately, he resolved to become a mathematician and enrolled at the Academy in Münster in 1839 to become a secondary school teacher. There his genius attracted the attention of Professor Gudermann, who gave him strong encouragement. While a secondary teacher for some 15 years, Weierstrass continued mathematical research and wrote several papers on elliptic functions. His 1854 paper on Abelian functions brought him widespread recognition, an honorary doctoral degree, and an offer of appointment at any university of his choice in Austria. He chose to wait for an appointment at the University of Berlin, which he received in 1856. There he lectured on analytic functions (complex variables), elliptic functions (his main area of research), Fourier series and integrals, calculus of variations, and applications to geometry and mathematical physics. His lectures attracted students from all over the world. His insistence on rigorous standards of proof is still a dominating force in the teaching of analysis today.

8. Riemann's Zeta function has important applications in physics and mathematics. It is used in number theory in the study of the distribution of prime numbers.

AN EVERYWHERE CONTINUOUS, NOWHERE DIFFERENTIABLE FUNCTION

In 1872, Weierstrass presented a paper to the Academy in Berlin in which he showed that the function[9]

$$f(x) = \sum_{k=1}^{\infty} b^k \cos(a^k \pi x), \tag{12}$$

where a is an odd natural number, $b \in [0,1)$, and $ab > 1 + \frac{3\pi}{2}$, is continuous everywhere on $(-\infty, \infty)$ yet is differentiable nowhere. This example created a sensation in the mathematical world because it challenged the then-widespread belief among mathematicians that continuous functions had to be differentiable everywhere except possibly at isolated "singular" points. In fact, A. Ampère had even published a "proof" of this "fact" in 1806. Weierstrass' example thus brought to an end a long string of futile attempts to show that differentiability somehow follows from continuity. It is perhaps even more remarkable that each term of the series (12) is infinitely differentiable everywhere, yet the sum function is not differentiable anywhere.

Weierstrass was not the first to claim the existence of an everywhere continuous, nowhere differentiable function, but he was the first to provide a rigorous proof. About 1830, Bolzano had constructed a similar example but was unable to prove that it was nowhere differentiable. Bolzano was primarily a philosopher-theologian rather than a mathematician, and lived in Prague, which was not a major center of mathematical research. Although he did not publish his example, it has subsequently been proved correct.

In 1860, the Swiss mathematician C. Cellérier gave another example,

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(a^k x)}{a^k}, \tag{13}$$

which is nowhere differentiable when a is a sufficiently large positive integer. His result was not published, however, until 1890, whereas Weierstrass' result was published in 1875.

According to Weierstrass, Riemann claimed in his lectures in 1861 that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(k^2 x)}{k^2}, \tag{14}$$

is nowhere differentiable, or at least nondifferentiable on a dense subset of $\mathbb{R}$. Weierstrass was unable to verify nondifferentiability of Riemann's function

9. The historical notes concerning this function are derived from Chapter 6 of [66], Section 6.4 of [21], Section III.9 of [61], pages 955–6 of [75], and pages 44–47 of [62], which are all recommended for further reading.

(14) and subsequently constructed his own example, the function (12) above. In 1970, J. Gerver proved that, in fact, Riemann's function (14) is differentiable at infinitely many points; namely, when $x = a\pi$, where $a = \frac{2m+1}{2k+1}$, for $m, n \in \mathbb{N}$.

The function we are going to consider is a clever simplification of Weierstrass' example, published by Davidson and Donsig in [31]:[10]

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(10^k \pi x)}{2^k}. \tag{15}$$

The graph of the sum of the first three terms is shown in Figure 9.12 below.

The idea behind (15) is to use a uniformly convergent series of continuous functions to ensure that the limit function is continuous, and use cosine functions of smaller and smaller periods as summands to ensure that the limit function has infinitely many oscillations in every open interval, causing it to be nowhere differentiable.

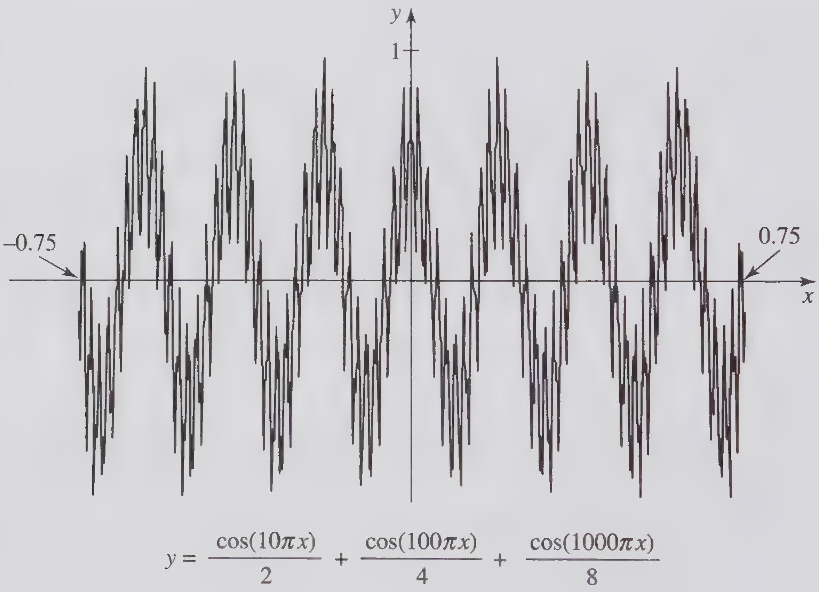

Figure 9.12

Theorem 9.4.1 (***Weierstrass***): *There exists a function that is continuous everywhere on $(-\infty, \infty)$, but differentiable nowhere.*

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10. Proofs using Weierstrass' function may be found in [63] as Theorem (17.7), in Chapter 9 of [34], and in Section 6.4 of [21]. Most authors use an entirely different kind of function, based on one devised by B. L. van der Waerden in 1930. See Example 8.13 of [30] for details.

Proof. Define the function $f(x) = \sum_{k=1}^{\infty} f_k(x)$, where $f_k(x) = \dfrac{\cos(10^k \pi x)}{2^k}$.

Claim #1: f is continuous everywhere.
Proof: Use the Weierstrass M-test and Corollary 9.3.7.

Claim #2: f is differentiable nowhere.

Proof: Let $x \in \mathbb{R}$. We shall prove that f is not differentiable at x by using the sequential criterion. We shall produce a sequence $\{x_n\}$ such that $x_n \to x$ but $\left\{\dfrac{f(x_n) - f(x)}{x_n - x}\right\}$ diverges.

First, represent x using a "decimal":

$$x = N + 0.d_1 d_2 d_3 \cdots d_k \cdots = N + \sum_{k=1}^{\infty} \frac{d_k}{10^k},$$

where N is an integer and $\forall k \in \mathbb{N}$, $d_k \in \{0, 1, 2, 3, \cdots, 9\}$.

Part 1: Let n be a fixed positive integer, and let

$$a_n = N + 0.d_1 d_2 d_3 \cdots d_n \text{ and } b_n = a_n + \frac{1}{10^n}.$$

Then $a_n \leq x \leq b_n$.

From $|f(a_n) - f(b_n)| = \left| f_n(a_n) - f_n(b_n) + \sum_{k \neq n} [f_k(a_n) - f_k(b_n)] \right|$ show that

$$|f(a_n) - f(b_n)| \geq |f_n(a_n) - f_n(b_n)| - \sum_{k \neq n} |f_k(a_n) - f_k(b_n)|.$$

Show that for $k \geq n$, $|f_k(a_n) - f_k(b_n)| = 0$, and hence,

$$|f(a_n) - f(b_n)| \geq |f_n(a_n) - f_n(b_n)| - \sum_{k=1}^{n-1} |f_k(a_n) - f_k(b_n)|. \tag{16}$$

Also show that $|f_n(a_n) - f_n(b_n)| = \dfrac{1}{2^{n-1}}$. Apply this to Inequality (16) to get

$$|f(a_n) - f(b_n)| \geq \frac{1}{2^{n-1}} - \sum_{k=1}^{n-1} |f_k(a_n) - f_k(b_n)|. \tag{17}$$

Show that, for $1 \leq k \leq n$, the mean value theorem guarantees that

$$|f_k(a_n) - f_k(b_n)| \leq \frac{10^k \pi}{2^k} \cdot \frac{1}{10^n} = \frac{5^{k-n} \pi}{2^n}.$$

Apply this to Inequality (17) to get

$$|f(a_n) - f(b_n)| \geq \frac{1}{2^{n-1}} - \sum_{k=1}^{n-1} \frac{5^{k-n}\pi}{2^n} > \frac{1}{2^{n-1}} - \frac{\pi}{2^n} \cdot \frac{1}{4} > \frac{1}{2^n}.$$

Apply the triangle inequality to get

$$|f(a_n) - f(x)| + |f(x) - f(b_n)| \geq |f(a_n) - f(b_n)| > \frac{1}{2^n}.$$

Thus, either $|f(a_n) - f(x)| > \dfrac{1}{2^{n+1}}$ or $|f(x) - f(b_n)| > \dfrac{1}{2^{n+1}}$. (Why?) Choose $x_n = a_n$ or b_n, so that $|f(x_n) - f(x)| > \dfrac{1}{2^{n+1}}$.

Part 2: Consider the sequence $\{x_n\}$ constructed in Part 1. Show that $|x_n - x| \leq \dfrac{1}{10^n}$, and so $x_n \to x$. But, show that $\left\{\dfrac{f(x_n) - f(x)}{x_n - x}\right\}$ diverges. Therefore, f is not differentiable at x. (Why?) ■

THE WEIERSTRASS APPROXIMATION THEOREM

By the time of Weierstrass (1815–1897), mathematicians had been making routine use of infinite series (especially power series) since the creation of calculus some two centuries earlier. They had achieved spectacular successes using series to represent functions. Mathematicians of Weierstrass' time depended so heavily on the use of infinite series in their work that they had come to believe that all functions of any worth could be analyzed using power series.

For a function to have a power series representation it must, of course, have derivatives of all orders. Even the existence of the Taylor polynomial $T_n(x)$ requires a function to have derivative of order n somewhere. Thus, it became common for mathematicians to believe that functions, in general, must be differentiable at least somewhere. So it is easy to see why Weierstrass' example of an everywhere continuous, nowhere differentiable function was so disturbing, even to Weierstrass himself.

In his **Approximation Theorem**,[11] Weierstrass used a completely original approach to show that, given an arbitrary continuous function f on a compact interval $[a, b]$ and an arbitrary $\varepsilon > 0$, there exists a polynomial p such that

$$\forall x \in [a, b],\ |f(x) - p(x)| < \varepsilon.$$

That is, relative to $[a, b]$, $\|f - p\| < \varepsilon$. To paraphrase, Weierstrass' theorem guarantees that for any continuous f on $[a, b]$ and any $\varepsilon > 0$, there is a polynomial that stays "ε-close" to f on $[a, b]$. For any prescribed accuracy, there is a polynomial whose values can be used to approximate the values of $f(x)$ with that accuracy over a given compact interval.

This was a partial vindication of those who advocated the use of (power) series to represent functions. For in calculations they frequently truncated series

11. Published in 1885, when Weierstrass was 70 years old, thirteen years after he announced his continuous, nowhere differentiable function.

after a convenient number of terms, and thus were really using polynomials to approximate functions. They relied on theorems such as Taylor series to address the accuracy of these polynomial approximations. Weierstrass' theorem guarantees that arbitrary continuous functions can indeed be approximated by polynomials, but both the polynomials and the approximating behavior are completely different from the familiar Taylor polynomials.

This is a deep theorem, and it has no "easy" proof. Weierstrass' original proof used "singular integrals," a topic well beyond the scope of this course. In subsequent decades intense work by many mathematicians produced quite a few alternative proofs, most of which are also beyond the scope of this course.[12] The proof we give here is a modified version of a proof suggested by H. Lebesgue in 1898 and included in [99]. I find it to be the most straightforward and most easily understood, as well as intuitively appealing.

We begin our proof with the following:

Definition 9.4.2 We say that a function $f : [a,b] \to \mathbb{R}$ **can be approximated by polynomials** if $\forall \varepsilon > 0$, there exists a polynomial p such that $\forall x \in [a,b]$, $|f(x) - p(x)| < \varepsilon$; that is, relative to $[a,b]$, $\|f - p\| < \varepsilon$.

Lemma 9.4.3 *A function $f : [a,b] \to \mathbb{R}$ **can be approximated by polynomials** iff $\exists$ sequence $\{p_n\}$ of polynomials such that $p_n \to f$ uniformly on $[a,b]$.*

Lemma 9.4.4 *The set $P[a,b]$ of polynomials with domain $[a,b]$ is a vector space [in fact, a subspace of $C[a,b]$].*

Lemma 9.4.5 *The set of all continuous $f : [a,b] \to \mathbb{R}$ that can be approximated by polynomials is a vector space [a subspace of $C[a,b]$]. Let us (temporarily) call this space $CAP[a,b]$.*

Definition 9.4.6 A subset S of $C[a,b]$ is said to be **dense**[13] in $C[a,b]$ if $\forall f \in C[a,b]$ and $\forall \varepsilon > 0$, $\exists\, g \in S \ni \|f - g\| < \varepsilon$.

Lemma 9.4.7 *A subset S of $C[a,b]$ is dense[14] in $C[a,b]$ iff $\forall f \in C[a,b]$, $\exists$ sequence $\{s_n\}$ of elements of S that converges uniformly to f on $[a,b]$.*

Thus, our task is to prove that the polynomials are dense in $C[a,b]$. We begin by obtaining an intermediate result, which requires another definition.

12. For a summary of the various proofs see the survey paper [107] by Pinkus.

13. Compare this with Exercise 3.2.29.

14. Compare this with Theorem 3.2.21.

Definition 9.4.8 A continuous function $f : [a, b] \to \mathbb{R}$ is said to be **piecewise linear**[15] if $\exists$ partition $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ of $[a, b]$ and constants $a_1, a_2, \cdots, a_n$, $m_1, m_2, \cdots, m_n \in \mathbb{R}$ such that $\forall t = 1, 2, \cdots, n$,

$$t \in [x_{i-1}, x_i] \Rightarrow f(t) = a_i + m_i(t - x_{i-1}).$$

Remarks 9.4.9 For a given piecewise linear, continuous f defined in 9.4.8,

(a) $\forall\, i = 1, 2, \cdots, n,\ \ f(x_{i-1}) = a_i$ and $f(x_i) = f(a) + \sum\limits_{k=1}^{i} m_k(x_k - x_{k-1})$.

(b) $\forall\, i = 1, 2, \cdots, n,\ \ m_i = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$.

(c) For convenience later, we define $m_0 = 0$.

(d) Geometrically, the graph of f consists of line segments connecting the endpoints $(x_0, f(x_0)), (x_1, f(x_1)), \cdots, (x_n, f(x_n))$, with slopes $m_1, m_2, \cdots, m_n$ respectively.

Thus, the graph of a piecewise linear continuous function is a **polygonal arc**, and such functions are often called **polygonal functions**. It was Lebesgue's genius to see that continuous functions could be approximated by polygonal functions, and that they in turn can be approximated by polynomials.

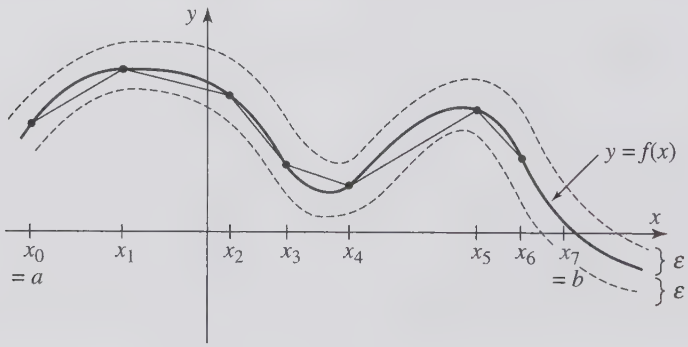

Figure 9.13

Theorem 9.4.10 *If f is continuous on $[a, b]$, then $\forall\, \varepsilon > 0$, $\exists$ polygonal g on $[a, b] \ni \|f - g\| < \varepsilon$. (The polygonal functions are dense in $C[a, b]$.)*

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15. In approximation theory, a piecewise linear continuous function is called a "spline of degree one."

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is continuous and $\varepsilon > 0$. Since f is *uniformly* continuous (why?) on $[a,b]$, $\exists\, \delta > 0 \ni \forall\, x_1, x_2 \in [a,b]$, $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$. Choose $n \in \mathbb{N} \ni \frac{b-a}{n} < \delta$, and let $\mathcal{P}_n = \{x_0, x_1, x_2, \cdots, x_n\}$ be the regular partition of $[a,b]$ into n subintervals of length $\frac{b-a}{n}$. Define $g : [a,b] \to \mathbb{R}$ by

$$\forall\, t \in [x_{i-1}, x_i],\ \ g(t) = f(x_{i-1}) + \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(t - x_{i-1}).$$

Then g is a polygonal function on $[a,b]$ and $\forall\, i = 1, 2, \cdots, n,\ \ g(x_i) = f(x_i)$. (Prove that g is continuous on $[a,b]$.)

Let $t \in [a,b]$. Then $t \in [x_{i-1}, x_i]$ for some i. Since g is piecewise linear, $g(t)$ is between $g(x_{i-1})$ and $g(x_i)$; that is, between $f(x_{i-1})$ and $f(x_i)$. Thus, by the intermediate value theorem,

$$\exists\, c \in [x_{i-1}, x_i] \ni f(c) = g(t).$$

Show that this implies $|g(t) - f(t)| < \varepsilon$.

Since this holds $\forall\, t \in [a,b]$, $\|f - g\| < \varepsilon$ on $[a,b]$. ■

The next step is to express polygonal functions as sums of simpler functions, which we shall be able to approximate with polynomial functions.

Theorem 9.4.11 *Every polygonal $f : [a,b] \to \mathbb{R}$ is a constant plus a linear combination of functions of the form*

$$(x - c)^+ = \max\{0, x - c\} = \begin{cases} 0 & \text{if } x \le c; \\ x - c & \text{if } x \ge c. \end{cases}$$

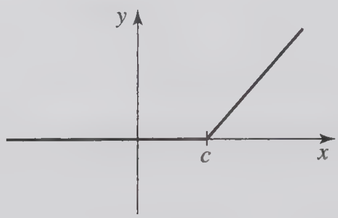


Figure 9.14

Proof. Suppose f is a polygonal function on $[a,b]$. Using the notation of Definition 9.4.8, show that

(a) $\forall\, t \in [x_0, x_1],\ f(t) = f(x_0) + (m_1 - m_0)(t - x_0)$
$$= f(a) + \sum_{i=1}^{n} (m_i - m_{i-1})(t - x_{i-1})^+.$$

(b) $\forall\, t \in [x_1, x_2],\ \ f(t) = f(x_1) + (m_1 - m_0)(t - x_0) + (m_2 - m_1)(t - x_1)$
$$= f(a) + \sum_{i=1}^{n} (m_i - m_{i-1})(t - x_{i-1})^+.$$

(c) $\forall\, t \in [x_{i-1}, x_i],\;\; f(t) = f(a) + \sum_{i=1}^{n} (m_i - m_{i-1})(t - x_{i-1})^+.$

Since this formula applies in every subinterval $[x_{i-1}, x_i]$, we conclude that

$$\forall\, t \in [a,b],\;\; f(t) = f(a) + \sum_{i=1}^{n} (m_i - m_{i-1})(t - x_{i-1})^+. \quad \blacksquare$$

We now need a few more technical results that will lead us to a proof of Weierstrass' theorem.

Lemma 9.4.12 *$\forall c \in \mathbb{R}$, the function $(x-c)^+$ is expressible in the form*

$$(x-c)^+ = \frac{(x-c) + |x-c|}{2}.$$

This suggests that we should now focus our attention on approximating $|x-c|$ by polynomials.

Theorem 9.4.13 *The function $f(x) = 1 - |x|$ can be approximated by polynomials on $[-1,1]$.*

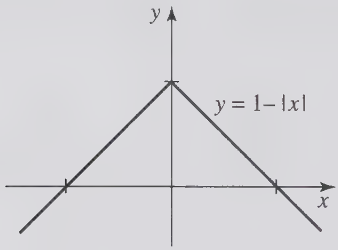


Figure 9.15

Proof. We shall use Lemma 9.4.3. Let $y = 1 - |x|$ on $[-1,1]$. Show that this is equivalent to

$$x^2 = 1 - 2y + y^2,\; 0 \le y \le 1,$$

which is equivalent to

$$y = \frac{y^2 + (1 - x^2)}{2}.$$

Define a sequence of polynomials $\{p_n\}$ recursively by

$$\begin{cases} p_0(x) = 1, \;\; \text{and } \forall n \in \mathbb{N}, \\ p_n(x) = \dfrac{p_{n-1}^2(x) + (1 - x^2)}{2}. \end{cases}$$

Write out the first three polynomials in this sequence to see what they look like.

Show that $p_n(x)$ is a polynomial in x^2 of degree 2^{n-1}.

Use mathematical induction to prove that $\forall x \in [-1,1]$, $0 \leq p_n(x) \leq 1$.

Show that $\forall x \in [-1,1]$, $\{p_n(x)\}$ is monotone decreasing.

Therefore, $\forall x \in [-1,1]$, $\{p_n(x)\}$ converges (why?).

Define the function $L : [-1,1] \to \mathbb{R}$ by $L(x) = \lim_{n\to\infty} p_n(x)$, and prove that $L(x) = 1 - |x|$.

Show that Dini's uniform convergence theorem[16] (9.3.14) guarantees that $p_n(x)$ converges uniformly to $1 - |x|$ on $[-1,1]$. ■

Corollary 9.4.14 *The function $|x|$ can be approximated by polynomials on $[-1,1]$.*

Proof. Apply Lemma 9.4.5 and Theorem 9.4.13. ■

Corollary 9.4.15 *Given any $c \in \mathbb{R}$, the function $|x - c|$ can be approximated by polynomials on any compact interval $[a,b]$.*

Proof. Suppose $a < b$ and $c \in \mathbb{R}$. Choose any $d > 0$ such that $[a,b] \subseteq [c-d, c+d]$. Let $\{q_n(x)\}$ be a sequence of polynomials converging uniformly to $|x|$ on $[-1,1]$ guaranteed by Corollary 9.4.14.

$$\text{Now } \forall\, x \in [c-d, c+d], \text{ let } t = \frac{x-c}{d}; \text{ show that } -1 \leq t \leq 1.$$

Define the polynomials $r_n(x)$ on $[c-d, c+d]$ by $r_n(x) = d\,q_n\left(\frac{x-c}{d}\right) = d\,q_n(t)$.

Show that $\{r_n\}$ converges uniformly to $|x - c|$ on $[c-d, c+d]$, and hence on $[a,b]$. ■

Corollary 9.4.16 *$\forall c \in \mathbb{R}$, the function $(x-c)^+$ can be approximated by polynomials on any compact interval $[a,b]$.*

Corollary 9.4.17 *Every polygonal $f : [a,b] \to \mathbb{R}$ can be approximated by polynomials on $[a,b]$.*

Theorem 9.4.18 (***Weierstrass' Approximation Theorem***) *Every continuous function $f : [a,b] \to \mathbb{R}$ can be approximated by polynomials on $[a,b]$.*

Corollary 9.4.19 *Given any continuous $f : [a,b] \to \mathbb{R}$, there is a sequence of polynomials converging uniformly to f on $[a,b]$.*

---

16. I bet you thought you would never use that theorem!

Corollary 9.4.20 *Given any continuous $f : [a, b] \to \mathbb{R}$, there is a series $\sum_{k=1}^{\infty} p_n(x)$ of polynomials converging uniformly to f on $[a, b]$.*

Proof. See Exercise 8.1.6. ■

Weierstrass' approximation theorem may be regarded as the fundamental theorem of "approximation theory," an area of mathematical research that came into prominence with the rise of computing technology. This area of study seeks to calculate the values of complicated functions by using simpler (more easily calculated) functions called "approximations." Polynomials are often used as approximations because of their relative simplicity. Although mathematicians of our time have developed a greater variety of approximation methods, polynomials remain a valuable tool in their arsenal.

A more popular approach to proving the Weierstrass approximation theorem uses "Bernstein polynomials." Recommended readings for this approach are Douglass [32] pp. 168–172, Gaskill and Narayanaswami [47] pp. 428–435, and Pugh [110] pp. 217–222. Another approach using "Dirac sequences" may be found in Douglass [32] pp. 281–285, Hairer and Wanner [61] pp. 265–269, and Stoll [128] pp. 346–352. For other approaches see the comprehensive survey paper by Allan Pinkus [107].

9.5 *A Glimpse Beyond the Horizon

It is time to confess that a thorough understanding of the Elements of Real Analysis is not an end in itself. It is rather a *beginning.* It opens the door to further study of a wide panorama of areas of modern analysis, such as multivariable real analysis, general integration theories, complex analysis, functional analysis, Fourier analysis, approximation theory, and various areas of applied mathematics.

We close this course with a few tantalizing ideas that may whet your appetite for further study in real and functional analysis. To do justice to these ideas would require several chapters, so we shall have to settle for a quick glimpse beyond the horizon.

Definition 9.5.1 A **normed vector space** is a vector space $\mathcal{V}$ together with a "norm" $\|\cdot\|$; that is, a function from $\mathcal{V}$ to $\mathbb{R}$ such that $\forall u, v \in \mathcal{V}$, and $\forall r \in \mathbb{R}$,

(a) $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$;

(b) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality);

(c) $\|ru\| = |r| \; \|u\|$.

In Theorem 9.2.2 we saw that the "sup norm" defined in 9.2.1 has these properties on the space of bounded functions on $[a, b]$, as well as on subspaces

such as $C[a, b]$. But there are other norms commonly in use on $C[a, b]$ and its subspaces; for example,

$$\|f\|_1 = \int_a^b |f|, \text{ and}$$

$$\|f\|_2 = \sqrt{\int_a^b |f|^2}.$$

Each of these norms on $C[a, b]$ is useful for certain purposes, such as in Fourier analysis or approximation theory. You may wish to check for yourself that these are indeed norms on $[a, b]$. It is fairly easy to understand why $\|\cdot\|_1$ might be useful as a norm, when you remember that the $\|f\|_1$ is intended to indicate the distance of f from the 0 function, and that $\|f - g\|$ is intended to represent the distance between f and g. The *area between their graphs* is a reasonable measure of the distance between two functions. See Figure 9.16.

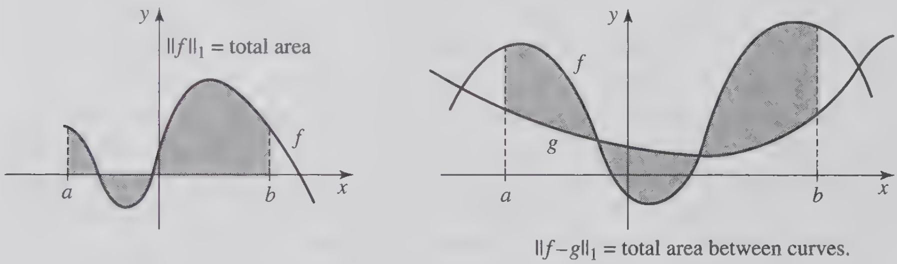


Figure 9.16

The importance of the second norm, $\|\cdot\|_2$, can perhaps be best understood when it is seen as more closely resembling the familiar "Euclidean" norm on $\mathbb{R}^n$,

$$\|(x_1, x_2, \cdots, x_n)\| = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

This norm, $\|\cdot\|_2$, is commonly used in approximation theory and Fourier analysis, although it is not the only one.

Many of the concepts learned in this course generalize to normed vector spaces. For example, a sequence $\{v_n\}$ in a normed vector space $\mathcal{V}$ is said to **converge** to an element $v \in \mathcal{V}$ if

$$\forall\, \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \|v_n - v\| < \varepsilon \text{ (i.e., } \|v_n - v\| \to 0).$$

A sequence $\{v_n\}$ in a normed vector space $\mathcal{V}$ is said to be a **Cauchy sequence** if

$$\forall\, \varepsilon > 0, \exists\, n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow \|v_m - v_n\| < \varepsilon.$$

Thus, a sequence $\{f_n\}$ of functions in the space $C[a, b]$ converges to a function f by this definition iff $f_n \to f$ *uniformly* on $[a, b]$. Theorems 9.2.7 and

9.3.6 tell us that a sequence $\{f_n\}$ in $C[a,b]$ converges to some $f \in C[a,b]$ iff it is a Cauchy sequence. Thus, the theory of norm-convergence of functions in $C[a,b]$ closely parallels the corresponding theory of convergence of sequences of real numbers.

Definition 9.5.2 A normed vector space $\mathcal{V}$ is said to be **complete** if every Cauchy sequence in $\mathcal{V}$ converges to an element of $\mathcal{V}$. (From Theorem 2.7.7 we know that, for Archimedean ordered fields, this condition is equivalent to completeness as we defined it in Chapter 1.)

For example, $C[a,b]$ (with the sup norm) is complete, by Theorems 9.2.7 and 9.3.6. Complete normed vector spaces are also called **Banach spaces** and are the subject of an entire area of contemporary mathematical research.

There are many normed vector spaces that are not complete. To see this, observe that in a normed vector space every convergent sequence is also a Cauchy sequence. By the Weierstrass approximation theorem there are sequences of polynomials that converge (relative to the sup norm) to a non-polynomial. Thus there are Cauchy sequences of polynomials in $P[a,b]$ that do not converge to an element of $P[a,b]$. Therefore, $P[a,b]$ is not complete. Similar arguments show that the space of piecewise linear functions on $[a,b]$, and the space of differentiable functions on $[a,b]$ are not complete relative to the sup norm. This is not the place to discuss completeness relative to the other norms.

We can redo much of Chapters 2, 3, and 5 in the context of a normed vector space $\mathcal{V}$. For example, we define the ε-neighborhood of an element $v \in \mathcal{V}$ to be the set

$$N_\varepsilon(v) = \{u \in \mathcal{V} : \|u - v\| < \varepsilon\}.$$

We say a set $A \subseteq \mathcal{V}$ is **open** if $\forall\, a \in A,\ \exists\, \varepsilon > 0 \ni N_\varepsilon(a) \subseteq A$, and we say a set A is **closed** if A^c is open. Similarly, we define an element $v \in \mathcal{V}$ to be a **cluster point** of a set $A \subseteq \mathcal{V}$ if every neighborhood of v contains a member of A other than v. We define the **closure** $\overline{A}$ of a set $A \subseteq \mathcal{V}$ to be the union of A and the set of all its cluster points, and prove that $\overline{A}$ is the smallest closed set containing A. A set $A \subseteq \mathcal{V}$ is **dense** in $\mathcal{V}$ if every neighborhood of every point of $\mathcal{V}$ contains a member of A. Theorems and proofs concerning these concepts are virtually the same as the those found in Chapter 3.

Equipped with these preliminaries, we bring our course to an end with a little razzle dazzle. Don't worry about all the details; just enjoy the ride. As in Definition 3.4.16, we define a set $A \subseteq \mathcal{V}$ to be **nowhere dense** in $\mathcal{V}$ if its closure, $\overline{A}$, contains no nonempty open sets. This definition allows us to discuss first and second category sets, just as we did in Section 5.7 but now in the context of normed vector spaces. We say that a set $A \subseteq \mathcal{V}$ is of **first category** (or "meager") in $\mathcal{V}$ if it is the union of a countable collection of nowhere dense sets; otherwise, we say it is of **second category**. Of significance here is the following deep theorem, whose proof we must omit.

Theorem 9.5.3 *Every complete normed vector space is of second category.*

The reason for including this theorem here is the following amazing discovery:

Theorem 9.5.4 *The set of all functions in $C[a, b]$ that are differentiable somewhere in $[a, b]$ is of first category in $C[a, b]$.*

By definition, the union of two first category sets must be a first category set. Also, $C[a, b]$ with the sup norm is a complete normed vector space, and so must be of second category. These statements cannot both be true unless the set of all everywhere continuous, nowhere differentiable functions on $[a, b]$ is of second category $C[a, b]$. This leads us to two remarkable conclusions:

(1) We have a proof of the *existence* of continuous, nowhere differentiable functions on $[a, b]$ that is valid without ever producing a single example.

(2) Among all functions that are continuous everywhere on $[a, b]$, those that are differentiable somewhere form a much smaller set than those that are differentiable nowhere. (First category sets are much "smaller" than second category sets.) This is reminiscent of the situation in the real number system: The rational (nice) numbers are far outnumbered by the irrational (not so nice) numbers, since the former form a countable set while the latter form an uncountable set. Similarly, the algebraic numbers form a countable set and so are far outnumbered by the transcendental numbers, which form an uncountable set.

With these rather unsettling results,[17] we take our leave. May you enjoy further study in real analysis. The subject is rich with treasures to discover!

17. For more details consult pages 63–65 and 70–72 of [16]

Appendix A

Logic and Proofs

A.1 The Logic of Propositions

> The theory of deductive logic, as it applies to mathematics, divides naturally into two main areas:
>
> – the calculus of propositions, and
> – the calculus of propositional functions.

Mathematics is expressed in language; indeed, some would say that it *is* a language. Mathematical truth is expressed in sentences. Even though these sentences may be abstract or symbolic, they must be clear and unambiguous. Moreover, some sentences taken together imply other sentences. Logic enables us to clarify these relationships and to distinguish valid implications from invalid ones. It should not be surprising, in view of the role of proofs in mathematics, that the principles of logic play a significant role in this book.

There are two types of sentences that occur frequently in mathematics: (1) propositions, and (2) propositional functions. We discuss (1) here, and defer (2) to Section A.2.

Definition A.1.1 A **proposition** is a declarative sentence that is either true or false, but not both.

(We say that a proposition has a definite "truth-value," either T or F, but not both.)

For example, "$3 + 5 = 10$" is a proposition; it has truth-value F, but it is still a proposition. On the other hand, "$x + y = 10$" is *not* a proposition; it is neither true nor false. Indeed, there is no way of knowing whether it is true or false because x and y are unspecified.

The actual words used in uttering a proposition are not important in this context; it is the *meaning* of the sentence that is important. Thus, two different

sentences may express the same proposition. For example, "John is five feet eleven inches tall" is the same proposition as "If John were one inch taller he would be exactly six feet tall." Moreover, translating the sentence into a different language would not make it a different proposition.

Thus, the notion of a proposition is really an abstraction. The words we use are merely a tangible representation of the proposition. They bear the same relationship to the (abstract) proposition as a tangible triangle drawn on paper or chalkboard bears to the (abstract) triangle it represents. In fact, all mathematical concepts are abstractions!

COMPOUND PROPOSITIONS

Simple propositions are often combined together to make compound propositions. Logicians have identified five **logical connectives** commonly used for this purpose:

THE FIVE LOGICAL CONNECTIVES

Connective	Name	Example	Symbolic form
and	conjunction	P and Q	$P \wedge Q$
or	disjunction	P or Q	$P \vee Q$
not	negation	not P	$\sim P$
if ... then	implication	if P, then Q	$P \Rightarrow Q$
if and only if	bi-implication	P if and only if Q	$P \Leftrightarrow Q$

In the study of logic we analyze the relationship between the truth-values of simple propositions and the truth-values of related compound propositions.

Definition A.1.2 The **conjunction "P and Q"** is symbolized $\boldsymbol{P \wedge Q}$ and is defined by the truth-table:

Table A.1

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Table A.1 shows that $\boldsymbol{P \wedge Q}$ is true only when P and Q are both true; it is false in all other cases.

Examples A.1.3 Some conjunctions:

(a) I ate a snack and then I went to bed.

(b) "I like cheese and crackers" is a conjunction when it is intended to mean "I like cheese and I like crackers," but it is not a conjunction when it is intended to mean "I like cheese with crackers."

(c) "$1 < 3 < 7$" is the conjunction "$1 < 3$ and $3 < 7$." □

Definition A.1.4 The **disjunction** "$\boldsymbol{P}$ **or** $\boldsymbol{Q}$" is symbolized $\boldsymbol{P} \vee \boldsymbol{Q}$ and is defined by the truth-table:

Table A.2

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Notice that, according to Table A.2, $\boldsymbol{P} \vee \boldsymbol{Q}$ is true when both P and Q are true. This tells us that we are using the *inclusive* "or" here. That is, $\boldsymbol{P} \vee \boldsymbol{Q}$ includes the possibility that P and Q are both true. It is false only when both P and Q are false. Sometimes in everyday usage, "or" is used in a different sense, the *exclusive* sense, to exclude that possibility. For example, a person saying, "Either you believe me or you don't" would most likely be intending the exclusive meaning of "or." In ordinary conversation it is up to the user of the word "or" to decide what he or she means. But in logic and mathematics, we cannot allow this ambiguity. Thus, we agree to use only the inclusive "or." For us "or" will always be equivalent to "and/or."

Examples A.1.5 Some disjunctions:

(a) I will see you tonight or I will phone you. (The inclusive "or" allows that I could do both.)

(b) $7 \geq 3$. (This is the disjunction "$7 > 3$ or $7 = 3$.")

(c) $x = 1$ or $x = 2$. (Observe that the solution of the equation "$x^2 - 3x + 2 = 0$" is the disjunction "$x = 1$ or $x = 2$," *not* the conjunction "$x = 1$ and $x = 2$." Many students use "and" here, when the proper connective is "or." In fact, the conjunction here would be incorrect, even false.) □

Definition A.1.6 The **implication** "If P, then Q" (or "P implies Q") is symbolized $\boldsymbol{P \Rightarrow Q}$ and is defined by the truth-table:

Table A.3

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

In the compound proposition $P \Rightarrow Q$, the proposition P is called the **hypothesis** and the proposition Q is called the **conclusion**. Notice that an implication is false only when its hypothesis is true and its conclusion is false. An implication with a false hypothesis is true regardless of the conclusion.

Examples A.1.7 Some implications:

(a) If you can't come tonight, then we'll cancel the party.

(b) If you hit me, I'll scream.

(c) "$A \subseteq B$"; [1] that is, "A is a subset of B." This statement is intended to mean "any x belonging to A must belong to B." That is, if $x \in A$, then $x \in B$. In symbols, $x \in A \Rightarrow x \in B$.

(d) Because of (c), the empty set[1] $\varnothing$ is a subset of every set B. That is, $\varnothing \subseteq B$, for all sets B. This is because $x \in \varnothing \Rightarrow x \in B$ is always true, since $x \in \varnothing$ is always false.

CAUTION: An implication is one-directional, in the sense that $Q \Rightarrow P$ is not the same as $P \Rightarrow Q$. For example,

$$\text{“If } x > 2, \text{ then } x > 1\text{”}$$

is not the same statement as

$$\text{“If } x > 1, \text{ then } x > 2.\text{”} \quad \square$$

---

1. See Appendix B.1, where the notions of sets, subsets, and the empty set are discussed.

ALTERNATIVE WAYS OF SAYING $P \Rightarrow Q$

The English language provides a variety of expressions, all of which are equivalent to $P \Rightarrow Q$:

If P, then Q.
If P, Q.
P implies Q.
Q if P.
P only if Q.
P is a sufficient condition for Q.
P is sufficient for Q.
Q is a necessary condition for P.
Q is necessary for P.

Examples A.1.8 The following are statements of the form $P \Rightarrow Q$. They are all equivalent (assuming x is a known number).

(a) If $x > 5$ then $x > 1$.
(b) If $x > 5$, $x > 1$.
(c) $x > 5$ implies $x > 1$.
(d) $x > 1$ if $x > 5$.
(e) $x > 5$ only if $x > 1$.
(f) $x > 5$ is a sufficient condition for $x > 1$.
(g) $x > 5$ is sufficient for $x > 1$.
(h) $x > 1$ is a necessary condition for $x > 5$.
(i) $x > 1$ is necessary for $x > 5$. $\square$

Notes: (a) $x = 1$ is a sufficient, but not necessary, condition, for $x^2 = 1$.

(b) $\cos x = 1$ is a necessary, but not sufficient, condition for $x = 0$.

(c) In any triangle, having two congruent sides is a necessary and sufficient condition for having two congruent angles.

Definition A.1.9 The **bi-implication** "***P* if and only if *Q***" is symbolized $\boldsymbol{P \Leftrightarrow Q}$ and is defined by the truth-table:

Table A.4

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Thus, $P \Leftrightarrow Q$ is true only when P and Q *have the same truth value.* Sometimes we say, "P is necessary and sufficient for Q." Mathematicians have invented the short word "**iff**"[2] to stand for "if and only if." That is,

$$\text{“}P \textbf{ iff } Q\text{” means “}P \Leftrightarrow Q\text{.”}$$

We shall use "iff" frequently in this course.

Examples A.1.10 Some true bi-implications:

(a) $x^2 = 4 \Leftrightarrow x = 2$ or $x = -2$.
(b) $|x - 3| < 1$ iff $2 < x < 4$.
(c) A triangle is isosceles if and only if two of its angles are congruent. □

CAUTION: In ordinary conversation, people usually avoid the "if and only if" phrase, because it is awkward. They often say "if" when they really mean "if and only if." For example, a person who says, "If you mow my yard, I'll give you \$25" probably means "I'll give your \$25 if and only if you mow my yard." (The \$25 will not likely be paid if the yard is not mowed!)

In mathematics we must be very careful not to write "if" when we mean "if and only if." That is perhaps a good reason for using the word "iff."

Remark A.1.11 (Use of "if" in Definitions) *When stating definitions,* mathematicians usually use "if" to mean "if and only if." It is logically incorrect usage of "if" but sanctioned by long-standing practice. According to this practice, definitions such as the following are quite common:

> **Definition:** A triangle is isosceles if it has (at least) two congruent sides.

While it would seem to be bad style to use "if" instead of the more correct "iff" in definitions, mathematicians seem to be incurable of this habit.

Definition A.1.12 The **negation** of a proposition P is the proposition "**not-**P." It is symbolized $\sim P$ and is defined by the truth-table:

Table A.5

P	$\sim P$
T	F
F	T

2. One bit of mathematical folklore suggests that the word "iff" was coined in the 1950s when mathematicians first conjoined "if" with "fi" (backwards "if") to form the biconditional "iffi," and then dropped the final "i" in the interest of simplicity.

Thus, $\sim P$ has the opposite truth-value from P.

Examples A.1.13 Some negations:

(a) The negation of "This car can go 97 mph." is "This car can't go 97 mph."

(b) The negation of "He's taking English and math" is "He is not taking both English and math (although he might be taking one of them)." A smoother way of saying that is, "Either he is not taking English or he is not taking math." We shall say more about this when we discuss de Morgan's law below. □

EXERCISE SET A.1-A

PART A: In Exercises 1–12, a compound proposition is given. Assign variables, like P, Q, etc. to the constituent propositions and translate the given verbal sentence into logical symbols using the logical connectives we have studied.

Example: I will come to your party only if I don't have to work and I'm feeling better.

Solution: Let C denote "I will come to your party;" W denote "I have to work;" and F denote "I'm feeling better." Then the given proposition is symbolized:

$$C \Rightarrow (\sim W \wedge F).$$

1. 9 is neither an even number nor a prime number.

2. I like peanuts but not walnuts. [Note: here, "but" is a conjunction used to mean "and."]

3. I like ham and beans but not for breakfast.

4. You will get an A only if you work hard.

5. Analysis is an interesting subject, and I will get an A if the professor is easy.

6. A triangle is isosceles if it is equilateral.

7. If you don't show up for the final exam, or if you fail it, you will not pass the course.

8. To receive a passing grade for this course you must pay your bill, come to class, do all the assignments, and pass the tests.

9. To be elected President, you must have strong party backing and lots of money, but cannot have a record of dishonesty or be seen as ignorant of the issues.

10. A necessary condition for an integer to be divisible by 12 is that it be divisible by both 4 and 3.

11. A sufficient condition for an integer to be divisible by 12 is that it be divisible by both 4 and 3.

12. It isn't true that if you go on diet X you will get sick and not lose weight.

PART B:

13. Suppose you know that $P \wedge \sim Q$, $Q \vee R$, and $R \Rightarrow S$ are all true. Find the only possible truth-values of P, Q, R, and S.

14. Suppose you know that $P \vee R$, $(P \vee Q) \Rightarrow S$, $(R \wedge P) \Rightarrow T$, and $\sim S$ are all true. Find the only possible truth-values of P, Q, R, S, and T.

15. Let A denote "The assignment is easy," W denote "I work hard," U denote "I understand the basic ideas," and E denote "I enjoy mathematics." Translate the following statements into symbolic form:

    (a) The assignment is easy only if I understand the basic ideas.

    (b) Enjoying mathematics is a necessary condition for me to work hard.

    (c) The assignment isn't easy if I don't work hard or do not understand the basic ideas.

    (d) The assignment is easy only if I work hard, but it is not true that my enjoying mathematics is a sufficient condition for my understanding the basic ideas.

    Translate the following symbolic forms into verbal statements:

    (e) $(U \wedge W) \Rightarrow E$.

    (f) $\sim A \Rightarrow (\sim U \vee \sim W)$.

    (g) $\sim (E \Rightarrow (W \Rightarrow A))$.

LOGICALLY EQUIVALENT PROPOSITIONS

Definition A.1.14 A **tautology** is a compound proposition that always has truth-value T, regardless of the truth-values of its constituent parts.

Examples A.1.15 Some obvious tautologies:

(a) $P \vee \sim P$	(d) $(P \wedge Q) \Rightarrow Q$
(b) $P \Rightarrow P$	(e) $\sim (P \wedge \sim P)$ (law of excluded middle)
(c) $P \Rightarrow (P \vee Q)$	(f) $\sim (\sim P) \Leftrightarrow P$ (double negation)

We can use truth-tables to verify (prove) that these are tautologies. For example, to verify (c) we construct the following truth table:

Table A.6

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Notice that the last column of this truth-table consists of all T's, indicating that the compound proposition heading that column is a tautology. $\square$

Definition A.1.16 Two compound propositions "P" and "Q" are **logically equivalent** if and only if the assertion "$P \Leftrightarrow Q$" is a tautology; that is, P and Q always have the same truth-value. To denote that P and Q are logically equivalent we shall write

$$P \equiv Q.$$

Examples A.1.17 Some obvious logical equivalences:

(a) $P \equiv (P \wedge P)$; $P \equiv (P \vee P)$;

(b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$;

(c) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$. $\square$

Examples A.1.18 Prove **de Morgan's laws:**

(a) $\sim (P \wedge Q) \equiv \sim P \vee \sim Q$;

(b) $\sim (P \vee Q) \equiv \sim P \wedge \sim Q$.

Proof. We prove (a) using the following truth-table, and leave the proof of (b) as Exercise 11 below.

Table A.7

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Observe that the fourth and seventh columns of Table A.7 are identical. That is, the propositions $\sim (P \wedge Q)$ and $\sim P \vee \sim Q$ have the same truth-value,

regardless of the truth-values of P and Q. By Definition A.1.16, this means that these compound propositions are logically equivalent. □

NOTE: de Morgan's laws illustrate a very important principle: The negation of an "and" is not another "and;" rather, it is an "or." Similarly, the negation of an "or" is not another "or;" it is an "and." This should be pondered and remembered, so that it is not a stumbling block for you.

Example A.1.19 Equivalent form of $\Rightarrow$: $(P \Rightarrow Q) \equiv \sim P \vee Q$.

Proof. We use the following truth-table:

Table A.8

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since columns three and five are identical, we have proved that $P \Rightarrow Q$ and $\sim P \vee Q$ are equivalent. □

Example A.1.20 Equivalence of contrapositive: $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$.

Proof. As in the previous examples, we use a truth-table:

Table A.9

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The proof is completed by observing that columns three and six are identical. □

CONVERSE, INVERSE AND CONTRAPOSITIVE

Associated with a given implication, $P \Rightarrow Q$, there are three related implications:

Given implication:	$P \Rightarrow Q$
Its **inverse**:	$\sim P \Rightarrow \ \sim Q$
Its **converse**:	$Q \Rightarrow P$ (sometimes written $P \Leftarrow Q$)
Its **contrapositive**:	$\sim Q \Rightarrow \ \sim P$.

These are not all logically equivalent, but from Example A.1.20 we see the following two equivalences:

Theorem A.1.21 ***Converse, Inverse, and Contrapositive*:**

(a) *An implication is logically equivalent to its contrapositive.*

(b) *The inverse and converse of an implication are logically equivalent.*

Contrary to what many people expect, the **negation of an implication** is *not* another implication. In fact, it is quite different. From Example A.1.19, we have $(P \Rightarrow Q) \equiv \sim P \vee Q$. Thus, using de Morgan's law,

$$\begin{aligned} \sim (P \Rightarrow Q) &\equiv \sim (\sim P \vee \ Q) \\ &\equiv \sim (\sim P) \wedge \sim Q \\ &\equiv P \wedge \sim Q. \end{aligned}$$

Therefore, we have proved the following theorem.

Theorem A.1.22 ***Negation of an Implication*:**
$\sim (P \Rightarrow Q) \equiv P \wedge \sim Q$.

Examples A.1.23 Prove the **distributive laws**:
(a) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$;
(b) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

We prove (a) and leave (b) as Exercise 20.

Table A.10

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Observe that the fifth and eighth columns of this table are identical. That is, the propositions $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent. □

EXERCISE SET A.1-B

In Exercises 1–10 use a truth-table to determine whether the given compound proposition is a tautology.

1. $P \vee \sim P$
2. $P \Rightarrow P$
3. $P \Rightarrow (P \vee Q)$
4. $(P \wedge Q) \Rightarrow Q$
5. $\sim (P \wedge \sim P)$
6. $(P \vee Q) \Rightarrow P$
7. $(P \vee Q) \Rightarrow (P \vee R)$
8. $P \Rightarrow (Q \Rightarrow P)$
9. $(P \Rightarrow Q) \Rightarrow P$
10. $[(P \Rightarrow Q) \wedge \sim Q] \Rightarrow \sim P$

In Exercises 11–20 verify the given equivalence using a truth-table.

11. $\sim (P \vee Q) \equiv (\sim P \wedge \sim Q)$
12. $P \Rightarrow Q \equiv \sim (P \wedge \sim Q)$
13. $P \vee Q \equiv \sim Q \Rightarrow P$
14. $P \vee Q \equiv \sim P \Rightarrow Q$
15. $\sim (P \wedge Q) \equiv Q \Rightarrow \sim P$
16. $P \Rightarrow (Q \Rightarrow R) \equiv (P \wedge Q) \Rightarrow R$
17. $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$
18. $\sim (P \Leftrightarrow Q) \equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$
19. $\sim (P \Leftrightarrow Q) \equiv (P \Leftrightarrow \sim Q)$
20. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

In Exercises 21 28 state the negation of the given symbolic statement, and use the principles of this section to "simplify" where appropriate.

21. $P \vee \sim Q$
22. $P \wedge (Q \vee R)$
23. $(P \vee Q) \Rightarrow R$
24. $P \vee (Q \Rightarrow R)$
25. $P \Rightarrow (Q \Rightarrow R)$
26. $(P \vee Q) \vee R$
27. $(Q \wedge P) \Rightarrow (R \vee S)$
28. $(P \Rightarrow Q) \Rightarrow R$

In Exercises 29–40 translate the given sentence into logical symbols, use the rules of this section to obtain the negation, and translate the result back into smooth English.

29. John is innocent and Mary's charge is a lie.
30. If it rains tomorrow there will be no picnic.
31. I'll order pizza if you come tonight.
32. I like you but I don't like the clothes you wear.
33. I am going either to the concert or to the football game.
34. Either we win soon or I'll quit the team.
35. This statement is true only if I can prove it.
36. If $x \neq 0$, then $x^2 > 0$.
37. The number 1 is neither a prime number nor a composite number.
38. f is continuous at x_0 if f is differentiable at x_0.
39. If you come to my house for dinner, I'll either grill a steak or make stir fry.
40. If you look in the right location tonight you will see the comet if the sky is clear.

A.2 The Logic of Predicates and Quantifiers

Many statements commonly made in mathematics, such as "$f'(x) = x^2 - 5\sin 2x$" have the *form* of a proposition, but are not propositions because they contain variables. Their truth-values cannot be determined as long as the values of the variables are unknown. The variables in the above equation are f and x. We call such statements "propositional functions," or "predicates."

Definition A.2.1 A **propositional function** (or **predicate**) is a declarative sentence containing one or more variables, which becomes a proposition when the variables are replaced by constants.

Such statements are sometimes called "open sentences"—the variables reserve open spaces in the sentence into which we may later put specific constants. Logicians prefer to call such statements "predicates." For example, in the statement "$x > 3$," we can think of x as the subject, and " > 3" as the predicate.

Examples A.2.2 Some propositional functions (open sentences):

(a) $5x^2 + 2x - 7 = 0$.

(b) $\sin 4x = 0.5$.

(c) n is divisible by 5.

(d) $3x - 4y = 0$.

(e) $x^2 - 4 = (x-2)(x+2)$.

(f) $\tan x = \dfrac{\sin x}{\cos x}$.

(g) $A \subseteq B$. □

Each variable appearing in a propositional function has a **domain**, the set of all constants (concrete objects) that may be substituted in place of that variable. In Example A.2.2, the domain of x would be the set of real numbers, the domain of n, the set of integers, and the domain of A and B, a collection of sets. The domain of a variable is allowed to change from one application to another, depending upon the intention of the user.

A propositional function may be true for some values of its variable(s) and false for others. The **truth set** of a propositional function in one variable is the set of all values in the domain of that variable that make it a true proposition.

Examples A.2.3 In each of the following, assume that the domain of x is the set of all real numbers.

Propositional Function	Truth Set
$x^2 - 3x + 2 = 0$	$\{1, 2\}$
$x^2 - 3x + 2 > 0$	$(-\infty, 1) \cup (2, \infty)$
$x^2 + 2x + 1 \leq 0$	$[-1, 2]$
$x^2 - 4 = (x-2)(x+2)$	$(-\infty, \infty)$
$\sin^2 x + \cos^2 x = 1$	$(-\infty, \infty)$
$\sin x = 3$	$\varnothing$, the empty set[3]

□

Notation: A propositional function in one variable will be symbolized $P(x)$, $Q(y)$, etc. Similarly, a propositional function in two variables will be

3. The empty set is discussed in Appendix B.1.

symbolized $P(x,y)$, $Q(u,v)$, etc. Of course, we can have a propositional function in any number of variables. For example, we could have:

$$P(x) \equiv 7x - 24 = \sin 3x$$
$$Q(x,y) \equiv 5x - 11 > 8y$$
$$R(x,y,z) \equiv \frac{3x - 4y}{x^2 + 15z} = 17$$

Using Logical Connectives: Even though propositional functions are not propositions, we will allow them to be joined together into compound expressions using the five logical connectives discussed in Section A.1. Thus, we will allow such expressions as

$$P(x) \Rightarrow Q(x)$$
$$P(x) \wedge [Q(x,y) \vee \sim R(y,z)].$$

Of course, these are not propositions, since they do not have a truth-value as long as the variables are unknown. They *become* propositions whenever the variables are replaced by constants from their respective domains. They can also be made into propositions by a process called "quantification," to which we now turn our attention.

UNIVERSAL QUANTIFICATION

Definition A.2.4 (**Universal Quantification**) Suppose a propositional function $P(x)$ is true for all values of x in its domain. That fact is itself a proposition that we denote

$$\forall x,\ P(x).$$

This is read "for all x, $P(x)$." The symbol "$\forall x$" is called the **universal quantifier**. Note again that "$\forall x$, $P(x)$" is a proposition. It is true or false, even though $P(x)$ is not by itself a proposition.

Examples A.2.5 Some universally quantified statements. (Assume the domain is the set of all real numbers.)

(a) $\forall x,\ \sin^2 x + \cos^2 x = 1$ (true);

(b) $\forall x,\ |\sin x| \leq 1$ (true);

(c) $\forall x,\ 6x + 11 = 7$ (false). □

Examples A.2.6 Express each of the following English sentences in symbolic form:

(a) Everyone must breathe and eat, or die.

(b) Every analysis student is intelligent and good-looking.

(c) The square of every nonzero real number is positive.

Solution:

(a) Let the domain of x be the set of all people, $B(x) \equiv x$ breathes, $E(x) \equiv x$ eats, and $D(x) \equiv x$ dies. Then the given statement is

$$\forall x, \{[B(x) \wedge E(x)] \vee D(x)\}.$$

(b) Let the domain of x be the set of all people, $A(x) \equiv x$ is an analysis student, $I(x) \equiv x$ is intelligent, and $G(x) \equiv x$ is good-looking. The given statement is

$$\forall x, \{A(x) \Rightarrow [I(x) \wedge G(x)]\}.$$

(c) Let the domain of x be the set of all real numbers. The given statement is

$$\forall x, [x \neq 0 \Rightarrow x^2 > 0]. \quad \square$$

Definition A.2.7 Restricted Universal Quantification: If S is a subset of the domain of a variable x, then the statement "For all x in the set S, $P(x)$ is true," is symbolized

$$\forall x \in S, P(x).$$

Example A.2.8 If S denotes the set of all analysis students then the statement, "Every analysis student is intelligent and good-looking" could be symbolized

$$\forall x \in S, [I(x) \wedge G(x)].$$

Compare this with Example A.2.6(b) above. $\square$

EXISTENTIAL QUANTIFICATION

Definition A.2.9 (Existential Quantification) Suppose a propositional function $P(x)$ is true for some (at least one) value of x in its domain. That fact is itself a proposition, which we denote

$$\exists x \ni P(x).$$

This is read "there exists an x such that $P(x)$." The symbol "$\exists x$" is called the **existential quantifier**, and the (optional) symbol $\ni$ is present to indicate the phrase "such that." Note that "$\exists x \ni P(x)$" is a proposition; it is true or false, even though $P(x)$ is not by itself a proposition.

We can also use **restricted existential quantification** to symbolize "there exists an x in the set S such that $P(x)$ is true" as

$$\exists x \in S \ni P(x).$$

Examples A.2.10 Some existentially quantified statements:

(a) $\exists x \ni \sin x = 1$ (true);
(b) $\exists x \ni \sin x \geq 2$ (false);
(c) $\exists x \ni 6x + 11 = 7$ (true). □

Examples A.2.11 Express each of the following English sentences in symbolic form:

(a) Somebody stole my wallet.
(b) Some analysis students are intelligent and good-looking.
(c) The equation $x^2 - x - 6 = 0$ has a solution in the real number system.
(d) The equation $x^2 + x + 1 = 0$ has no real number solution.

Solution:

(a) Let the domain of x be the set of all people, and $S(x) \equiv x$ stole my wallet. Then the given statement is

$$\exists x \ni x \text{ stole my wallet, or}$$
$$\exists x \ni S(x).$$

(b) Let the domain of x be the set of all people, $A(x) \equiv x$ is an analysis student, $I(x) \equiv x$ is intelligent, and $G(x) \equiv x$ is good-looking. The given statement is

$$\exists x \ni \{A(x) \wedge [I(x) \wedge G(x)]\}.$$

Alternatively, we could let $S =$ the set of all analysis students; then the statement becomes

$$\exists x \in S \ni [I(x) \wedge G(x)].$$

(c) Let the domain of x be the set of all real numbers. The given statement is

$$\exists x \ni x^2 - x - 6 = 0.$$

The statement is true, since $x^2 - x - 6 = 0$ is true when $x = -2$ or $x = 3$.

(d) Let the domain of x be the set of all real numbers. The given statement is

$$\sim \exists x \ni x^2 + x + 1 = 0.$$

The statement is true for this domain, since the discriminant of this quadratic expression is negative. If we change the domain of x to be the set of *complex* numbers, the statement would change its truth-value to false, since the quadratic formula tells us that $x^2 + x + 1 = 0$ when $x = \dfrac{-1 \pm i\sqrt{3}}{2}$. □

QUANTIFIER NEGATION

Quantified statements occur so frequently in mathematics that it is important to be able to form their negations correctly. The procedure may seem tricky at first, but the idea is very simple, and with a little practice you will become quite good at it. The following principle is extremely important.

Principle of Quantifier Negation:

(1) $\sim (\forall x, P(x)) \equiv \exists x \ni \ \sim P(x)$.

(2) $\sim (\exists x \ni P(x)) \equiv \forall x, \ \sim P(x)$.

To form the negation of a quantified statement, we simply change the quantifier (universal to existential, or vice versa) and negate the statement that follows the quantifier. A few examples will help make this principle clear.

Examples A.2.12 Negations of quantified statements:

(a) $\sim \forall x, [P(x) \wedge Q(x)] \equiv \exists x \ni \sim [P(x) \wedge Q(x)]$ (quantifier negation)
$\equiv \exists x \ni [\sim P(x) \vee \sim Q(x)]$ (de Morgan's law)

(b) $\sim \exists x, [P(x) \Rightarrow Q(x)] \equiv \forall x, \sim [P(x) \Rightarrow Q(x)]$ (quantifier negation)
$\equiv \forall x, [P(x) \wedge \sim Q(x)]$ (negation of $\Rightarrow$, A.1.22) □

Examples A.2.13 Form the negation of each given statement, first in symbolic form, then in English where appropriate (see Examples A.2.5–A.2.11):

(a) $\forall x, \sin^2 x + \cos^2 x = 1$.
(b) $\forall x, |\sin x| \leq 1$.
(c) Every analysis student is intelligent and good-looking.
(d) The square of every nonzero real number is positive.
(e) $\exists x \ni \sin x = 1$.

(f) Somebody stole my wallet.

(g) The equation $x^2 - x - 6 = 0$ has a solution in the real number system.

Solution: In each case we apply the principle of quantifier negation.

(a) The negation is $\exists\, x \ni \sim [\sin^2 x + \cos^2 x = 1]$, which is equivalent to $\exists\, x \ni \sin^2 x + \cos^2 x \neq 1$.

(b) The negation is $\exists\, x \ni \sim |\sin x| \leq 1$, which is equivalent to $\exists\, x \ni |\sin x| > 1$.

(c) From Example A.2.6 (b), we want $\sim [\forall x, \{A(x) \Rightarrow [I(x) \wedge G(x)]\}]$. This translates directly into English as, "It is not true that every mathematics student is intelligent and good-looking." Using the principle of quantifier negation and other rules of logic, we can continue:

$$\begin{array}{ll} \exists\, x \ni \sim \{A(x) \Rightarrow [I(x) \wedge G(x)]\} & \text{(quantifier negation)} \\ \exists\, x \ni \{A(x) \wedge \sim [I(x) \wedge G(x)]\} & \text{(negation of implication—A.1.22)} \\ \exists\, x \ni \{A(x) \wedge [\sim I(x) \vee \sim G(x)]\} & \text{(de Morgan's law—A.1.18)} \end{array}$$

In English, the resulting negation is, "There is an analysis student who is either not intelligent or not good-looking." (This is false, of course!)

(d) From Example A.2.6 (c), we want $\sim \big[\forall x, [x \neq 0 \Rightarrow x^2 > 0]\big]$, which is equivalent to

$$\begin{array}{ll} \exists\, x \ni \sim [x \neq 0 \Rightarrow x^2 > 0] & \text{(quantifier negation)} \\ \exists\, x \ni [x \neq 0 \wedge \sim (x^2 > 0)] & \text{(negation of implication—A.1.22)} \\ \exists\, x \ni [x \neq 0 \ \wedge \ x^2 \leq 0] & \text{(property of real numbers)} \end{array}$$

In English, "There is a nonzero real number whose square is not positive."

(e) The negation, $\sim (\exists\, x \ni \sin x = 1)$, is equivalent to $\forall x, \sin x \neq 1$.

(f) From Example A.2.11 (a), we want $\sim [\exists\, x \ni S(x)]$, which is equivalent to $\forall x, \sim S(x)$. In smooth English we say simply, "Nobody stole my wallet."

(g) From Example A.2.11 (c), we want

$$\begin{aligned} \sim \big[\exists\, x \ni x^2 - x - 6 = 0\big] &\equiv \forall x, \sim \big(x^2 - x - 6 = 0\big) \ \text{(quantifier negation)} \\ &\equiv \forall x,\ x^2 - x - 6 \neq 0. \end{aligned}$$

In smooth English, it is best to say simply, "There is no real number x that satisfies the equation $x^2 - x - 6 = 0$." □

CATEGORICAL PROPOSITIONS

Before the creation of modern symbolic logic around the beginning of the last century, classical logic had identified four "categorical propositions." Not

having modern logical symbolism available, logicians expressed these propositions in words:

All A's are B's	–**Universal positive**	**(UP)**
Some A's are B's	–**Existential positive**	**(EP)**
No A's are B's	–**Universal negative**	**(UN)**
Some A's are not B's	–**Existential negative**	**(EN)**

Statements of this type are very common in mathematics as well as in ordinary conversation. Our knowledge of the principles of quantification and negation will enable us to understand these statements better and to avoid much confusion.

Examples A.2.14 Some categorical propositions:

(a) All math courses are interesting. **(UP)**
(b) Some cars are too expensive. **(EP)**
(c) No wars are justifiable. **(UN)**
(d) Some functions are not differentiable. **(EN)** □

The four categorical propositions are special cases of quantified statements. Let $A(x) \equiv$ "x is an A," and $B(x) \equiv$ "x is a B." Then it is easy to translate the categorical propositions into the symbolism of symbolic logic, as shown in Table A.11:

Table A.11

Categorical Proposition	Name	Translation
All A's are B's	UP	$\forall x,\ [A(x) \Rightarrow B(x)]$
Some A's are B's	EP	$\exists\, x \ni [A(x) \wedge B(x)]$
No A's are B's	UN	$\forall x,\ [A(x) \Rightarrow \sim B(x)]$
Some A's are not B's	EN	$\exists\, x \ni [A(x) \wedge \sim B(x)]$

Note: The translation of **UN** used in Table A.11 needs a word of explanation. An exact literal translation of **UN** would be "$\sim \exists\, x \ni [A(x) \wedge B(x)]$" However, by the principle of quantifier negation, this is equivalent to the form used in Table A.11. (See Theorem A.2.15.)

The importance of Table A.11 is to show that the classical categorical propositions can be handled within the context of quantified propositional functions. In this context, it is easy to derive the negations of the categorical propositions.

Theorem A.2.15 *Negations of Categorical Propositions:*

(a) $\sim \mathbf{UP} \equiv \mathbf{EN}$
(b) $\sim \mathbf{EP} \equiv \mathbf{UN}$
(c) $\sim \mathbf{UN} \equiv \mathbf{EP}$
(d) $\sim \mathbf{EN} \equiv \mathbf{UP}$

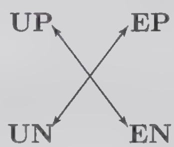
Proof: (a)
$$\begin{aligned} \sim \mathbf{UP} &\equiv \sim \forall x, [A(x) \Rightarrow B(x)] \\ &\equiv \exists x \ni \ \sim [A(x) \Rightarrow B(x)] \\ &\equiv \exists x \ni [A(x) \wedge \sim B(x)] \ \equiv \ \mathbf{EN}. \end{aligned}$$

(b)
$$\begin{aligned} \sim \mathbf{EP} &\equiv \sim (\exists x \ni [A(x) \wedge \ B(x)]) \\ &\equiv \forall x, \ \sim [A(x) \wedge \ B(x)] \\ &\equiv \forall x, [\sim A(x) \vee \sim B(x)] \\ & \ \ \forall x, [A(x) \Rightarrow \ \sim B(x)] \ \equiv \ \mathbf{UN}. \end{aligned}$$

(c) Using (b), we have $\sim \mathbf{UN} \equiv \sim (\sim \mathbf{EP}) \ \equiv \ \mathbf{EP}$.

(c) Using (a), we have $\sim \mathbf{EN} \equiv \sim (\sim \mathbf{UP}) \ \equiv \ \mathbf{UP}$. ∎

Negation of the categorical propositions is easy to remember using the following rectangular diagram, where the diagonals represent negations:



Examples A.2.16 Negations of categorical propositions:

(a) ~(All eagles are graceful) ≡ Some eagles are not graceful.

(b) ~(Someone stole my wallet) ≡ No one stole my wallet.

(c) ~(No one can jump that high) ≡ Someone can jump that high.

(d) ~(Someone will not pass the exam.) ≡ Everyone will pass the exam.

□

MULTIPLY QUANTIFIED STATEMENTS

It is quite common in mathematics for statements to contain more than one quantified variable. When translating such statements into logical symbolism, great care must be taken to capture the precise meaning intended by

the original statement. Incorrect quantification can completely destroy the intended meaning. Of special importance is the *order* in which we write multiple quantifiers.

Example A.2.17 Consider the propositional function $M(x, y) \equiv$ "x is the mother of y." There are six different statements that can be made by applying quantifiers to this statement:

(a) $\forall x, \forall y, M(x,y) \equiv$ Everyone is mother to everyone.

(b) $\forall x, \exists y \ni M(x,y) \equiv$ Everyone is a mother.

(c) $\exists x \ni \forall y, M(x,y) \equiv$ Someone is "mother-of-all."

(d) $\forall y, \exists x \ni M(x,y) \equiv$ Everyone has a mother.

(e) $\exists y \ni \forall x, M(x,y) \equiv$ Someone has everyone as mother.

(f) $\exists x, \exists y \ni M(x,y) \equiv$ Someone is a mother.

Note that a slight change in the quantifiers used, or in the order in which they are written, can have a drastic effect on the meaning of the statement. □

EXERCISE SET A.2

PART A: In Exercises 1–28, define appropriate propositional functions, specify the domain(s) of the variable(s), and translate the given statement into symbolic form.

Example: All unicorns have four legs and one horn.

Solution: Let the domain of x be the set of all animals. Let $U(x) \equiv$ "x is a unicorn," $L(x) \equiv$ "x has four legs," and $H(x) \equiv$ "x has one horn." Then the given proposition is symbolized:

$$\forall x, [U(x) \Rightarrow \{L(x) \land H(x)\}].$$

1. Lawyers are all wealthy.
2. Anyone who wants to succeed must work hard.
3. No one who wants an A in this course can afford to miss an assignment. (Two equivalent ways—see note following Table A.11.)
4. Someone in the class will win the raffle.
5. Someone in this room is guilty, but no one in this room will be charged.

6. No one in this class will fail if he does the assignments regularly.
7. Someone waiting in line for the show will not get in.
8. Some integers are even and some are odd.
9. If anyone in this room is guilty, he or she should confess now.
10. If anyone in this room is guilty, we are all doomed.
11. The equation $x^2 + 3x - 1 = 0$ has no real number solutions.
12. The equation $x^2 + 3x + 2 = 0$ has two real number solutions.
13. I can't agree with all of your ideas.
14. I like some flavors of ice cream, but not all flavors.
15. Everyone is eligible to try, but not all will succeed.
16. No one can lose weight if he or she doesn't try.
17. No matter who is elected, someone will be unhappy.
18. All months have at least 28 days, but none have more than 31 days.

    NOTE: Exercises 19–28 require more than one variable. Be careful—they can be tricky!

19. All men are brothers.
20. Some people have trouble getting along with each other.
21. No two people look exactly alike.
22. If someone loves another person, he or she wants to be with that person.
23. Some triangles are similar but not congruent.
24. Two people doing the same job should be paid at the same rate.
25. One number in this set is smaller than all the rest.
26. Every set of positive integers contains a smallest element.
27. There is no largest real number. [Use two variables.]
28. You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time. (Who said this? Notice that the first two claims are ambiguous; each can be interpreted in two different ways.)

PART B: Write the negation of each proposition in Part A, first in symbols, and then in words. Give a complete analysis in each case.

PART C:

1. Let I denote the set of integers, E denote the set of even integers, and O denote the set of odd integers. Translate each of the following into words, and tell whether it is true or false. Then state the negation, first in symbols and then in words.

   (a) $\forall x \in I, \exists y \in O \ni x + y \in E$.

   (b) $\forall x \in O, \exists y \in O \ni x + y \in E$.

   (c) $\exists y \in I \ni \forall x \in I \ni x + y = x$.

   (d) $\forall x \in I, \exists y \in I \ni x + y = 0$.

2. For each of the following, tell whether the statement is true or false, and then state the negation.

   (a) For each positive real number x, there is a positive real number y such that y is less than x.

   (b) For each positive integer x, there is a positive integer y such that y is less than x.

   (c) There is a real number x such that for each real number y, $x + y = 0$.

A.3 Strategies of Proving Theorems

There is, of course, no such thing as a general method that works in proving all theorems. Proving theorems is a creative art that defies all attempts to reduce it to a routine procedure. Nevertheless, it is possible to make some helpful suggestions and describe some useful strategies. That is about all we attempt in this section.

First, a **theorem** usually has hypotheses and always has a conclusion. The theorem claims that the **hypotheses** provide sufficient evidence to guarantee the truth of the **conclusion**. Schematically, a theorem takes the following form:

$$\left\{\begin{array}{c} \text{Hypothesis \#1} \\ \text{Hypothesis \#2} \\ \vdots \\ \underline{\text{Hypothesis \#}n} \\ \therefore \text{Conclusion} \end{array}\right. \qquad \text{or, symbolically,} \qquad \left\{\begin{array}{c} H_1 \\ H_2 \\ \vdots \\ \underline{H_n} \\ \therefore C \end{array}\right.$$

We shall find it convenient to refer to this schematic in horizontal form:

Table A.12

Theorem: $H_1, H_2, \cdots, H_n, \therefore C$

To **prove** a theorem is to demonstrate that its hypotheses do, in fact, provide sufficient evidence to *guarantee* the truth of the conclusion. In addition to the hypotheses, a proof may use any agreed-upon rules of logic, definitions of the terms involved, theorems previously proved, and perhaps even results from other agreed-upon areas of mathematics. In one sense, proving a theorem is like presenting an argument in a court of law. In the case of proving a theorem, however, we must show that our evidence is totally convincing—not just beyond reasonable doubt, but beyond *all* doubt. There is something *absolute* about a proof in mathematics. In a mathematical proof we must show that if all our assumptions are true, then our conclusion *must* be true (beyond any doubt). No wonder we are studying a little logic as part of this course in real analysis! Proving a theorem is an awesome responsibility.

Second, a theorem does not exist in isolation, but as part of a larger, deductive system. It is derived from results established previously within the system, uses terms defined within the system, and in turn is used in deriving later results of the system. Thus, a theorem is like a node in a network of results in which information flows in one direction only. In proving a theorem, only information preceding it in the flow may be used. Beginning students often have difficulty understanding that information to come later in the flow is *never* allowed as a step, or a justification for a step, in the proof of a theorem.

We now present some proof strategies. We shall not give examples or exercises in this section. Indeed, this entire book serves as a set of examples of these strategies. We set them forth here as reference for your use as you need them.

PROOF STRATEGIES: PS-1–PS-7

(PS-1) DIRECT PROOF of Theorem: $H_1, H_2, \cdots, H_n, \therefore C$.

"Direct proof" does not indicate a specific strategy as much as it indicates that we are not using any of the specialized strategies to be described below. We first gather together all that we know about the consequences of all the hypotheses being true. We think about conditions that would make the conclusion true. We try to link the former to the latter. We describe this below as a formal procedure. First, we need a theorem.

Theorem A.3.1 (***Transitivity of Implication***) *If $P \Rightarrow Q$ and $Q \Rightarrow R$, then $P \Rightarrow R$.*

Proof. Use a truth table to show that $[P \Rightarrow Q \text{ and } Q \Rightarrow R] \Rightarrow [P \Rightarrow R]$ is a tautology. ■

Here's how we use Theorem A.3.1 to give a **direct proof** of the theorem: $H_1, H_2, \cdots, H_n, \therefore C$:

Let H denote the conjunction of all the hypotheses; that is, $H \equiv (H_1 \text{ and } H_2 \text{ and } \cdots \text{ and } H_n)$. In a direct proof, we try to string together a sequence of implications:

$$\begin{aligned} H &\Rightarrow P_1 \\ P_1 &\Rightarrow P_2 \\ P_2 &\Rightarrow P_3 \\ &\vdots \\ P_{n-1} &\Rightarrow P_n \\ P_n &\Rightarrow C. \end{aligned}$$

Then, by the transitivity of implication, $H \Rightarrow C$, and we have proved the theorem.

(PS-2) TO PROVE AN IMPLICATION (CONDITIONAL PROOF):

To prove a theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore P \Rightarrow Q,$$

we prove the equivalent theorem:

$$H_1, H_2, \cdots, H_n, P, \ \therefore Q.$$

That is, to prove $P \Rightarrow Q$, add P as a hypothesis and prove Q.

(PS-3) TO PROVE AN IMPLICATION BY ITS CONTRAPOSITIVE:

To prove a theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore P \Rightarrow Q,$$

we prove the equivalent theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore \sim Q \Rightarrow \sim P.$$

That is, to prove $P \Rightarrow Q$, we prove the contrapositive, $\sim Q \Rightarrow \sim P$.

(PS-3′) TO PROVE A THEOREM BY ITS CONTRAPOSITIVE:

To prove a theorem:

$$H, \therefore C,$$

we prove the equivalent theorem:

$$\sim C, \ \therefore \sim H.$$

(PS-4) TO PROVE A DISJUNCTION:

To prove a theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore P \vee Q,$$

we prove the equivalent theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore \sim P \Rightarrow Q.$$

That is, to prove P or Q, we prove the implication $\sim P \Rightarrow Q$. Alternatively, we could prove the equivalent theorem:

$$H_1, H_2, \cdots, H_n, \sim P, \ \therefore Q.$$

(PS-5) INDIRECT PROOF (PROOF BY CONTRADICTION):

By a **contradiction**, we mean any proposition of the form "$R \wedge \sim R$," which always has truth-value F. By the truth table defining "$\Rightarrow$," any implication of the form $H \Rightarrow [R \wedge \sim R]$ is true only when H is false. Thus, if we are able to prove that $\sim P \Rightarrow [R \wedge \sim R]$, we know that P is true. That is the basis of "proof by contradiction."

To prove a theorem:

$$H_1, H_2, \cdots, H_n, \ \therefore P,$$

we prove the equivalent theorem:

$$H_1, H_2, \cdots, H_n, \quad \therefore \sim P \Rightarrow [R \wedge \sim R],$$

or

$$H_1, H_2, \cdots, H_n, \sim P, \ \therefore [R \wedge \sim R].$$

That is, to prove P we add $\sim P$ as a hypothesis, and show that this leads to a contradiction.

(PS-6) PROOF BY CASES:

We begin with the following observation:

Theorem A.3.2 *$[(P \vee Q) \wedge \{(P \Rightarrow R) \wedge (Q \Rightarrow R)\}] \Rightarrow R$ is a tautology.*

Logicians call this tautology "constructive dilemma." It can be proved by a truth table. In words, it says

$$\left[\begin{array}{l}\text{If we know that at least one of two possibilities} \\ \text{(``cases'') must occur, and if each of these cases} \\ \text{implies } R\text{, then } R \text{ must be true.}\end{array}\right]$$

PROOF BY CASES:

To prove the theorem

$$H_1, H_2, \cdots, H_n, \ \therefore \mathrm{C},$$

we can first prove

$$(H_1 \wedge H_2 \wedge \cdots \wedge H_n) \Rightarrow (P_1 \vee P_2)$$

and then prove both

$$\begin{array}{l}\text{Case 1: } [(H_1 \ \wedge \ H_2 \ \wedge \ \cdots \ \wedge \ H_n) \ \wedge \ P_1] \Rightarrow C, \text{ and} \\ \text{Case 2: } [(H_1 \ \wedge \ H_2 \ \wedge \cdots \ \wedge \ H_n) \ \wedge \ P_2] \Rightarrow C.\end{array}$$

Of course, proof by cases can be extended to more than two cases. That is, we can apply this procedure when we have $(P_1 \vee P_2 \vee \cdots \vee P_n)$ rather than just $(P_1 \vee P_2)$. Then our proof would break down into Case 1, Case 2, $\cdots$, Case n.

(PS-6′) PROOF BY CASES—Using Constructive Dilemma:

To prove the theorem

$$H_1, H_2, \cdots, H_n, \therefore C,$$

we need only prove

$$\begin{array}{l}\text{Case 1: } [(H_1 \ \wedge \ H_2 \ \wedge \ \cdots \ \wedge \ H_n) \ \wedge \ P] \Rightarrow C, \text{ and} \\ \text{Case 2: } [(H_1 \ \wedge \ H_2 \ \wedge \ \cdots \ \wedge \ H_n) \ \wedge \ \sim P] \Rightarrow C,\end{array}$$

where P is any proposition of your choosing.

(PS-7) PROVING THAT SEVERAL STATEMENTS ARE EQUIVALENT:

To prove the

$$\text{Theorem: } P_1, P_2, \cdots, P_n, \text{ are all equivalent,}$$

we prove the equivalent

$$\text{Theorem: } P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_n \Rightarrow P_1.$$

Proving the latter establishes a "circular" relationship, showing that any one of these propositions will imply all of the others.

Notice that neither of these theorems claims that the statements are all *true*, only that they are equivalent. That means that, if any one of them is true, all the others are true as well. If any one of them is false, all of them are false.

A.4 Properties of Equality

Students sometimes ask how we justify the familiar manipulation of equality (the "equals sign") in proofs. In mathematics, as elsewhere, the term "equals" is understood to mean "is the same as." Thus, for example, when we write

$$\{8, 5, 4\} = \{5, 8, 5, 4\}$$[4]

we are indicating that the expressions on either side of the equals sign represent the same set; the two sets are indistinguishable as sets. Of course, there is *something* distinguishable about the two sides of the equation, or else our eyes could not perceive that the equation has two sides, and we would have no reason to ask whether they are the same. Perhaps a more mature understanding of "=" is that it means "is, for our present purposes, the same as."

RULES OF EQUALITY

The familiar rules of equality used most frequently in proofs are

(1) The **reflexive property:** $\forall x,\ x = x$.

(2) The **symmetric property:** $\forall x, y,\ x = y \Rightarrow y = x$.

(3) The **transitive property:** $\forall x, y, z,\ (x = y \text{ and } y = z) \Rightarrow x = z$.

(4) The **replacement property:** In any context in which $x = y$, either x or y can be replaced by the other whenever it occurs.

The replacement property has subtle applications. For example, the familiar assertions

(1) if $x = y$, then $\forall z,\ x + z = y + z$, and

(2) if $a = b$ and $c = d$, then $a + c = b + d$

are justified by the replacement property of equality. These principles are more commonly stated, "when equals are added to equals, the results are equal." They are not axioms about addition, but are properties of equality.

4. See Appendix B.1 for a discussion of "set" notation.

Appendix B

Sets and Functions

B.1 Sets and the Algebra of Sets

The concept of *set* is the most basic of all mathematical concepts. Indeed, it even precedes counting. In order to be able to count "how many," one must be able to conceive of the objects being counted as somehow separated from all objects *not* being counted. Thus, it is natural that the algebra of sets is logically placed before the algebra of numbers. This section will review the basic concepts, operations, and relations of sets.

We first point out that the word "set" is an undefined term in our context. That is, we assume that the reader has an understanding of the word without further definition. The notion of "set" conforms to certain axioms, but to describe these axioms is beyond the scope of this book. One of them, however, states that a set is completely determined by its *elements* (or its *members*); that is, by what *belongs to* it.

NOTATION: We usually denote sets by capital letters and their elements by lowercase letters.

The symbol "$\in$" is used to denote "is an element of." Thus,

$$x \in A$$

is the statement "x is an element of A," or "x belongs to A."

To describe a set it is sufficient to list its members, in any order. When we do so, it is customary to enclose its member in "braces." Thus, for example, the set of integers from 1 through 10 can be denoted

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \text{ or } A = \{5, 2, 8, 10, 4, 3, 9, 7, 1, 6\}.$$

If a set has too many members to list, we can use an ellipsis (three dots), as in $\{1, 2, 3, ..., 100\}$, or resort to *describing* its elements within braces, using the following technique.

Definition B.1.1 If $p(x)$ is any proposition about a variable x, then $\{x : p(x)\}$ denotes the set of all values of x for which $p(x)$ is true. It is sometimes called the "truth set" of $p(x)$.

Thus, the set of even positive integers less than 100 could be written $A = \{2, 4, 6, ..., 98\} = \{x : x \text{ is an even positive integer less than } 100\}$.

Definition B.1.2 In any particular context in which variables are used, there is a **universal set**, $\mathcal{U}$, from which the variables draw their values. This $\mathcal{U}$ is often understood without explicit mention. For example, when you see an equation like $3x^2 + 7x - 10 = 0$, you assume that $\mathcal{U}$ is a set of numbers, usually either the set of all real numbers or the set of all complex numbers. To **solve** an equation is to find all values in $\mathcal{U}$ that, when substituted for the unknown(s), make the equation true.

The **empty set**, $\varnothing$, is the set that has no members. Thus, for example, $\varnothing = \{x : x \neq x\}$.

Definition B.1.3 Some Special Sets. Although the official definitions of natural numbers, integers, rational numbers, and so on are not given until Chapter 1, we shall use the following symbols for the sets of these familiar types of numbers:

$$\mathbb{N} = \{\text{all natural numbers}\} = \{1, 2, 3, 4, \cdots\};$$

$$\mathbb{Z} = \{\text{all integers}\} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\};$$

$$\mathbb{Q} = \{\text{all rational numbers}\} = \left\{\tfrac{m}{n} : m, n \in \mathbb{Z}, \text{ and } n \neq 0\right\};$$

$$\mathbb{R} = \{\text{all real numbers}\} = \{\text{all numbers located on a "number line"}\}.$$

We also use the interval notation familiar from calculus: $\forall a, b \in \mathbb{R}$,

$$\begin{array}{ll}
(a, b) = \{x \in \mathbb{R} : a < x < b\}; & (-\infty, b) = \{x \in \mathbb{R} : x < b\}; \\
[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}; & (-\infty, b] = \{x \in \mathbb{R} : x \leq b\}; \\
(a, b] = \{x \in \mathbb{R} : a < x \leq b\}; & (a, +\infty) = \{x \in \mathbb{R} : x > a\}; \\
[a, b) = \{x \in \mathbb{R} : a \leq x < b\}; & [a, +\infty) = \{x \in \mathbb{R} : x \geq a\};
\end{array}$$
$$(-\infty, \infty) = \mathbb{R}.$$

Definition B.1.4 Let A and B be sets. Then

(a) The **union** of A and B is the set $\mathbf{A} \cup \mathbf{B} = \{x : x \in A \text{ or } x \in B\}$.

(b) The **intersection** of A and B is the set $\mathbf{A} \cap \mathbf{B} = \{x : x \in A \text{ and } x \in B\}$.

(c) The **complement** of A is the set $\mathbf{A}^c = \{x \in \mathcal{U} : x \notin A\}$.

(d) The **relative complement** of A in B is the set $\mathbf{B} - \mathbf{A} = \{x \in B : x \notin A\}$.

These sets are conveniently illustrated in the following "Venn diagrams:"

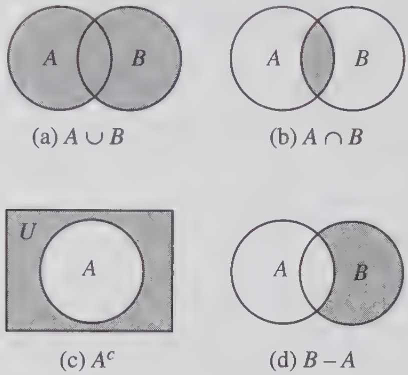


Figure B.1

Examples B.1.5 Let $\mathcal{U} = \{\text{real numbers}\}$. Then

(a) $[0,3] \cup [1,5] = [0,5]$

(b) $[0,3] \cap [1,5] = [1,3]$

(c) $[0,3]^c = (-\infty, 0) \cup (3, +\infty)$

(d) $[0,3] - [1,5] = [0,1)$

(e) $[1,5] - [0,3] = (3,5]$ $\square$

Definition B.1.6 We say that $\mathbf{A} \subseteq \mathbf{B}$ (A is a **subset** of B) iff every element of A is also an element of B.

For example, $\{1,2,3\} \subseteq \mathbb{N}$ and $\{x : x^2 - 3x + 2 = 0\} \subseteq \mathbb{N}$, but $[1,3] \nsubseteq \mathbb{N}$.

Theorem B.1.7 (*Algebra of Sets*) *For any sets $A, B, C \in \mathcal{U}$,*

(a) $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$.

(b) $\varnothing \subseteq A$, $A \subseteq A$, *and* $A \subseteq \mathcal{U}$.

(c) $A \cap B \subseteq A$ *and* $A \cap B \subseteq B$.

(d) $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$.

(e) $A \cup B = A$ *iff* $B \subseteq A$.

(f) $A \cap B = A$ *iff* $A \subseteq B$.

(g) $(A \cup B)^c = A^c \cap B^c$. *(de Morgan's law)*

(h) $(A \cap B)^c = A^c \cup B^c$. *(de Morgan's law)*

(i) $A^c = \mathcal{U} - A$.

(j) $\mathcal{U}^c = \varnothing$; *and* $\varnothing^c = \mathcal{U}$.

(k) $B - A = B \cap A^c$.

(l) $A^{cc} = A$.

(m) $A \cup (B \cup C) = (A \cup B) \cup C$. *(associative law for* $\cup$*)*

(n) $A \cap (B \cap C) = (A \cap B) \cap C$. *(associative law for* $\cap$*)*

(o) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(distributive law)*

(p) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(distributive law)*

Proof of (g):

Part 1: Suppose $x \in (A \cup B)^c$. Then $x \in \mathcal{U}$ but $x \notin A \cup B$. Then it is not true that $x \in A$ or $x \in B$. By de Morgan's rule in logic, this means $x \notin A$ and $x \notin B$. That is, $x \in A^c$ and $x \in B^c$; i.e., $x \in A^c \cap B^c$. Therefore, by B.1.6, $(A \cup B)^c \subseteq A^c \cap B^c$.

Part 2: Suppose $x \in A^c \cap B^c$. Then $x \in A^c$ and $x \in B^c$; i.e., $x \notin A$ and $x \notin B$. By de Morgan's rule in logic, this means it is not true that $x \in A$ or $x \in B$. Then $x \in \mathcal{U}$ but $x \notin A \cup B$; i.e., $x \in (A \cup B)^c$. Therefore, by B.1.6, $A^c \cap B^c \subseteq (A \cup B)^c$.

By Parts 1 and 2, together with Part (a), $(A \cup B)^c = A^c \cap B^c$.

Proof of (o):

Part 1: Suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Then $x \in A$ and ($x \in B$ or $x \in C$). By the distributive law in logic (Theorem A.1.23, (a)) this means ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$). That is, $x \in A \cap B$ or $x \in A \cap C$. Thus, $x \in (A \cap B) \cup (A \cap C)$.

Part 2: Exercise 5. ■

While the above theorem summarizes the algebra of sets when only several sets are involved, we need algebraic rules to cover situations in which many, even infinitely many, sets are involved. The following definition and theorem cover these situations.

Definition B.1.8 (Operations on Collections of Sets)
Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets, "indexed" by some set Λ of "indices" λ. Then

(a) $\cap\mathcal{C} = \bigcap_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for every } \lambda \in \Lambda\}$.

(b) $\cup\mathcal{C} = \bigcup_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for at least one } \lambda \in \Lambda\}$. □

Examples B.1.9 (a) $\bigcap\{(-\frac{1}{n}, 1+\frac{1}{n}) : n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} (-\frac{1}{n}, 1+\frac{1}{n}) = [0, 1]$.

(b) $\bigcup\{(-\frac{1}{n}, 1+\frac{1}{n}) : n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} (-\frac{1}{n}, 1+\frac{1}{n}) = (-1, 2)$.

Figure B.2

Theorem B.1.10 (*Algebra of Collections of Sets*) *Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets and let B be any set. Then*

(a) $\left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} A_\lambda^c$. (*de Morgan's law*)

(b) $\left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c = \bigcap_{\lambda\in\Lambda} A_\lambda^c$. (*de Morgan's law*)

(d) $B \cup \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (B \cup A_\lambda)$. (*distributive law*)

(d) $B \cap \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (B \cap A_\lambda)$. (*distributive law*)

(e) $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$. (*de Morgan's law*)

(f) $B - \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) = \bigcap_{\lambda\in\Lambda} (B - A_\lambda)$. (*de Morgan's law*)

Proof of (e): Let $\mathcal{C} = \{A_\lambda : \lambda \in \Lambda\}$ be a collection of sets and let B be any set.

Part 1: Suppose $x \in B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Then $x \in B$ but $x \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. That is, $x \in B$ but $\sim \forall\lambda \in \Lambda,\ x \in A_\lambda$. By "quantifier negation,"[5] this means $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \notin A_\lambda$. That is, $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \in A_\lambda^c$. Equivalently, $\exists\,\lambda \in \Lambda \ni x \in B \cap A_\lambda^c$. Equivalently, $\exists\,\lambda \in \Lambda \ni x \in B - A_\lambda$. But that means $x \in \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$. Therefore, $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) \subseteq \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$.

Part 2: Suppose $x \in \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$. Then $\exists\,\lambda \in \Lambda \ni x \in B - A_\lambda$. That is, $\exists\,\lambda \in \Lambda \ni x \in B \cap A_\lambda^c$. Equivalently, $x \in B$ but $\exists\,\lambda \in \Lambda \ni x \in A_\lambda^c$. By quantifier negation, this means $x \in B$ but $\sim \forall\lambda \in \Lambda,\ x \in A_\lambda$. That is, $x \in B$ but $x \notin \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$; i.e., $x \in B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right)$. Therefore, $\bigcup_{\lambda\in\Lambda} (B - A_\lambda) \subseteq \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$.

By Parts 1, 2, and Theorem B.1.7 (a), $B - \left(\bigcap_{\lambda\in\Lambda} A_\lambda\right) = \bigcup_{\lambda\in\Lambda} (B - A_\lambda)$. ■

EXERCISE SET B.1

1. In each of the following, a universal set $\mathcal{U}$ and sets A, B, and C are given. Find $A \cap B$, $A \cup B$, A^c, B^c, $A - B$, $B - A$, $A \cup (B \cap C)$, and $A \cap (B \cup C)$.
   (a) $\mathcal{U} = \{1, 2, 3, \cdots, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7\}$, and $C = \{3, 4, 5\}$.
   (b) $\mathcal{U} = \{1, 2, 3, \cdots, 10\}$, $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{2, 4, 6, 8, 10\}$.
   (c) $\mathcal{U} = \{\text{all real numbers}\}$, $A = (0, 4)$, $B = [3, 6]$, $C = (2, 5)$.
   (d) $\mathcal{U} = \{\text{all real numbers}\}$, $A = (-\infty, 2)$, $B = [1, +\infty)$, $C = (-1, 1)$.

2. Prove Theorem B.1.7 (e).

3. Prove Theorem B.1.7 (h).

4. Prove Theorem B.1.7 (k).

5. Finish the proof of Theorem B.1.7 (o) by proving "Part 2."

6. Prove Theorem B.1.7 (p).

7. In each of the following, a collection of sets $\{A_\lambda : \lambda \in \Lambda\}$ is given. Assume $\mathcal{U} = \{\text{all real numbers}\}$. Find $\bigcap_{\lambda\in\Lambda} A_\lambda$, $\bigcup_{\lambda\in\Lambda} A_\lambda$, $\bigcup_{\lambda\in\Lambda} A_\lambda^c$, and $\bigcup_{\lambda\in\Lambda} A_\lambda^c$. In each case, verify Theorem B.1.10 (a) or (b).
   (a) $\{A_\lambda : \lambda \in \Lambda\} = \{(-n, n) : n \in \mathbb{N}\}$.
   (b) $\{A_\lambda : \lambda \in \Lambda\} = \{(-\infty, n) : n \in \mathbb{N}\}$.
   (c) $\{A_\lambda : \lambda \in \Lambda\} = \left\{\left(-\frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}$.

5. See section A.2.

(d) $\{A_\lambda : \lambda \in \Lambda\} = \left\{\left[-2+\frac{1}{n}, 2-\frac{1}{n}\right] : n \in \mathbb{N}\right\}$.

(e) $\{A_\lambda : \lambda \in \Lambda\} = \{(n, n+1) : n \in \mathbb{N}\}$.

8. Prove Theorem B.1.10 (a).

9. Prove Theorem B.1.10 (b).

10. Prove Theorem B.1.10 (c).

11. Prove Theorem B.1.10 (d).

12. Prove Theorem B.1.10 (f).

B.2 Functions

BASIC CONCEPTS OF FUNCTIONS

Definition B.2.1 If A and B are sets, a **function** f from A to B is any rule of correspondence that associates to each element $a \in A$ a unique element $f(a) \in B$. The set A is called the **domain** of f, and the set B is called the **codomain** of f. The set $\mathcal{R}(f) = \{f(a) : a \in A\}$ is called the **range** of f. We often denote the domain of f by $\mathcal{D}(f)$. The range of a function is a subset of its codomain.

The notational phrase

$$f : A \to B$$

is often used as a sentence saying that "f is a function from set A to set B." It is also used as a noun, referring to "the function f from A to B." Context will determine which of the two uses is intended.

A function $f : A \to B$ may be viewed intuitively as an input/output relation. To each input $a \in A$ there corresponds a unique output $f(a) \in B$. The set of all inputs is A, or $\mathcal{D}(f)$, and the set of all outputs is $\mathcal{R}(f)$.

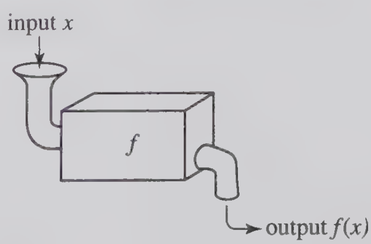


Figure B.3

Definition B.2.2 Two functions $f : A \to B$ and $g : A' \to B'$ are said to be **equal** if $A = A'$, $B = B'$, and $\forall x \in A$, $f(x) = g(x)$.

Definition B.2.3 A function $f : A \to B$ is **one-to-one** (or **1-1**) if $\forall a, a' \in A$, $f(a) = f(a') \Rightarrow a = a'$. Equivalently, $a \neq a' \Rightarrow f(a) \neq f(a')$. That is, a function is 1-1 iff [6] different inputs always result in different outputs.

Definition B.2.4 A function $f : A \to B$ is **onto** B if $\mathcal{R}(f) = B$. That is, $f : A \to B$ is onto B iff every element of B is an output of f.

Example B.2.5 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is
(a) not 1-1, since $f(1) = 1$ and $f(-1) = 1$;
(b) not onto $\mathbb{R}$, since $\mathcal{R}(f) = [0, +\infty) \neq \mathbb{R}$. □

Example B.2.6 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is
(a) 1-1, since

$$\begin{aligned} f(a) = f(b) &\Rightarrow a^3 = b^3 \\ &\Rightarrow a^3 - b^3 = 0 \\ &\Rightarrow (a - b)(a^2 + ab + b^2) = 0 \\ &\Rightarrow a - b = 0, \text{ since } a^2 + ab + b^2 \neq 0 \text{ (if } b \neq 0) \\ &\Rightarrow a = b. \end{aligned}$$

Note that the reason $a^2 + ab + b^2 \neq 0$ when $b \neq 0$ lies in the fact that a and b are real numbers, and the discriminant of the quadratic function $g(a) = a^2 + ab + b^2$ is $D = b^2 - 4(1)(b^2) = -3b^2 < 0$.

(b) onto $\mathbb{R}$, since $\forall x \in \mathbb{R}$, $\exists \sqrt[3]{x} \in \mathbb{R}$ and $f(\sqrt[3]{x}) = x$, so $x \in \mathcal{R}(f)$. □

Definition B.2.7 A function $f : A \to B$ that is both 1-1 and onto is said to be a **1-1 correspondence**.

Definition B.2.8 Two sets A and B are said to have the **same cardinal number** (of elements) if $\exists$ 1-1 correspondence $f : A \to B$.

IMAGES AND INVERSE IMAGES OF SETS

Definition B.2.9 Suppose $f : A \to B$ is a function and $C \subseteq A$ and $D \subseteq B$. Then

$$f(C) = \{f(x) : x \in C\};$$

$$f^{-1}(D) = \{x : f(x) \in D\}.$$

6. For the definition of "iff" see Definition A.1.9.

The set $f(C)$ is called the **image of** C under f and the set $f^{-1}(D)$ is called the **inverse image of** D under f. When we write $f^{-1}(D)$ we must be careful not to assume that f^{-1} is a function. Sometimes f^{-1} is a function, but that is a separate issue, to be discussed later.

Notice that $\mathcal{D}(f) = f^{-1}(B)$ and $\mathcal{R}(f) = f(A)$.

Example B.2.10 Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 3$. Let $C = [-1, 2]$ and $D = [0, 7]$. Then (see Figure B.4)

$$f(C) = [3, 7] \text{ and } f^{-1}(D) = [-2, 2].$$

Observe that

$$f([-1, 2]) = f([0, 2]) = f([-2, 2]) = [3, 7],$$

and that

$$f^{-1}([0, 7]) = f^{-1}([3, 7]) = f^{-1}([-11, 7]) = [-2, 2].$$

Further, $f(\{2\}) = f(\{-2, 2\} = \{7\}$, while $f^{-1}(\{7\}) = f^{-1}(\{-3, 0, 7\} = \{-2, 2\}$.
□

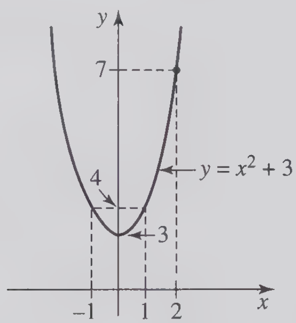


Figure B.4

Theorem B.2.11 (***Functions and Sets***) *Suppose $f : A \to B$ is a function. Then*

(a) $\forall C_1, C_2 \subseteq A$, $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

(b) $\forall C_1, C_2 \subseteq A$, $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$.
It is possible that $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$. (See B.2.12 (b) *below.)*

(c) $\forall C_1, C_2 \subseteq A$, $f(C_1) - f(C_2) \subseteq f(C_1 - C_2)$.
It is possible that $f(C_1) - f(C_2) \neq f(C_1 - C_2)$. (See B.2.12 (c) *below.)*

(d) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(e) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(f) $\forall D_1, D_2 \subseteq B$, $f^{-1}(D_1 - D_2) = f^{-1}(D_1) - f^{-1}(D_2)$.

Note that f^{-1} satisfies certain universal properties that f does not.

Proof of (a): Suppose $f : A \to B$ is a function, and $C_1, C_2 \subseteq A$.

Part 1: Let $y \in f(C_1 \cup C_2)$. By definition, this means $\exists x \in C_1 \cup C_2 \ni f(x) = y$. But then, $\exists x \in C_1 \ni f(x) = y$, or $\exists x \in C_2 \ni f(x) = y$. That is, $y \in f(C_1)$ or $y \in f(C_2)$. Thus, $y \in f(C_1) \cup f(C_2)$. Therefore, $f(C_1 \cup C_2) \subseteq f(C_1) \cup f(C_2)$.

Part 2: Let $y \in f(C_1) \cup f(C_2)$. Then $y \in f(C_1)$ or $y \in f(C_2)$. That is, $\exists x \in C_1 \ni f(x) = y$, or $\exists x \in C_2 \ni f(x) = y$. In both of these two cases, we can say $\exists x \in C_1 \cup C_2 \ni f(x) = y$. Thus, $y \in f(C_1 \cup C_2)$. Therefore, $f(C_1) \cup f(C_2) \subseteq f(C_1 \cup C_2)$.

By Parts 1 and 2 and Theorem B.1.7 (a), $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

Proof of (e): The above proof is in two parts. We can often combine the two parts into a single-part proof using a string of equivalent statements, as we do here. Suppose $f : A \to B$ is a function, and $D_1, D_2 \subseteq B$. Then

$$\begin{aligned} x \in f^{-1}(D_1 \cap D_2) &\Leftrightarrow f(x) \in D_1 \cap D_2 \\ &\Leftrightarrow f(x) \in D_1 \text{ and } f(x) \in D_2 \\ &\Leftrightarrow x \in f^{-1}(D_1) \text{ and } x \in f^{-1}(D_2) \\ &\Leftrightarrow x \in f^{-1}(D_1) \cap f^{-1}(D_2). \end{aligned}$$

Therefore, $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$. ■

Example B.2.12 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 3$. Then (see Figure B.5).

(a) $f((-1,0) \cup (0,2)) = f((-1,2) - \{0\}) = [3,7) - \{3\} = (3,7)$, and
$f(-1,0) \cup f(0,2) = (3,4) \cup (3,7) = (3,7)$.

(b) $f((-1,0) \cap (0,2)) = f(\varnothing) = \varnothing$, while
$f(-1,0) \cap f(0,2) = (3,4) \cap (3,7) = (3,4)$.
Note: in this case, $f(C_1 \cap C_2) \neq f(C_1) \cap f(C_2)$.

(c) $f((-1,1) - (0,1))) = f(-1,0] = [3,4)$, while
$f(-1,1) - f(0,1) = [3,4) - (3,4) = \{3\}$.
Note: in this case, $f(C_1 - C_2) \neq f(C_1) - f(C_2)$.

(d) $f^{-1}((-\infty,3] \cup (2,4)) = f^{-1}(-\infty,4) = (-1,1)$, and
$f^{-1}(-\infty,3] \cup f^{-1}(2,4) = \{0\} \cup (-1,1) = (-1,1)$.

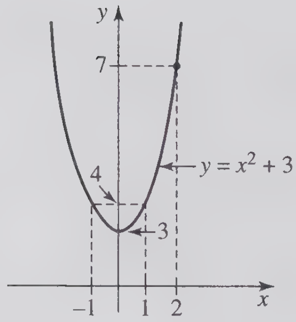


Figure B.5

(e) $f^{-1}((-\infty, 3] \cap (2,4)) = f^{-1}(2,3] = \{0\}$, and
$f^{-1}(-\infty, 3] \cap f^{-1}(2,4) = \{0\} \cap (-1,1) = \{0\}$.

(f) $f^{-1}((-\infty, 3] - (2,4)) = f^{-1}(-\infty, 2] = \varnothing$, and
$f^{-1}(-\infty, 3] - f^{-1}(2,4) = \{0\} - (-1,1) = \varnothing$. □

The following theorem generalizes Theorem B.2.11 to families of sets, even infinitely many sets.

Theorem B.2.13 (***Functions and Collections of Sets***) *Suppose $f : A \to B$ is a function. Then*

(a) *If $\{C_\lambda : \lambda \in \Lambda\}$ is a family of subsets of A, then*

(1) $$f\left(\bigcup_{\lambda \in \Lambda} C_\lambda\right) = \bigcup_{\lambda \in \Lambda} f(C_\lambda) \text{ and}$$

(2) $$f\left(\bigcap_{\lambda \in \Lambda} C_\lambda\right) \subseteq \bigcap_{\lambda \in \Lambda} f(C_\lambda).$$

(b) *If $\{D_\lambda : \lambda \in \Lambda\}$ is a family of subsets of B, then*

(1) $$f^{-1}\left(\bigcup_{\lambda \in \Lambda} D_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(D_\lambda) \text{ and}$$

(2) $$f^{-1}\left(\bigcap_{\lambda \in \Lambda} D_\lambda\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(D_\lambda).$$

Proof. Exercises 10–13. ■

GRAPHS OF FUNCTIONS $f : \mathbb{R} \longrightarrow \mathbb{R}$

Definition B.2.14 The **graph** of a function $f : A \to B$, where $A, B \subseteq \mathbb{R}$, is the set of all points (x, y) in the Cartesian (rectangular) coordinate system for which $y = f(x)$. That is,

$$\text{graph}(f) = \{(x, f(x) : x \in \mathcal{D}(f)\}.$$

Thus, a function $f : A \to B$, where $A, B \subseteq \mathbb{R}$,

- must pass the **vertical line test**:

  (a) no vertical line may intersect its graph in more than one point;

  (b) every vertical line that intersects the set A on the x-axis also intersects its graph.

- is **1-1** iff it passes the **horizontal line test**: no horizontal line may intersect its graph at more than one point.

- is **onto** B iff every horizontal line that intersects B on the y-axis also intersects its graph.

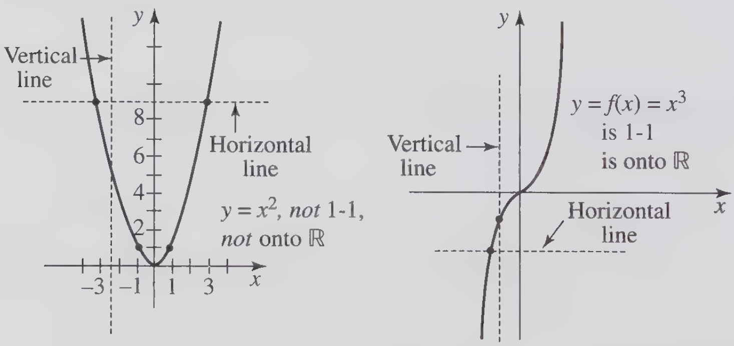


Figure B.6

EXERCISE SET B.2

1. For each of the following functions f, find (the largest possible subsets of $\mathbb{R}$ that could be) $\mathcal{D}(f)$ and $\mathcal{R}(f)$, and tell whether or not f is 1-1 and/or onto $\mathbb{R}$.

   (a) $f(x) = 2x - 3$ (b) $f(x) = |x| - 2$

   (c) $f(x) = \sqrt{3x - 4}$ (d) $f(x) = x^2 + 2x + 4$

   (e) $f(x) = \dfrac{|x|}{x}$ (f) $f(x) = \dfrac{1}{x^2 + 1}$

2. Let $f(x) = 4 - x^2$. Find

(a) $\mathcal{D}(f)$	(b) $\mathcal{R}(f)$
(c) $f[0,1]$	(d) $f^{-1}[0,1]$
(e) $f(0,2)$	(f) $f^{-1}(0,4)$
(g) $f^{-1}[2,4]$	(h) $f^{-1}[-4,0]$
(i) $f^{-1}(0,\infty)$	(j) $f^{-1}(-\infty,2]$
(k) $f^{-1}(\{0\})$	(l) $f^{-1}(\{-1\})$

3. Let $f(x) = 2^x$. Find

(a) $\mathcal{D}(f)$	(b) $\mathcal{R}(f)$
(c) $f[0,1]$	(d) $f(-\infty,2)$
(e) $f(0,\infty)$	(f) $f[-1,\frac{1}{2}]$
(g) $f^{-1}[1,2]$	(h) $f^{-1}(2,8)$
(i) $f^{-1}[-1,1)$	(j) $f^{-1}(-\infty,0)$

4. Redo Example B.2.12 using the function $f(x) = 4 - x^2$ instead of the function given there.

5. Redo Example B.2.12 using the function $f(x) = x^3 - 3x^2$ instead of the function given there.

6. Prove Theorem B.2.11 (b).

7. Prove Theorem B.2.11 (c).

8. Prove Theorem B.2.11 (d).

9. Prove Theorem B.2.11 (f).

10. Prove Theorem B.2.13 (a) (1).

11. Prove Theorem B.2.13 (a) (2).

12. Prove Theorem B.2.13 (b) (1).

13. Prove Theorem B.2.13 (b) (2).

B.3 Algebra of Real-Valued Functions

Definition B.3.1 Let $\mathcal{S}$ denote an arbitrary set. Any function $f : \mathcal{S} \to \mathbb{R}$ is called a **real-valued function** on $\mathcal{S}$. We shall consider the set of all such functions,

$$\mathcal{F}(\mathcal{S}, \mathbb{R}) = \{\text{all functions } f : \mathcal{S} \to \mathbb{R}\}.$$

We define algebraic operations on this set $\mathcal{F}(\mathcal{S}, \mathbb{R})$. In particular, we define

(a) **Addition:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $f+g$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(f+g)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{f(x)+g(x)}_{\text{in } \mathbb{R}}.$$

(b) **Multiplication by "scalars" $r \in \mathbb{R}$:** $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, and $\forall r \in \mathbb{R}$, we define the function rf by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(rf)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{r \cdot f(x)}_{\text{in } \mathbb{R}}.$$

(c) **Multiplication:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function fg by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{(fg)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{f(x) \cdot g(x)}_{\text{in } \mathbb{R}}.$$

(d) **Division:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\dfrac{f}{g}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\left(\frac{f}{g}\right)}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\left(\frac{f(x)}{g(x)}\right)}_{\text{in } \mathbb{R}}$$

Notice that f/g is not necessarily in $\mathcal{F}(\mathcal{S}, \mathbb{R})$, since we do not know if the denominator is ever 0 without knowing the specific function $g(x)$. The domain of f/g may be different from $\mathcal{S}$.

(e) **Absolute value:** $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $|f|$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{|f|}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{|f(x)|}_{\text{in } \mathbb{R}}.$$

(f) **Maximum:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\max\{f, g\}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\max\{f,g\}}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\max\{f(x), g(x)\}}_{\text{in } \mathbb{R}}.$$

(g) **Minimum:** $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$, we define the function $\min\{f, g\}$ by specifying that $\forall x \in \mathcal{S}$,

$$\underbrace{\min\{f, g\}}_{\text{in } \mathcal{F}(\mathcal{S},\mathbb{R})}(x) = \underbrace{\min\{f(x), g(x)\}}_{\text{in } \mathbb{R}}.$$

Example B.3.2 Consider the functions $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ given by $f(x) = 3x+2$ and $g(x) = \dfrac{1}{x-1}$. Then $\forall x \in \mathbb{R}$,

(a) $(f+g)(x) = f(x) + g(x) = 3x + 2 + \dfrac{1}{x-1}$.

(b) $2f(x) = 2(3x+2) = 6x+4$.

(c) $(fg)(x) = f(x)g(x) = (3x+2)\left(\dfrac{1}{x-1}\right) = \dfrac{3x+2}{x-1}$.

(d) $|f|(x) = |f(x)| = |3x+2|$.

(e) $\max\{f, g\}(x) = \max\left\{3x+2, \dfrac{1}{x-1}\right\}$; for example, $\max\{f, g\}(0) = 2$ and $\max\{f, g\}(-1) = -\dfrac{1}{3}$.

(f) $\min\{f, g\}(x) = \min\left\{3x+2, \dfrac{1}{x-1}\right\}$; for example, $\min\{f, g\}(0) = -\dfrac{1}{2}$ and $\min\{f, g\}(-1) = -1$. $\square$

Theorem B.3.3 (***Algebra of Functions***) *Let $\mathcal{S}$ denote an arbitrary nonempty set. Then $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the operations* (a)–(c) *specified in Definition B.3.1 above, satisfies the following properties:*

(1) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f+g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(2) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + (g+h) = (f+g) + h$;

(3) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f+g = g+f$;

(4) $\exists 0 \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni \forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f + 0 = 0 + f = f$;

(5) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \exists -f \in \mathcal{F}(\mathcal{S}, \mathbb{R}) \ni f + (-f) = 0$;

(6) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\, r \in \mathbb{R},\ rf \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(7) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\, r \in \mathbb{R},\ r(f+g) = rf + rg$;

(8) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\, r, s \in \mathbb{R},\ (r+s)(f) = rf + sf$;

(9) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \forall\, r, s \in \mathbb{R},\ r(sf) = (rs)f = s(rf)$;

(10) $\forall f \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ 1f = f$;

(11) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ fg \in \mathcal{F}(\mathcal{S}, \mathbb{R})$;

(12) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f(gh) = (fg)h$;

(13) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ fg = gf$;

(14) $\forall f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ f(g+h) = fg + fh$;

(15) $\forall f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R}),\ \ and\ \forall r \in \mathbb{R},\ \ r(fg) = (rf)g = f(rg)$;

Proof. (1) Let $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. By Definition B.3.1 (a), $f + g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$.

(2) Let $f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}[f + (g+h)](x) &= f(x) + (g+h)(x) \text{ by definition of } f + (g+h)\\ &= f(x) + [g(x) + h(x)] \text{ by definition of } g + h\\ &= [f(x) + g(x)] + h(x)] \text{ by axiom (A2) of } \mathbb{R}\\ &= (f+g)(x) + h(x) \text{ by definition of } f + g\\ &= [(f+g) + h](x) \text{ by definition of } (f+g) + h.\end{aligned}$$

Thus, by Definition B.2.2, $f + (g+h) = (f+g) + h$.

(3) and (13) Exercise 5.

(4) Define the function $0 \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ by the rule: $\forall x \in \mathcal{S}$, $0(x) = 0 \in \mathbb{R}$. Let $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}(0 + f)(x) &= 0(x) + f(x) \text{ by definition of } 0 + f\\ &= 0 + f(x) \text{ by definition of the 0 function}\\ &= f(x) \text{ by axiom (A3) of } \mathbb{R}.\end{aligned}$$

Thus, by Definition B.2.2, $0 + f = f$. Also, $f + 0 = f$, by Part (c) above.

(5) Exercise 6.
(6) See proof of (1) above.
(7) Exercise 7.
(8) Let $f, \in \mathcal{F}(\mathcal{S}, \mathbb{R})$ and $r, s \in \mathbb{R}$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}[(r+s)f](x) &= (r+s) \cdot f(x) \text{ by definition of } (r+s)f\\ &= r \cdot f(x) + s \cdot f(x) \text{ by axiom (D) of } \mathbb{R}\\ &= (rf + sf)(x) \text{ by definition of } rf + sf.\end{aligned}$$

Thus, by Definition B.2.2, $(r+s)f = fr + sf$.

(9) Exercise 8.
(10) Exercise 9.
(11) Exercise 10.

(12) Let $f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$,

$$\begin{aligned}
[f(gh)](x) &= f(x) \cdot (gh)(x) \quad \text{by definition of } f(gh) \\
&= f(x) \cdot [g(x) \cdot h(x)] \quad \text{by definition of } gh \\
&= [f(x) \cdot g(x)] \cdot h(x) \quad \text{by axiom (M2) of } \mathbb{R} \\
&= (fg)(x) \cdot h(x) \quad \text{by definition of } fg \\
&= [(fg)h](x) \quad \text{by definition of } (fg)h.
\end{aligned}$$

Thus, by Definition B.2.2, $f(gh) = (fg)h$.

(13) See Exercise 5.
(14) Exercise 11.
(15) Exercise 12. ■

Students who have had a course in linear algebra will observe that Properties (1)–(10) say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication by "scalars" in Definition B.3.1, is a **vector space**. Students who have also had a course in abstract algebra will observe that properties (1)–(5) and (11)–(14) say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication in Definition B.3.1, is a **commutative ring**. They may be interested in proving that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition and multiplication in Definition B.3.1, is **not** an integral domain.

Properties (1)–(15) taken together, say that $\mathcal{F}(\mathcal{S}, \mathbb{R})$, together with the addition, multiplication by "scalars," and multiplication in Definition B.3.1, is a **commutative algebra**. The theory of "algebras" is important in advanced analysis, but not in this course.

COMPOSITE FUNCTIONS AND INVERSES

Definition B.3.4 If $f : A \to B$ and $g : B \to C$, then the **composite function** $g \circ f$ is defined by the rule

$$\forall x \in \mathcal{S}, \ (g \circ f)(x) = g(f(x)).$$

The following schematic diagram may be helpful in giving an intuitive understanding of $g \circ f$:

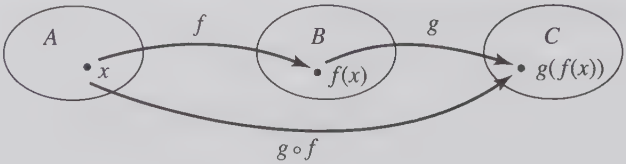


Figure B.7

BEWARE! Notice the reversal of orientation: In the schematic drawing, the function f is drawn to the left of g, but in the composite function notation $g \circ f$, g is written to the left of f. When the composite function $g \circ f$ operates on the element x, f operates first and then g operates on the result, despite the fact that when we write "$g \circ f$" we write g first. Care must be exercised to avoid confusion.

ALSO, BEWARE: In general,

$$f \circ g \neq g \circ f,$$

although sometimes they are equal.

Example B.3.5 For the functions $f(x) = 3x + 2$ and $g(x) = \dfrac{1}{x-2}$,

$$(g \circ f)(x) = g(3x - 2) = \frac{1}{(3x+2)-2} = \frac{1}{3x}, \text{ whereas}$$

$$(f \circ g)(x) = f\left(\frac{1}{x-2}\right) = 3\left(\frac{1}{x-2}\right) + 2 = \frac{2x-1}{x-2}.$$

In this example, $f \circ g \neq g \circ f$. □

Theorem B.3.6 *Composite functions obey the associative law. That is, if* $f : A \to B$, $g : B \to C$, *and* $h : C \to D$, *then*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof. Suppose $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then $\forall x \in A$,

$$\begin{aligned} [h \circ (f \circ g)](x) &= h[(g \circ f)(x)] \text{ by definition of } h \circ (g \circ f) \\ &= h[g(f(x))] \text{ by definition of } g \circ f \\ &= (h \circ g)[f(x)] \text{ by definition of } h \circ g \\ &= [(h \circ g) \circ f](x) \text{ by definition of } (h \circ g) \circ f. \end{aligned}$$

Thus, by Definition B.2.2, $h \circ (g \circ f) = (h \circ g) \circ f$. ■

Theorem B.3.7 *Suppose* $f : A \to B$ *and* $g : B \to C$.

(a) *If f and g are both 1-1, then so is $g \circ f$.*

(b) *If f and g are both onto, then so is $g \circ f$.*

(c) *If f and g are both 1-1 correspondences, then so is $g \circ f$.*

Proof. Exercise 14. ■

Definition B.3.8 Let A denote an arbitrary set. The **identity function on** A is the function $i_A : A \to A$ defined by the rule

$$\forall x \in A,\ i_A(x) = x.$$

Note: The identity function is a 1-1 correspondence.

Theorem B.3.9 *Let A and B be arbitrary sets. The identity function on A, $i_A : A \to A$, satisfies the following rules:*

(a) $\forall f : A \to B,\ f \circ i_A = f$;

(b) $\forall g : B \to A,\ i_A \circ g = g$.

Proof. Exercise 15. ■

Theorem B.3.10 *Suppose $f : A \to B$.*

(a) *If $\exists\, g : B \to C \ni g \circ f$ is 1-1, then f is 1-1.*

(b) *If $\exists\, h : D \to A \ni f \circ h$ is onto B, then f is onto B.*

Proof. (a) Suppose $\exists\, g : B \to C \ni g \circ f$ is 1-1. Let $a \neq a'$ in A. Since $g \circ f$ is 1-1,

$$(g \circ f)(a) \neq (g \circ f)(a'), \text{ which means that}$$
$$g(f(a)) \neq g(f(a')).$$

This cannot happen unless $f(a) \neq f(a')$. Therefore, f is 1-1.

(b) Suppose $\exists\, h : D \to A \ni f \circ h$ is onto B. Let $b \in B$. Since $f \circ h$ is onto B, $\exists\, d \in D \ni (f \circ h)(d) = b$. Let $a = h(d)$. Then $a \in A$ and $f(a) = f(h(d)) = b$. That is, $\forall\, b \in B$, $b \in \mathcal{R}(f)$. Therefore, f is onto B. ■

Definition B.3.11 Suppose $f : A \to B$. If $\exists$ function $g : B \to A \ni g \circ f = i_A$ and $f \circ g = i_B$, then we say that f is **invertible**, and we say that the function g is the **inverse function** of f. In symbols,

$$g = f^{-1}.$$

Theorem B.3.12 *A function $f : A \to B$ is* ***invertible*** *iff f is 1-1 and onto B (that is, f is a 1-1 correspondence). Moreover, if $g = f^{-1}$, then $f = g^{-1}$.*

Proof. First, the $\Rightarrow$ direction. Suppose $f : A \to B$ is invertible. Then $\exists$ function $g : B \to A \ni g \circ f = i_A$ and $f \circ g = i_B$. Since i_A is 1-1, Theorem B.3.10 (a) says that f is 1-1. Since i_B is onto B, Theorem B.3.10 (b) says that f is onto B.

Next, the $\Leftarrow$ direction. Suppose $f : A \to B$ is 1-1 and onto B. Let $b \in B$. Since f is onto B, $\exists a \in A \ni f(a) = b$. Moreover, since f is 1-1, there is no more than one such a. Define

$$g(b) = a.$$

Then $\forall a \in A$, $(g \circ f)(a) = g(f(a)) = g(b) = a = i_A(a)$. Also, $\forall b \in B$, $(f \circ g)(b) = f(g(b)) = f(a) = b = i_B(b)$. That is, $g \circ f = i_A$ and $f \circ g = i_B$. Therefore, by Definition B.3.11, f is invertible.

Finally, the proof of $g = f^{-1} \Rightarrow f = g^{-1}$ is Exercise 16. ■

Corollary B.3.13 *If $f : A \to B$ is a 1-1 correspondence, so is $f^{-1} : B \to A$.*

Proof. Apply Theorem B.3.10. ■

EXERCISE SET B.3

1. Let $f(x) = 2x+1$ and $g(x) = x^2-2$. Find $(f+g)(x)$, $(f-g)(x)$, $f(x+2)$, $f(x)+2$, $g(x+2)$, $g(x)+2$, $3f(x)$, $f(3x)$, $3g(x)$, $g(3x)$, $(fg)(x)$, $(f/g)(x)$, $|f|(x)$, $\max\{f,g\}(x)$, $\min\{f,g\}(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$.

2. Repeat Exercise 1 with $f(x) = \dfrac{x}{3x+4}$ and $g(x) = \dfrac{1}{x}$.

3. For each of the following functions, f, find $\mathcal{D}(f)$ and $\mathcal{R}(f)$, and tell whether f is 1-1:

   (a) $f(x) = 7x + 8$ (b) $f(x) = \sqrt{x+1}$
   (c) $f(x) = \sqrt{x^2-1}$ (d) $f(x) = \ln x$
   (e) $f(x) = e^x$ (f) $f(x) = \dfrac{x}{x+1}$
   (g) $f(x) = \sin x$ (h) $f(x) = x^3 + 2$

4. Which of the functions given in Exercise 3, viewed as $f : \mathcal{D}(f) \to \mathcal{R}(f)$, are invertible. Find f^{-1} where possible.

5. Prove Theorem B.3.3, (3) and (13).

6. Prove Theorem B.3.3, (5). Note: you must define $-f$.

7. Prove Theorem B.3.3, (6).

8. Prove Theorem B.3.3, (9).

9. Prove Theorem B.3.3, (10).

10. Prove Theorem B.3.3, (11).

11. Prove Theorem B.3.3, (14).

12. Prove Theorem B.3.3, (15).

13. Find functions $f \neq g, f \neq g^{-1}$, such that $g \circ f = f \circ g$.

14. Prove Theorem B.3.7.

15. Prove Theorem B.3.9.

16. Finish proving Theorem B.3.12, by proving that $g = f^{-1} \Rightarrow f = g^{-1}$.

Appendix C

Answers & Hints for Selected Exercises

Caveat: The answers and hints provided here are intentionally brief. Complete solutions and proofs will require more fully developed explanations. Individual instructors will have their own standards of completeness and will make them known to their students. Most solutions given here will need considerable amplification to meet these standards.

Some abbreviations used here are Thm. for Theorem, Cor. for Corollary, Defn. for Definition, and Ex. for Example.

Chapter 1

EXERCISE SET 1.1-A

1. A0, A1, A2, M0, M1, M2, M3, D.
3. All field axioms.
5. A0, A1, A2, A3, A4, but not M0. M1–M4 and D are not relevant.
7. A0, A1, A2, A3, A4, M0, D.
9. All field axioms.

EXERCISE SET 1.1-B

1. $x \neq 0 \Rightarrow \exists u \in F \ni xu = ux = 1$. Then $xy = xz \Rightarrow u(xy) = u(xz) \Rightarrow (ux)y = (ux)z \Rightarrow 1y = 1z \Rightarrow y = z$.

3. Suppose $x \neq 0$ and u, u' both have property described in (M4); i.e., $xu = 1$, and $xu' = 1$. Then $u = u1 = u(xu') = (ux)u' = 1u' = u'$.

5. Suppose $x \neq 0$. Then $xx^{-1} = 1 \neq 0$, so by (d), $x^{-1} \neq 0$.

7. Using (h) and (M2), $(-x)y = [(-1)x]y = (-1)(xy) = -(xy)$. Similarly, $x(-y) = x[(-1)y] = [x(-1)]y = [(-1)x]y = (-x)y$.

9. Using (i), $(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$ by (b).

11. By Thm. 1.1.4 (h), $-(x+y) = (-1)(x+y) = (-1)x+(-1)y = -x+(-y) = -x-y$ by Defn. 1.1.5.

13. By Defn. 1.6, $x \neq 0 \Rightarrow 0 \div x = 0x^{-1} = x^{-1}0 = 0$ by Thm. 1.1.4 (d).

15. By Thm. 1.1.4, $(-x)(-x^{-1}) = xx^{-1} = 1$. Apply Thm. 1.1.3 (d).

17. Suppose $b, d \neq 0$. Then (supply reasons) $\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1})(cd^{-1}) = [(ab^{-1})c]d^{-1} = [a(b^{-1}c)]d^{-1} = [a(cb^{-1})]d^{-1} = [(ac)b^{-1}]d^{-1} = (ac)(b^{-1}d^{-1}) = (ac)(bd)^{-1} = \frac{ac}{bd}$.

19. $a, b \neq 0 \Rightarrow \frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = (ab)(ab)^{-1} = 1$. Apply Thm. 1.1.3 (d).

EXERCISE SET 1.2-A

1. #3, #8

3. By (O3) one & only one is true: $y - x \in \mathcal{P}$, $x - y \in \mathcal{P}$, $y - x = 0$.

5. x, y negative$\Rightarrow -x, -y \in \mathcal{P} \Rightarrow (-x)(-y) \in \mathcal{P} \Rightarrow xy \in \mathcal{P}$.

7. Suppose $xy > 0$. Then $x, y \neq 0$. If x, y do not have the same sign, then one must be positive and the other negative. Then, by (d), $xy < 0$. Contradiction.

9. $1 = 1^2$ and $1 \neq 0$. Apply Thm. 1.2.6 (c).

11. Suppose $z < 0$. Then $x < y \Rightarrow y - x \in \mathcal{P}, -z \in \mathcal{P} \Rightarrow -z(y-x) \in \mathcal{P} \Rightarrow xz - yx \in \mathcal{P} \Rightarrow xy > xz$.

13. (a) $xx^{-1} = 1 > 0$, so by Thm. 1.2.6 (e), x, x^{-1} have the same sign. To prove (b) and (c), apply Part (a) to Thm. 1.2.8 (c) and (d).

15. $x < y, u < v \Rightarrow (y-x), (v-u) \in \mathcal{P} \Rightarrow (y-x)+(v-u) \in \mathcal{P} \Rightarrow (y+v)-(x+u) \in \mathcal{P} \Rightarrow x+u < y+v$.

17. By (b), $x < y \Rightarrow x + x < y + x = x + y < y + y \Rightarrow 2x < x + y < 2y$, so by Cor. 1.2.9 (b), $x < \frac{x+y}{2} < y$.

19. (O1) $a+b\sqrt{2}, c+d\sqrt{2} \in \mathcal{P}' \Rightarrow a > b\sqrt{2}, c > d\sqrt{2} \Rightarrow (a+c) > (b+d)\sqrt{2} \Rightarrow (a+c)+(b+d)\sqrt{2} \in \mathcal{P}' \Rightarrow (a+b\sqrt{2})+(c+d)\sqrt{2} \in \mathcal{P}'$.

(O2) $a > b\sqrt{2}$, $c > d\sqrt{2} \Rightarrow (a - b\sqrt{2})(c-d)\sqrt{2} > 0 \Rightarrow (ac+2bd) > (ad+bc)\sqrt{2} \Rightarrow (ac+2bd)+(ad+bc)\sqrt{2} \in \mathcal{P}' \Rightarrow (a+b\sqrt{2})(c+d\sqrt{2}) \in \mathcal{P}'$.

(O3) Given $a, b \in \mathbb{Q}$, exactly one is true: $a > b\sqrt{2}$, $a = b\sqrt{2}$, $a < b\sqrt{2}$.
Case 1: $a > b\sqrt{2} \Rightarrow a + b\sqrt{2} \in \mathcal{P}'$.

Case 2: $a = b\sqrt{2} \Rightarrow a = b = 0$, since otherwise $\frac{a}{b} = \sqrt{2}$, which would tell us that $\sqrt{2}$ is rational. So, in this case $a + b\sqrt{2} = 0$.

Case 3: $a < b\sqrt{2} \Rightarrow (-a) > (-b)\sqrt{2} \Rightarrow (-a)+(-b)\sqrt{2} \in \mathcal{P}' \Rightarrow -(a+b\sqrt{2}) \in \mathcal{P}'$.

21. $i^2 = -1 < 0$ by Cor. 1.2.7, which would contradict Thm. 1.2.6 (c).

EXERCISE SET 1.2-B

1. (a) By (O3), $x \geq 0$ or $x < 0$. In the former case $|x| = x \geq 0$ and in the latter, $|x| = -x \geq 0$.

(d) $|x - y| = |-(y - x)| = |y - x|$ by (b).

(e) We have four cases:

(1) $x \geq 0, y \geq 0$. Then $xy \geq 0$ and $|xy| = xy = |x||y|$.

(2) $x \geq 0, y < 0$. Then $xy \leq 0$ and $|xy| = -xy = x(-y) = |x||y|$.

(3) $x < 0, y \geq 0$. Then $xy \leq 0$ and $|xy| = -xy = (-x)y = |x||y|$.

(4) $x < 0, y < 0$. Then $xy > 0$ and $|xy| = xy = (-x)(-y) = |x||y|$.

3. (c) By (b), $|x| - |y| \leq |x - y|$ and $|y| - |x| \leq |x - y|$. Since $||x| - |y|| =$ either $|x| - |y|$ or $|y| - |x|$, the desired result follows.

5. Let $A = \cup\{[y, z] : y, z \in I\}$. Show $A = I$.

$x \in A \Rightarrow x \in [y, z]$ for some $y, z \in I \Rightarrow x \in I$ since I is an interval. Thus, $A \subseteq I$.

$x \in I \Rightarrow [x, x] \subseteq I \Rightarrow x \in A$. Thus, $I \subseteq A$.

7. $x \leq y \Rightarrow \min\{x, y\} = x = -(-x) = -\max\{-x, -y\}$ since $-y \leq -x$.

9. Multiply both sides of the given inequality by the lowest common denominator. Prove the resulting inequality and then divide both sides by the LCD.

EXERCISE SET 1.3

1. $1 - 1 \notin \mathbb{N}_F$; $1 \div 2 \notin \mathbb{N}_F$.

In Exercises 3–19 we show only the induction step, $P(k) \Rightarrow P(k+1)$. Begin by assuming $P(k)$. Then,

3. $1+2+3+\cdots+k+(k+1) = \frac{k(k+1)}{2} + (k+1) = (k+1)\left[\frac{k}{2}+1\right] = \frac{(k+1)(k+2)}{2}$.

5. $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + (k+1)\right]$
$= (k+1)^2 \frac{k^2+4k+4}{4} = \frac{(k+1)^2(k+2)^2}{2}$.

7. $1 + 4 + 7 + \cdots + [3(k+1) - 2] = 1 + 4 + 7 + \cdots + (3k-2) + [3k+1]$

$= \frac{k(3k-1)}{2} + 3k + 1 = \frac{3k^2+5k+2}{2} = \frac{(k+1)(3k+2)}{2} = \frac{(k+1)(3(k+1)-1)}{2}.$

9. Assume $k^5 - k = 5m$ for some $m \in \mathbb{Z}$. Then $(k+1)^5 - (k+1) =$
$k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)$
$= 5[m + k^4 + 2k^3 + 2k^2 + k]$.

11. $1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^{k+1}} = 1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^k} + \frac{1}{3^{k+1}} = \left[\frac{3}{2} - \frac{1}{2}\left(\frac{1}{3^k}\right)\right] + \frac{1}{3^{k+1}}$
$= \frac{3}{2} + \frac{-3+2}{2 \cdot 3^{k+1}} = \frac{3}{2} - \frac{1}{2}\frac{1}{3^{k+1}}.$

13. $2^{k+1} = 2^k 2 \leq (k+1)!2 \leq (k+1)!(k+2) = (k+2)!$.

15. $(1+x)^{k+1} = (1+x)^k(1+x) \geq [1 + kx + \frac{1}{2}k(k-1)x^2](1+x)$ since $x \geq 0$.
$= 1 + kx + \frac{1}{2}k(k-1)x^2 + x + kx^2 + \frac{1}{2}k(k-1)x^3$
$\geq 1 + (k+1)x + [\frac{1}{2}k(k-1) + k]x^2 = 1 + (k+1)x + [\frac{1}{2}k^2 + \frac{1}{2}k]x^2$
$= 1 + (k+1)x + \frac{1}{2}(k+1)kx^2.$

17. Assume $2^{2k-1} + 1 = 3m$ for some $m \in \mathbb{N}$. Then $2^{2(k+1)-1} + 1 = 2^{2k+1} + 1$
$= 2^2 2^{2k-1} + 1 = 4(3m-1) + 1 = 12m - 4 + 1 = 3(4m-1)$.

19. $x^{k+1} - y^{k+1} = x^k x - y^k y = x^k x - y^k x + y^k x - y^k y = x(x^k - y^k) + y^k(x-y)$
$= x(x-y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \cdots + xy^{k-2} + y^{k-1}) + y^k(x-y)$
$= (x-y)(x^k + x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1} + y^k)$.

EXERCISE SET 1.4

3. Let $n \in \mathbb{Z}$. n not divisible by $2 \Rightarrow \exists k \in \mathbb{Z} \ni n = 2k+1 \Rightarrow n^2 = 4k^2+4k+1 \Rightarrow$ n^2 not divisible by 2.

5. Suppose x is rational and y is irrational, and let $z = x + y$. Then $y = z - x$. If z is rational than so is y, which would be a contradiction.

7. Let x be irrational. Since $x + (-x) = 0$, Exercise 5 says $-x$ cannot be rational. Since $x(x^{-1}) = 1$, Ex. 6 says x^{-1} cannot be rational.

9. $\sqrt{2} \cdot \sqrt{2} = 2$.

10. $\forall n \in \mathbb{N}$, $n + \sqrt{2}$ is irrational, by Ex. 6. Moreover, $n + \sqrt{2} = m + \sqrt{2} \Rightarrow n = m$, so there are infinitely many such irrational numbers.

EXERCISE SET 1.5

1. $x \in \mathbb{Q} \Rightarrow x = \frac{a}{b}$ for some $a \in \mathbb{Z}$, $b \in \mathbb{N}$. Then $|a| + 1 > \frac{|a|}{b} \geq \frac{a}{b} = x$ and $|a| + 1 \in \mathbb{N}$.

3. Assume F has A.P., and $a > 0$. Then $\forall x \in F$, $\frac{x}{a} \in F$ so by A.P. $\exists n \in \mathbb{N} \ni n > \frac{x}{a}$. Since $a > 0$, this means $na > x$.

5. Suppose $x \in$ Archimedean F. First prove existence. If x is an integer, take $n = x + 1$. If x is not an integer then $x > 0$ or $x < 0$. The first case is covered

by Thm. 1.5.3. Suppose $x < 0$. Then $-x > 0$ so by Thm. 1.5.3, $\exists m \in \mathbb{N} \ni m - 1 < -x < m$. Then $-m < x < -m + 1$, so we may take $n = -m + 1$. Uniqueness follows by the argument given in Thm. 1.5.3.

7. Let $x < y$ in an ordered F. Then $x < \frac{x+y}{2} < y$ (Thm. 1.2.10 (d)).

9. Suppose that $\forall \varepsilon > 0$, $x \leq a + \varepsilon$. Then $\forall \varepsilon > 0$, $x - a \leq \varepsilon$, so by (a), $x - a \leq 0$; i.e., $x \leq a$.

11. Suppose that $\forall \varepsilon > 0$, $|a - b| \leq \varepsilon$. By (a), $|a - b| \leq 0$. But $|a - b| \geq 0$, so $|a - b| = 0$; i.e., $a = b$.

EXERCISE SET 1.6-A

1. (a) Yes; $3, 4, 86$; 3 (c) Yes; $4, 4.01, 86$; 3 (e) Yes; $0, 0.2, 86$; 0
(g) Yes; $-100, 0, 25$; none (i) No (k) Yes; $2, 3, 86$; 2
(m) Yes; $2, 3, 86$; 1.5 [Draw graph of $f(x) = 1 + \frac{1}{x}$ for $x > 2$.]

2. (a) Yes; $-1, -2, -100$; -1 (c) Yes; $1, 0, -20$; 1 (e) No
(g) Yes; $-100, 0, 25$; none
(i) Yes; $0, -1, -100$; 0 [Draw graph of $f(x) = \frac{1}{x}$.]
(k) Yes; $1, 0, -100$; 1 [Draw graph.] (m) Yes; $2, 1, -100$; 1 [Draw graph.]

3. Examples given in Exercises 1 and 2.

5. Alter the proof already given that shows S has a maximum element.

7. Alter the proof of Part (a) given.

9. If $u = \inf A \in A$, then $u \in A$ and $\forall a \in A$, $a \geq u$, so by defn., $u = \min A$.

11. Let F be Archimedean, $A \subseteq F$, and $u \in F$.

($\Rightarrow$) Suppose $u = \inf A$. Let $\varepsilon > 0$. Then $\forall x \in A$, $x \geq u > u - \varepsilon$. Also, $u + \varepsilon > \inf A$, so $u + \varepsilon$ is not a lower bound for A, so $\exists x \in A \ni x < u + \varepsilon$.

($\Leftarrow$) Suppose (a) and (b) hold. Then,

(1) $\forall x \in A$, $x > u - \varepsilon$. By Exercise 1.5.12, $x \geq u$.

(2) Suppose v is a lower bound for A. For contradiction, suppose $v > u$. Let $\varepsilon = v - u$. By (b), $\exists x \in A \ni x < u + \varepsilon = v$. Contradiction. Therefore, all lower bounds of A are $\leq u$.
By (1) and (2) together, $u = \inf A$.

EXERCISE SET 1.6-B

1. Suppose A is a nonempty set with a lower bound in a complete ordered F. By Exercise 1.6-A.12, the set $-A = \{-a : a \in A\}$ is bounded above. By completeness, $\exists u = \sup(-A)$. Then

(a) $\forall a \in A$, $-a \leq u$, so $a \geq -u$.

(b) If v is any lower bound for A, then by Exercise 1.6-A.12, $-v$ is an upper bound for $-A$, so $-v \leq u$. That is, $v \geq -u$.
By (a) and (b) together, $-u = \inf A$.

3. Let A be a nonempty subset of ordered F with an upper bound in F, and B be the set of all upper bounds of A in F. By completeness, $\exists u = \sup A$. Then $u \in B$ and $\forall b \in B$, $u \leq b$. $\therefore u = \min B$.

5. Let $a = \sup A$, $b = \sup B$ and $c = \max\{a, b\}$. Then
(a) Let $x \in A \cup B$. Then either $x \in A$ so $x \leq a \leq c$, or $x \in B$ so $x \leq b \leq c$. Thus $x \leq c$.
(b) If d is any upper bound for $A \cup B$ then d is any upper bound for A and d is any upper bound for B, so $d \geq a$ and $d \geq b$; thus $d \geq c$.
By (a) and (b) together, $c = \sup A \cup B$.

7. (a) $\forall x \in X$, $f(x) + g(x) \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$. Thus, $\sup\{f(x){:}x \in X\}+\sup\{g(x){:}x \in X\}$ is an upper bound for $\{f(x)+g(x){:}x \in X\}$.

Chapter 2

EXERCISE SET 2.1

1. (a) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{81}$ (g) $\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}$

2. (a) 23; 101; $n_0 > \sqrt{\frac{5}{\varepsilon}}$ (c) 700; 14,000; $n_0 > \frac{7}{\varepsilon}$

(e) 1,197; 23,997; $n_0 > \frac{12}{\varepsilon} - 4$ ($\frac{12}{\varepsilon}$ will do.)

(g) 64; 1,266; $n_0 > \frac{31}{49\varepsilon} + \frac{1}{7}$ ($\frac{1}{\varepsilon} + 1$ will do.)

(i) 501; 10,001; $n_0 > \frac{5}{\varepsilon}$ (k) 101; 2001; $n_0 > \max\{2, \frac{2}{\varepsilon}\}$

(m) 34; 668; $n_0 > \frac{2}{3\varepsilon} + 1$ (o) 304; 6004; $n_0 > \max\{10, \frac{4}{\varepsilon} + 1\}$

EXERCISE SET 2.2

1. Use Def. 2.1.1 directly. **3.** $\forall n \in \mathbb{N}$, $|x_n - c| = 0$.

5. $\{(-1)^n\}$ **7.** Take $a_n = n$, $b_n = \frac{1}{n}$.

9. When $c = 0$, $\{cx_n\}$ is a constant sequence.

10. Take $a_n = (-1)^n$, $b_n = (-1)^{n+1}$. Thm. 2.2.13 applies only when $\{a_n\}$ and $\{b_n\}$ both converge, so this example does not contradict that theorem.

13. By the algebra of limits, (a) 19; (b) $-\frac{7}{60}$; (c) $\frac{1}{8}$; (d) $\frac{1+\sqrt{5}}{2}$

15. We give only the numerical answers, except for (l), for which we give a complete solution. You must supply the reasons.
(a) 0; (c) $\frac{1}{3}$; (e) 25; (g) 0; (i) 0; (k) 3; (m) 0; (o) $\frac{1}{2}$; (q) -1

(l) Using algebra and Thm. 2.2.13,

$$\lim_{n\to\infty}\frac{4n^2-5n}{8n^2+3n-1}=\lim_{n\to\infty}\frac{4-\frac{5}{n}}{8+\frac{3}{n}-\frac{1}{n^2}}=\frac{\lim\limits_{n\to\infty}\left(4-\frac{5}{n}\right)}{\lim\limits_{n\to\infty}\left(8+\frac{3}{n}-\frac{1}{n^2}\right)}$$

$$=\frac{\lim\limits_{n\to\infty}4-\lim\limits_{n\to\infty}\frac{5}{n}}{\lim\limits_{n\to\infty}8+\lim\limits_{n\to\infty}\frac{3}{n}-\lim\limits_{n\to\infty}\frac{1}{n^2}}=\frac{4-5\lim\limits_{n\to\infty}\frac{1}{n}}{8+3\lim\limits_{n\to\infty}\frac{1}{n}-\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)^2}=\frac{1}{2}.$$

17. In each case, prove the sequence is unbounded. For example, in (c), $\frac{2n+3}{\sqrt{n}}>\frac{2n}{\sqrt{n}}=2\sqrt{n}$.

19. By Thm. 2.2.16, only the tail of $\{x_n\}$ matters.

21. Apply the algebra of limits to Exercise 1.2-B.6.

EXERCISE SET 2.3

1. Apply the first squeeze theorem with $a_n=0$, $b_n=|a_n-L|$ and $c_n=|b_n|$. Then apply Thm. 2.2.1 (b).

2. (a) $\left|\frac{6}{2+7n}-0\right|=\frac{6}{2+7n}<\frac{6}{7n}<\frac{1}{n}\to 0.$

(c) $\left|\frac{n}{3n+8}-\frac{1}{3}\right|=\left|\frac{3n-(3n+8)}{3n+8}\right|=\frac{8}{3n+8}<\frac{8}{3n}=\frac{8}{3}\cdot\frac{1}{n}\to 0.$

(e) $\left|\frac{10n-11}{7-2n}-(-5)\right|=\left|\frac{(10n-11)+5(7-2n)}{7-2n}\right|=\frac{24}{2n-7}<\frac{24}{2n-n}=\frac{24}{n}\to 0$ (for $n>7$).

(g) $\left|\frac{100}{n^2}-0\right|=\frac{100}{n^2}\to 0.$ (i) $0<\frac{n}{n^2+5}<\frac{n}{n^2}=\frac{1}{n}\to 0.$

(k) $\left|\frac{3n^2+n-5}{n^2+6n}-3\right|=\left|\frac{(3n^2+n-5)-3(n^2+6n)}{n^2+6n}\right|=\left|\frac{-17n-5}{n^2+6n}\right|=\frac{17n+5}{n^2+6n}<\frac{17n+n}{n^2}$ $=\frac{18}{n}\to 0.$ (m) $\left|\frac{5n}{2n^3-7}\right|=\frac{5n}{2n^3-7}<\frac{5n}{2n^3-n^3}=\frac{5}{2n^2}\to 0.$ (for $n\geq 2$)

(o) $\left|\frac{n^3-2n^2}{6-3n+2n^3}-\frac{1}{2}\right|=\left|\frac{2(n^3-2n^2)-(6-3n+2n^3)}{2(6-3n+2n^3)}\right|=\left|\frac{-4n^2+3n-6}{2(2n^3-3n+6)}\right|=\frac{|4n^2-3n+6|}{2|(2n^3-3n+6)|}$ $=\frac{|n(4n-3)+6|}{2|n(2n^2-3)+6|}=\frac{4n^2-3n+6}{2(2n^3-3n+6)}<\frac{4n^2}{2(2n^3-3n^2)}=\frac{4n^2}{2n^2(2n-3)}=\frac{2}{2n-3}\to 0.$ (for $n\geq 2$)

3. Modify the proof of (a) where appropriate.

5. (b) By Exercise 1.3.4, and the algebra of limits,

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3}=\lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6}\cdot\frac{1}{n^3}\right)=\lim_{n\to\infty}\left(\frac{n}{n}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\cdot\frac{1}{6}\right)$$
$$=1\cdot 1\cdot 2\cdot\frac{1}{6}=\frac{1}{3}.$$

7. (a) Let $p\in\mathbb{N}$ be fixed and $\forall n\in\mathbb{N}$, $x_n=\dfrac{c^n}{n^p}$. Note that when $|c|=1$,

$x_n = \dfrac{1}{n^p} \to 0$. (Apply Exercise 2.2.16.) When $|c| < 1$,
$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{c^{n+1}}{(n+1)^p} \cdot \frac{n^p}{c^n}\right| = |c|\left(\frac{n}{n+1}\right)^p = |c|\left(1+\frac{1}{n}\right)^p \to |c| < 1,$$ so by Thm. 2.3.10, $x_n \to 0$. (b) Do as in (a) but with $x_n = \dfrac{n^p}{c^n}$.

8. $0 \le a \le b \Rightarrow 0 \le b^n \le a^n + b^n \le 2b^n \Rightarrow b \le \sqrt[n]{a^n + b^n} \le \sqrt[n]{2}b$. Apply Example 2.3.9 and squeeze.

11. If $c = 0$, $\{\frac{c^n}{n!}\}$ is a constant sequence. If $c \neq 0$ let $x_n = \frac{c^n}{n!}$. Then
$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{c^{n+1}}{(n+1)!} \cdot \frac{n!}{c^n}\right| = \frac{|c|}{n+1} \to 0.$$ Apply Thm. 2.3.10.

13. Let $x > 1$, and $\forall n \in \mathbb{N}$, $x_n = \dfrac{n}{x^n}$. Then
$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{x^{n+1}} \cdot \frac{x^n}{n} = \frac{n+1}{n} \cdot \frac{1}{x} \to \frac{1}{x} < 1.$$ Apply Thm. 2.3.10.

15. 0; 0; −1; 0; 0; $\frac{1}{2}$; 0; 0; 0; 0

19. If $c = 0$, $\{n^p c^n\}$ is a constant sequence. If $0 < |c| < 1$, let $x_n = n^p c^n$.
Then $\left|\dfrac{x_{n+1}}{x_n}\right| = \left|\dfrac{(n+1)^p c^{n+1}}{n^p c^n}\right| = |c|\left(\dfrac{n+1}{n}\right)^p = |c|\left(1+\dfrac{1}{n}\right)^p \to |c| < 1$.
Apply Thm. 2.3.10.

20. (a) $cn - 1 < \lfloor cn \rfloor \le cn$, so $c - \frac{1}{n} < \frac{\lfloor cn \rfloor}{n} \le c$.

(b) Since $\frac{1}{cn} \to 0$, $\exists n_0 \in \mathbb{N} \ni n \ge n_0 \Rightarrow 0 < \frac{1}{cn} < 1 \Rightarrow \lfloor \frac{1}{cn} \rfloor = 0$.

EXERCISE SET 2.4

1. (a) $10{,}000$, M^2 (b) 101, $M+1$ (c) $10{,}201$, $(M+1)^2$
(d) 18, $\sqrt{3M+3}$.
Hint for the proof for (c): to achieve $\frac{1-n}{\sqrt{n}} < -M$, or $\frac{n-1}{\sqrt{n}} > M$, note that $\frac{n-1}{\sqrt{n}} > \frac{n-1}{\sqrt{n}+1} = \frac{(\sqrt{n}+1)(\sqrt{n}-1)}{\sqrt{n}+1} = \sqrt{n} - 1$, so take $\sqrt{n} - 1 > M$.

3. (a) Let $x_n = \dfrac{3^n + 8^n}{7^n}$. then $0 < \dfrac{1}{x_n} = \dfrac{7^n}{3^n + 8^n} < \dfrac{7^n}{8^n} < \left(\dfrac{7}{8}\right)^n \to 0$ by Thm. 2.3.7. Apply the squeeze principle and Thm. 2.4.4.
In (b) we show an alternative approach, using the comparison test (2.4.7).

(b) $\dfrac{n^3 + \sin(3n)}{n-1} \ge \dfrac{n^3 - 1}{n-1} = n^2 + n + 1 > n \to \infty$. Apply Thm. 2.4.7.

(c) Let $x_n = \dfrac{n!}{n^{100}}$ and $y_n = \dfrac{1}{x_n} = \dfrac{n^{100}}{n!}$. Then $\left|\dfrac{y_{n+1}}{y_n}\right| = \dfrac{(n+1)^{100}}{(n+1)!} \cdot \dfrac{n!}{n^{100}}$
$= \left(\frac{n+1}{n}\right)^{100} \cdot \dfrac{1}{n+1} \to 0$. By Thm. 2.3.10, $y_n \to 0$. Apply Thm. 2.4.4.

9. (a) $a_n = b_n = n$ (b) $a_n = 2n,\ b_n = n$ (c) $a_n = n,\ b_n = 2n$
(d) $a_n = n + L,\ b_n = n$

11. By Defn. 2.2.12, $\exists L > 0$ and $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow b_n = |b_n| > L$. Let $M > 0$. Since $a_n \to \infty$, $\exists n_2 \in \mathbb{N} \ni n \geq n_1 \Rightarrow a_n > \dfrac{M}{L}$. Then $n \geq \max\{n_1, n_2\} \Rightarrow a_n b_n > M$.

13. Use the given condition and Thm. 2.3.10 to show $\dfrac{1}{x_n} \to 0$. Apply Thm. 2.4.4.

15. Assume $r > 1$. For $a > 0$, $a + ar + ar^2 + \cdots + ar^n > na \to +\infty$.
For $a < 0$, $a + ar + ar^2 + \cdots + ar^n = a(1 + r + r^2 + \cdots + r^n) < na \to -\infty$.
Apply Thm. 2.4.7.

19. Suppose $a_n \to \infty$, $b_n \to L < 0$, and let $M > 0$. Then $\exists\ n_1, n_2 \in \mathbb{N} \ni$
$n \geq n_1 \Rightarrow b_n < \dfrac{L}{2}$ and $n \geq n_2 \Rightarrow a_n > \dfrac{2M}{-L}$. Then
$n \geq \max\{n_1, n_2\} \Rightarrow a_n(-b_n) > M \Rightarrow a_n b_n < -M$.

EXERCISE SET 2.5

2. In each of the following, x_n denotes the general term of the given sequence.

(a) Neither. $x_{2n-1} < x_{2n}$ while $x_{2n} > x_{2n+1}$

(c) Strictly increasing. $x_{n+1} - x_n = 1 + \dfrac{(-1)^{n+1}}{n+1} - \dfrac{(-1)^n}{n} \geq 1 - \dfrac{1}{n+1} - \dfrac{1}{n}$
$= 1 - \dfrac{2n+1}{n(n+1)} > 1 - \dfrac{2n+2}{n(n+1)} = 1 - \frac{2}{n} > 0$ when $n > 2$.

(e) Neither. $\{x_n\} = \{1, 3, 1, 3, 1, 3, \cdots, 1, 3, \cdots\}$.

(h) Both monotone increasing and monotone decreasing: $\{0, 0, 0, 0, 0, 0, \cdots\}$

(i) Strictly decreasing. $x_n > 0$ and $\dfrac{x_{n+1}}{x_n} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \left(\dfrac{n+1}{n}\right) \cdot \dfrac{1}{2} < 1$.

(k) Strictly increasing. As $n \to \infty$, $\dfrac{\pi}{2n}$ decreases to 0, so $\cos \dfrac{\pi}{2n}$ increases to 1.

(m) Strictly decreasing, since the function $f(x) = \dfrac{3x+5}{x^2 - x - 2}$ has negative derivative when $x > 2$.

3. Modify the proof of (a), replacing "increasing" by "decreasing," "sup" by "inf," etc.

5. $\{d_n\}$, where d_n is the decimal expansion of $\sqrt{2}$ to n decimal places.

7. First show $\{x_n\}$ monotone increasing, by math induction. Then, show $\{x_n\}$ bounded above (by 5), also by induction. By Thm. 2.5.3, $\{x_n\}$ converges, say to L. Then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{4x_n+5} \Rightarrow L = \sqrt{4L+5} \Rightarrow L^2 = 4L+5 \Rightarrow L^2 - 4L - 5 = 0 \Rightarrow L = 5$ or $L = -1$. By Thm. 2.3.12, $L \geq 0$. $\therefore L = 5$.

9. First show by induction that $\forall n \in \mathbb{N}$, $0 < x_n \leq 1$. For the general step, $0 < x_k \leq 1 \Rightarrow 0 < x_k^2 \leq x_k \leq 1$, and since $0 < \frac{k}{k+1} < 1$, it follows that $0 < \frac{k}{k+1}x_k^2 \leq 1$; i.e., $0 < x_{k+1} \leq 1$. To show that $\{x_n\}$ is monotone decreasing, the general induction step is $x_{k+2} = \dfrac{(k+1)x_{k+1}^2}{k+2} = \dfrac{k+1}{k+2} \cdot x_{k+1} \cdot x_{k+1} < x_{k+1}$. By the monotone convergence thm., $\exists L = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{n}{n+1}x_n^2 = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} x_n^2 = L^2$, so $L^2 - L = 0$; i.e., $L = 0$ or 1. Since $\{x_n\}$ is decreasing, $L \neq 1$, so $L = 0$.

11. Show by induction that $\{x_n\}$ is monotone increasing and bounded above, say by 2, so by the monotone convergence thm., it converges. Then $L = \lim_{n\to\infty} x_n \Rightarrow L = \sqrt[3]{L+6} \Rightarrow L^3 - L - 6 = 0 \Rightarrow (L-2)(L^2+2L+3) = 0 \Rightarrow L = 2$.

14. Note that $0 < x_{n+1} = x_n \cdot \frac{2n+1}{2n+2} < x_n$. Apply the monotone convergence theorem.

15. Take $x_1 = 3$ and calculate $x_1, x_2, x_3, \cdots$ until $x_n^2 < 10 + 1.5 \times 10^{-6}$. Now $x_4 = 3.1622776\cdots$ and $x_4^2 = 10.0000000\cdots$, so to 4 decimal places, $\sqrt{10} = 3.1623$.

17. $|a| < 1 \Rightarrow \{|a^n|\}$ is strictly decreasing. By the monotone convergence thm., $\{|a^n|\} \to L \geq 0$. Taking limit of both sides of $|a^{n+1}| = |a| \cdot |a|^n$ yields $L = L|a|$, so $L(1-|a|) = 0$. Since $|a| \neq 1$, $L = 0$. $\therefore |a^n| \to 0$, so $a^n \to 0$.

20. Between 13.12 and 14.12.

21. Since $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, $\sum_{k=1}^{n} \frac{1}{k(k+1)} = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1$.

23. (a) True. $x_n \leq x_{n+1}$, $y_n \leq y_{n+1} \Rightarrow x_n + y_n \leq x_{n+1} + y_{n+1}$.
(b) False. Take $x_n = n + \frac{1}{n}$ and $y_n = n$.
(c) True. $0 \leq x_n \leq x_{n+1}$, $0 \leq y_n \leq y_{n+1} \Rightarrow 0 \leq x_n y_n \leq x_{n+1} y_{n+1}$.
(d) False. Take $x_n = y_n = -\frac{1}{n}$.
(e) False. Take $x_n = n$ and $y_n = n^2$.

25. Use induction to show that the sequence is monotone increasing and bounded above by 2; thus, it converges, say to L. Then, since $x_{n+1} = \sqrt{2 + x_n}$, we have $L^2 = 2 + L$, from which we conclude $L = 2$ (supply reasons).

EXERCISE SET 2.6

4. (a) $\left(1+\frac{1}{n}\right)^{2n} = \left[\left(1+\frac{1}{n}\right)^n\right]^2 \to e^2$. (b) $\left(1+\frac{1}{3n}\right)^{2n} = \left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{\frac{1}{3}} \to e^{\frac{1}{3}}$ since $\left\{\left(1+\frac{1}{3n}\right)^{3n}\right\}$ is a subsequence of $\left\{\left(1+\frac{1}{n}\right)^n\right\}$.

(c) e (d) $\left(\frac{n}{n+1}\right)^n = \left(\frac{1}{\frac{n+1}{n}}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e}$. (e) $\frac{1}{e}$

5. (a) False. If $\exists n_1 \in \mathbb{N} \ni n \geq n_1 \Rightarrow x_n \in A$ and $\exists n_2 \in \mathbb{N} \ni n \geq n_2 \Rightarrow x_n \in B$, then $n \geq \max\{n_1, n_2\} \Rightarrow x_n \in A \cap B = \varnothing$, a contradiction.

(b) True. $\{(-1)^n\}$ is frequently in $(-2, 0)$ and frequently in $(0, 2)$.

7. (a) Suppose $x_n \to \infty$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Let $M > 0$. Then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_n > M$. So, $k \geq n_0 \Rightarrow n_k \geq k \geq n_0 \Rightarrow x_{n_k} > M$.
(b) Same as (a), with "$> M$" replaced by "$< -M$."

10. (a) $0, +\infty$; diverges (c) $-\infty, 0, +\infty$; diverges (e) $-5, 5$; diverges
(g) $0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1$; diverges (i) 0; converges
(k) $-1, 1$; diverges

13. $\{1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, 1, 2, 3, 4, 5, 6, 7, 1, \cdots\}$

15. $\{x_n\}$ not bounded above $\Rightarrow \forall k \in \mathbb{N}, \exists n_k \in \mathbb{N} \ni x_{n_k} > k \Rightarrow x_{n_k} \to +\infty$.
Similarly for $\{x_n\}$ not bounded below.

17. ($\Leftarrow$) Suppose every subsequence of $\{x_n\}$ has a subsequence converging to real number L. Then L is a cluster point of $\{x_n\}$. If L' is another, then $\{x_n\}$ has a subsequence converging to L', and this in turn has a subsequence converging to L. By Thm. 2.6.8 and the uniqueness of limits, $L = L'$. Thus, $\{x_n\}$ has one and only one cluster point. By Thm. 2.6.17, $x_n \to L$.

19. Redo Case 1, using $(L, +\infty)$, changing inequalities, etc.

21. Suppose $\{x_n\}$ is bounded and all its convergent subsequences converge to L. If $\{x_{n_k}\}$ is any subsequence, it is bounded, so by Bolzano-Weierstrass it has a convergent subsequence, which must converge to L. So, every subsequence of $\{x_n\}$ has a subsequence converging to L. By Exercise 2.6.17, $x_n \to L$.

EXERCISE SET 2.7

1. (a) $m, n > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{m} - \frac{1}{n}\right| < \frac{1}{\min\{m,n\}} - \frac{1}{\max\{m,n\}} < \frac{1}{\min\{m,n\}} < \varepsilon.$

(c) $n > m > \frac{1}{\varepsilon} \Rightarrow \left|\frac{m}{m^2+1} - \frac{n}{n^2+1}\right| = \left|\frac{mn^2+m-nm^2-n}{(m^2+1)(n^2+1)}\right| = \frac{|(n-m)(mn-1)|}{(m^2+1)(n^2+1)}$
$< \frac{n(mn)}{m^2n^2} = \frac{1}{m} < \varepsilon.$

2. (b) $\left|\frac{m^2+1}{m} - \frac{n^2+1}{n}\right| = \left|(m-n) + \left(\frac{1}{m} - \frac{1}{n}\right)\right|$. Note that if $n \geq 2$ and $m = n+k$ for some $k \geq 1$, then $\left|\frac{m^2+1}{m} - \frac{n^2+1}{n}\right| = k + \frac{1}{n+k} - \frac{1}{n} = k - \frac{k}{n(n+k)} > k - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}$. Thus, $\nexists n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow |x_n - x_m| < \frac{1}{2}$.

3. For $m < n$, $|x_m - x_n| \leq |x_m - x_{m+1}| + |x_{m+1} - x_{m+2}| + \cdots + |x_{n-1} - x_n|$
$< C^m + C^{m+1} + \cdots + C^n < C^m(1 + C + +C^{n-m}) < C^m\left(\frac{1}{1-C}\right)$ (see Exercise 2.3.6). By Thm. 2.3.7, $C^m \to 0$.

5. Suppose a Cauchy sequence $\{x_n\}$ has c as a cluster point. Then it has a subsequence $\{x_{n_k}\}$ converging to c. Follow the proof of Thm. 2.7.4 from the second paragraph, using c instead of L.

7. (a) Modify the proofs of Thm. 2.2.13. For example, for Cauchy sequences $\{x_n\}, \{y_n\}$, $|(x_m + y_m) - (x_n - y_n)| \leq |x_m - x_n| + |y_m - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. To see that $\left\{\frac{x_n}{y_n}\right\}$ is not necessarily Cauchy, take $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$.

(b) A sufficient condition: $\{y_n\} \not\to 0$.

9. (a) $|x_{n+2} - x_{n+1}| = \left|\frac{x_{n+1}+x_n}{2} - x_{n+1}\right| = \frac{|x_{n+1}+x_n-2x_{n+1}|}{2} = \frac{1}{2}|x_{n+1} - x_n|$.
(b) $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}x_{n-1} \Rightarrow x_{n+1} + \frac{1}{2}x_n = x_n + \frac{1}{2}x_{n-1} = \cdots = b + \frac{1}{2}a$.
(c) Let $x_n \to L$. Then $L + \frac{1}{2}L = b + \frac{1}{2}a$, so $L = \frac{a+2b}{3}$.

11. By hypothesis, $\exists c \ni 0 < c < 1$ and $\forall x, y \in I$, $|f(x) - f(y)| < c|x - y|$. Pick any $x_1 \in I$. $\forall n \in \mathbb{N}$, define $x_{n+1} = f(x_n)$. Then $|x_{n+2} - x_{n+1}| = |f(x_{n+1}) - f(x_n)| \leq c|x_{n+1} - x_n|$. By Ex. 8, $\{x_n\}$ converges; say $x_n \to L$. By Exercise 2.3.18, $L \in [a, b]$. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow |x_n - L| < \frac{\varepsilon}{2} < \frac{\varepsilon}{2c} \Rightarrow |f(L) - L| \leq |f(L) - f(x_n)| + |f(x_n) - L| \leq c|L - x_n| + |x_{n+1} - L| < c\frac{\varepsilon}{2c} + \frac{\varepsilon}{2} = \varepsilon$. By the forcing principle, $f(L) = L$.

If $f(L') = L'$ for some $L \neq L' \in A$, then $|L - L'| = |f(L) - f(L')| \leq c|L - L'| < |L - L'|$, a contradiction.

13. For $a \geq 1$ define $x_1 = a$ and $x_{n+1} = a + \frac{1}{x_n}$. First prove by induction that $a + \frac{1}{2a} \leq x_n \leq 2a$, and use this to prove that $\frac{1}{x_n} \leq \frac{2a}{2a^2+1} < \frac{1}{a} \leq 1$. Show that $|x_{n+1} - x_n| = \frac{|x_n - x_{n-1}|}{x_n x_{n-1}} \leq \left(\frac{2a}{2a^2+1}\right)^2 |x_n - x_{n-1}|$. Apply Exercise 8 to conclude that $\{x_n\}$ converges, say $x_n \to L$. Show $L = a + \frac{1}{L}$, and hence $L = \frac{a+\sqrt{a^2+4}}{2}$.

EXERCISE SET 2.8

1. Let $A = \{a_1, a_2, \cdots, a_n \cdots\}$ and $B = \{b_1, b_2, \cdots, b_n \cdots\}$. Define the function (sequence) $f : \mathbb{N} \to A \cup B$ by

$$f(x) = a_k \text{ if } n = 2k-1, \quad f(x) = b_k \text{ if } n = 2k \quad (k \in N).$$

If $A \cap B = \varnothing$, f is a 1-1 correspondence. If $A \cap B \neq \varnothing$, then deleting those b_k which $\in A$ will result in a subsequence that is a 1-1 correspondence $\mathbb{N} \leftrightarrow A \cup B$.

3. Let
$$\begin{array}{llllll} A_1 = \{a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots\ \} \\ A_2 = \{a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \cdots\ \} \\ A_3 = \{a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & \cdots\ \} \\ \vdots & & & \vdots & & \end{array}$$

If $A_1, A_2, \cdots A_n, \cdots$ are pairwise disjoint, then the function $f : \bigcup_{n=1}^{\infty} A_n$ given by the diagonal counting procedure used in Thm. 2.8.5 is a 1-1 correspondence. If they are not pairwise disjoint, then by omitting from the listing those members of $\bigcup_{n=1}^{\infty} A_n$ previously "counted" by the function f, the resulting subsequence of f is a 1-1 correspondence $\mathbb{N} \leftrightarrow \bigcup_{n=1}^{\infty} A_n$.

5. (a) The function $f : A \to A$ given by $f(x) = x$ is a 1-1 correspondence.
(b) If $f : A \to B$ is a 1-1 correspondence, then $f^{-1} : B \to A$ is a 1-1 corresp.
(c) If $f : A \to B, g : B \to C$ are 1-1 correspondences, then so is $g \circ f : A \to C$.
(For background on this exercise, see Appendix B, Thm. B.3.7.)

7. Suppose A is an infinite set and $B = \{x_1, x_2, \cdots, x_n\} \subseteq A$. Then $A - B$ has a denumerable subset $A' = \{x_{n+1}, x_{n+2}, \cdots\}$. The function $f : A \to A - B$ given by $f(x) = \left\{ \begin{array}{l} x \text{ if } x \notin A' \cup B \\ x_{n+k} \text{ if } x = x_n,\ n \in N \end{array} \right\}$ is a 1-1 correspondence.

9. Suppose A is an infinite set and $B = \{b_1, b_2, b_3, \cdots\}$ is a denumerable subset of A. (The case of finite B is covered by Ex. 7.) Then $A - B$ is infinite, so it has a denumerable subset $B' = \{b'_1, b'_2, b'_3, \cdots\}$. Then $f : A \to A - B$ given by

$$f(x) = \left\{ \begin{array}{l} x \text{ if } x \notin B \cup B' \\ b'_{2k} \text{ if } x = b_k,\ k \in \mathbb{N} \\ b'_{2k-1} \text{ if } x = b'_k,\ k \in \mathbb{N} \end{array} \right\} \text{ is a 1-1 correspondence.}$$

11. $f(x) = \frac{a(b-a)}{b-x}$ shows $(a, b) \cong (a, +\infty)$. [Draw graph.]

13. $f(x) = \tan x$ shows $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cong \mathbb{R}$. [Draw graph.] Since $(0,1) \cong \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cong \mathbb{R}$, transitivity implies $(0,1) \cong \mathbb{R}$.

15. If $f : A \to B$ and $g : B \to C$ are 1-1 correspondences, then $(f, g) : A \times B \to C \times D$ given by $(f, g)(x, y) = (f(x), g(y))$ is a 1-1 correspondence.

EXERCISE SET 2.9

1. (a) $0, -1$ (b) $1, -1$ (c) $+\infty, -\infty$ (d) $1, -1$ (e) $3, -1$

(f) $0, 0$ (g) $0, 1, +\infty$ (h) $1, 0$

3. Redo the proof of Thm. 2.9.7, making appropriate changes.

5. (a) and (b) False. Take $\{x_n\} = \{1, 0, 1, 0, \cdots\}$ and $\{y_n\} = \{0, 1, 0, 1, \cdots\}$.
(c) and (d) False. Take $\{x_n\} = \{-1, 0, -1, 0, \cdots\}$, $\{y_n\} = \{0, -1, 0, -1, \cdots\}$.

7. Redo #6, changing sup to inf, upper bars to lower bars, etc.

9. Suppose $\{x_n\}, \{y_n\} \geq 0$, $x_n \to x \neq 0$, and $\overline{\lim_{n\to\infty}} y_n = y$. Then y is a cluster point of $\{y_n\}$, so $\{y_n\}$ has a subsequence $y_{n_k} \to y$. Since $x_n \to x$, $x_{n_k} \to x$. So $x_{n_k}y_{n_k} \to xy$. $\therefore$ xy is a cluster point of $\{x_ny_n\}$.

Suppose z is a cluster point of $\{x_ny_n\}$. Then $\exists$ subsequence $x_{m_k}y_{m_k} \to z$. Since $x_n \to x \neq 0$, $\frac{1}{x_n} \to \frac{1}{x}$, so $y_{m_k} = \frac{x_{m_k}y_{m_k}}{x_{m_k}} \to \frac{z}{x}$. $\therefore$ $\frac{z}{x}$ is a cluster point of $\{y_n\}$. By Thm. 2.9.10, y is the largest cluster point of $\{y_n\}$, so $\frac{z}{x} \leq y$. i.e., $z \leq xy$. $\therefore$ xy is the largest cluster point of $\{x_ny_n\}$; i.e., $\overline{\lim_{n\to\infty}} x_ny_n = xy$.

Chapter 3

EXERCISE SET 3.1

2. Only (b), (d), and (h) are open.

3. A° is open, since the union of open sets is open. If U is any open subset of A, then U is a subset of the union of all open subsets of A; i.e., $U \subseteq A^\circ$.

5. $(\Rightarrow)$ Suppose A is open. Then $\forall x \in A$, $\exists \varepsilon_x > 0 \ni N_{\varepsilon_x}(x) \subseteq A$. Then,
$$A = \bigcup_{x\in A} N_{\varepsilon_x}(x).$$
$(\Leftarrow)$ The union of any family of open sets is open, by the "open set theorem."

6. (b) $A^\circ = (-\infty, 0) \cup (0, 1)$ $A^{ext} = (1, +\infty)$ $A^b = \{0, 1\}$
(d) $A^\circ = (-\infty, 1)$ $A^{ext} = (1, +\infty)$ $A^b = \{1\}$
(f) $A^\circ = (-\infty, 0) \cup (0, 1)$ $A^{ext} = (1, +\infty)$ $A^b = \{0, 1\}$
(h) $A^\circ = A$ $A^{ext} = \varnothing$ $A^b = \{1, 2, 3\}$
(j) $A^\circ = \varnothing$ $A^{ext} = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right) \cup (1, +\infty)$ $A^b = A$
(l) $A^\circ = \varnothing$ $A^{ext} = (-\infty, 0) \cup (1, +\infty)$ $A^b = \mathbb{Q} \cap [0, 1]$

7. $\forall x \in \mathbb{R}$, every nbd. of x contains irrational numbers, so $x \notin \mathbb{Q}^\circ$. $\therefore$ $\mathbb{Q}^\circ = \varnothing$.
$\forall x \in \mathbb{R}$, every nbd. of x contains rational numbers, so $x \notin \mathbb{Q}^{co}$. $\therefore$ $\mathbb{Q}^{ext} = \varnothing$.
$\forall x \in \mathbb{R}$, every nbd. of x contains both rational and irrational numbers, so $x \in \mathbb{Q}^b$. $\therefore$ $\mathbb{Q}^b = \mathbb{R}$.

9. ($\Rightarrow$) Suppose A is open. Let $x \in A$. Then $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$, so $N_\varepsilon(x)$ contains no points of A^c, so $x \notin A^b$. $\therefore$ A contains none of its boundary points.

($\Leftarrow$) Suppose A contains no boundary points. Then $\forall x \in A$, $x \notin A^b$, so some nbd. N of x contains no points of A^c; i.e., $N \subseteq A$. $\therefore$ A is open.

11. Apply Def. 3.1.15.

13. (a) 6 (b) none (c) $1, 2, 3, 4, 5, 6, 7, 8, 9$ (d) none (e) $\mathbb{Z}$
(f), (g), (h) none (i), (j) $\{\frac{1}{n} : n \in \mathbb{N}\}$ (k), (l) none

15. If x is an isolated point of A, then x has a nbd N containing no points of A other than x. Then N contains x, a point of A, and points of A^c, so $x \in A^b$. The converse is false since 1 is a boundary point of $[0,1]$ but not an isolated pt.

17. (a) $x \in (A \cap B)^\circ \Rightarrow \exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A \cap B \Rightarrow \exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A$ and $N_\varepsilon(x) \subseteq B \Rightarrow x \in A^\circ \cap B^\circ$.

$x \in A^\circ \cap B^\circ \Rightarrow \exists \varepsilon_1, \varepsilon_2 > 0 \ni N_{\varepsilon_1}(x) \subseteq A$ and $N_{\varepsilon_2}(x) \subseteq B$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $\varepsilon > 0$ and $N_\varepsilon(x) \subseteq A \cap B$, so $x \in (A \cap B)^\circ$.

(b) $x \in A^\circ \cup B^\circ \Rightarrow \exists \varepsilon_1 > 0 \ni N_{\varepsilon_1}(x) \subseteq A \subseteq A \cup B$ or $\exists \varepsilon_2 > 0 \ni N_{\varepsilon_2}(x) \subseteq B \subseteq A \cup B \Rightarrow \exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq A \cup B \Rightarrow x \in (A \cup B)^\circ$.

(c) Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $A^\circ \cup B^\circ = \varnothing$ while $(A \cup B)^\circ = \mathbb{R}$.

19. A is dense in $\mathbb{R} \Leftrightarrow \forall a < b \in \mathbb{R}$, $(a,b) \cap A \neq \varnothing$ (see Defn. 1.5.6). Note that $\forall x \in (a,b)$, $\exists \varepsilon > 0 \ni N_\varepsilon(x) \subseteq (a,b)$. Thus, A is dense in $\mathbb{R} \Leftrightarrow \forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $N_\varepsilon(x) \cap A \neq \varnothing$.

EXERCISE SET 3.2

1. $\varnothing^c = \mathbb{R}$, open; $\{a\}^c = (-\infty, a) \cup (a, +\infty)$, open; $(-\infty, a]^c = (a, +\infty)$, open; etc.

3. (b),(d),(h) open; (c),(e),(g),(j) closed; (a),(f),(i),(k),(l) neither.

5. (a) $[3,5] \cup \{6\}$ (b),(d),(f),(g) $(-\infty, 1]$ (c),(e) none (h),(k) $\mathbb{R}$ (i),(j) $\{0\}$
(l) $[0,1]$

7. No; e.g., $\sup\{1,2\} = 2$, but 2 is not a cluster point of $\{1,2\}$.

Suppose $u = \sup A \notin A$. By the ε-criterion for $\sup A$, $\forall \varepsilon > 0$, $\exists a \in A \ni u - \varepsilon < a < u$ (since $u \notin A$). Thus, every $N_\varepsilon(u)$ contains a point of A other than u.

9. (b) If $x \in A^b$ but $x \notin A$, then every nbd. of x contains a point of A other than x (since $x \notin A$).

(c) If x is a cluster point of A but $x \notin A$, then every nbd. of x contains a point of A and also contains a point (x) of A^c.

11. Let A be a finite set.

(a) Let $x \in \mathbb{R}$. Since A has only finitely many elements, x cannot satisfy the condition of Thm. 3.2.11. So, x is not a cluster point of A.

(b) A contains all its cluster points, since it doesn't have any. $\therefore$ A is closed.

13. Let $\mathcal{C}$ denote the collection of all closed sets containing A. Then

(b) $\forall C \in \mathcal{C}$, $A \subseteq C$ so $A \subseteq \cap\mathcal{C} = \overline{A}$.

(c) $\forall C \in \mathcal{C}$, $\cap\mathcal{C} \subseteq C$. $\therefore$ $\overline{A}$ is a subset of every closed subset of A; i.e., $\overline{A}$ is the smallest closed set containing A.

(d) A is closed $\Leftrightarrow$ A is the smallest closed set containing $A \Leftrightarrow \overline{A} = A$.

(e) The smallest closed set containing $\varnothing$ [or $\mathbb{R}$] is $\varnothing$ [or $\mathbb{R}$].

15. Suppose $A \neq \varnothing$ and A is bounded above. By Exercise 7, $\sup A \in A \cup A'$.

16. (a) $[3,5] \cup \{6\}$ (b), (d), (f), (g) $(-\infty, 1]$ (c) $\{1,2,3,4,5,6,7,8,9\}$ (e) $\mathbb{Z}$
(h), (k) $\mathbb{R}$ (i), (j) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ (l) $\mathbb{Q} \cap [0,1]$

17. $x \in \overline{A} \Leftrightarrow x \in A$ or $x \in A' \Leftrightarrow x \in A$ or every nbd. of x contains a point of A other than $x \Leftrightarrow$ every nbd. of x contains a point of A.

19. Suppose $A \subseteq B$. Then every closed set containing B also contains A. Thus, $A \subseteq \cap\{\text{all closed sets containing } B\} = \overline{B}$. Since $\overline{B}$ is a closed set containing A, $\overline{A} \subseteq \overline{B}$.

21. Let x be a cluster point of A', and $\varepsilon > 0$. Then $N_\varepsilon(x)$ contains a point $a' \in A'$ other than x. Then $a' \in (x-\varepsilon, x)$ or $(x, x+e)$. In either case, $\exists \delta > 0 \ni N_\delta(a') \subseteq N_\varepsilon(x)$ but $x \notin N_\delta(a')$ [draw figure]. Since a' is a cluster point of A, $N_\delta(a')$ contains a point of A other than a (or x). Thus, $N_\varepsilon(x)$ contains a point of A other than x; i.e., $x \in A'$. Therefore, A' contains all its cluster points, and so is closed.

23. (a) A° is open , so $(A^\circ)^\circ = A^\circ$. (Thm. 3.1.11)

(b) $\overline{A}$ is closed, so $\overline{\overline{A}} = \overline{A}$. (Thm. 3.2.15)

(c) $\mathbb{R} - A^{ext} = A^\circ \cup A^b$ (by Thm. 3.1.18) $= \overline{A}$ by Ex. 14.

(d) $\overline{A^c}$ is closed, so $\overline{A^c}^c$ is open. $A^c \subseteq \overline{A^c}$, so $\overline{A^c}^c \subseteq (A^c)^c = A$. Thus, $\overline{A^c}^c$ is an open subset of A. $\therefore$ $\overline{A^c}^c \subseteq A^\circ$.

$A^\circ \subseteq A$, so $A^c \subseteq A^{\circ c}$, which is closed. Thus, $\overline{A^c} \subseteq \overline{A^{\circ c}} = A^{\circ c}$. $\therefore$ $A^\circ = A^{\circ cc} \subseteq \overline{A^c}^c$. Therefore, $A^\circ = \overline{A^c}^c$.

25. ($\Rightarrow$) $x \in \overline{A} \Rightarrow x \in A$ or $x \in A'$ by Thm. 3.2.17. In the former case, take $\{x_n\} = \{x\}$; in the latter, Thm. 3.2.18 guarantees a sequence in A converging to x.

($\Leftarrow$) Suppose $\exists$ sequence in A converging to x. Suppose $x \notin A$. Then $\exists$ sequence of points in A other than x converging to x. Then $x \in A'$ by Thm. 3.2.18. Thus, $x \in A \cup A' = \overline{A}$.

27. ($\Rightarrow$) Suppose A dense in $\mathbb{R}$ by Defn. 1.5.6. Let $r \in \mathbb{R}$ and $\varepsilon > 0$. Then $\exists$ $x \in A \ni r < x < r + \varepsilon$, so $N_\varepsilon(x)$ contains a point of A other than r, so r is a cluster point of A.

($\Leftarrow$) Suppose every real number is a cluster point of A. Let $a < b$. Let $c = \frac{a+b}{2}$ and $\varepsilon = \frac{b-a}{2}$. Since c is a cluster point of A, $N_\varepsilon(c)$ must contain a point x of A other than C. Then $a < x < b$. [Draw figure.]

29. ($\Rightarrow$) Suppose A is dense in B. Let $b \in B$ and $\varepsilon > 0$. Then $b \in \overline{A}$, so by Ex. 25, $\exists$ sequence $\{a_n\}$ in A converging to b. Then every $N_\varepsilon(x)$ contains at least one point of A.

($\Leftarrow$) Suppose that $\forall b \in B$, every nbd of b contains a point of A. Let $b \in B$. Then $\forall n \in \mathbb{N}$, $\exists\ a_n \in A \ni a_n \in N_{\frac{1}{n}}(b)$, so $b \in \overline{A}$. Thus $B \subseteq \overline{A}$; i.e., A is dense in B.

EXERCISE SET 3.3

1. Let $\mathcal{U}$ be an open cover of a finite set $A = \{a_1, a_2, \cdots, a_n\}$. Then $\forall i = 1, 2, \cdots, n$, $\exists\ U_i \in \mathcal{U}$ containing a_i. Then $\{U_1, U_2, \cdots, U_n\}$ is a finite subcover.

3. Part (a):

(a) $\{(-n, n) : n \in \mathbb{N}\}$ (b) $\left\{\left(a + \frac{1}{n}, b - \frac{1}{n}\right) : n > \frac{2}{b-a}\right\}$

(c) $\left\{\left(a + \frac{1}{n}, b + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ (e) $\left\{\left(-\infty : a - \frac{1}{n}\right) : n \in \mathbb{N}\right\}$

(f) $\left\{\left(-n : a + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$ (i) $\left\{N_{\frac{1}{n+1}}\left(\frac{1}{n}\right) : n \in \mathbb{N}\right\}$

Part (b): Apply Thm. 3.3.6 or 3.3.8.

(a),(e),(f),(g),(h),(j),(k),(l) not bounded. (b),(c),(d),(e),(g),(i),(l) not closed.

5. If $\mathcal{U}$ is an open cover of the union of compact sets $A_1, A_2, \cdots, A_n$, then each set A_i can be covered by a finite number m_i of sets in $\mathcal{U}$. Then $\bigcup_{i=1}^n A_i$ can be covered by $m_1 + m_2 + \cdots + m_n$ (a finite number) of sets in $\mathcal{U}$.

7. Suppose $x_n \to L$, and let $S = \{x_n : n \in \mathbb{N}\} \cup \{L\}$. Since a convergent sequence is bounded, S is bounded, and contains its only cluster point, L. So, S is closed, and hence compact.

9. A is nonempty, compact $\Rightarrow$ A is bounded, nonempty $\Rightarrow \exists\ u = \inf A$ and $v = \sup A$. By Ex. 3.2.15, $u, v \in A$ since A is closed. Then $u = \min A$ and $v = \max A$ (Thm. 1.6.5).

11. Let A be compact and $B = \{|x - y| : x, y \in A\}$.

First show B compact using sequential criterion. Let $\{b_n\}$ be a sequence in B. Then $\forall n \in \mathbb{N}$, $\exists\ x_n, y_n \in A \ni b_n = |x_n - y_n|$. Since A compact, $\{x_n\}$ has a subsequence converging to a point of A; say $x_{n_k} \to x \in A$. Similarly $\{y_{n_k}\}$ has a subsequence $y_{n'_k} \to y \in A$. Since $\{n'_k\}$ is a subsequence of $\{n_k\}$, $x_{n'_k} \to x$. Then $b_{n'_k} = |x_{n'_k} - y_{n'_k}| \to |x - y| \in B$. $\therefore B$ is compact.

By Ex. 9, $\sup B \in B$. $\therefore \exists\ x_0, y_0 \in A \ni |x_0 - y_0| = \sup\{|x - y| : x, y \in A\} = d(A)$.

EXERCISE SET 3.4

1. $\mathbf{C} \subseteq [0,1]$, so it is bounded. $\mathbf{C}$ is the intersection of a family of closed sets, so it is closed.

3. $\mathbf{C}$ contains no open intervals, so $\mathbf{C}^\circ = \varnothing$. By Exercise 3.2.23, $\mathbf{C}^b = \overline{C} - \mathbf{C}^\circ$. Since $\mathbf{C}$ is closed $\overline{C} = C$. $\therefore C^b = C - \varnothing = C$.

5. In stage 1 of defining the Cantor set, C_1 is formed by removing from $[0,1]$ every number whose ternary decimal requires "1" as first digit. [e.g., $\frac{1}{3} = 0.10000\cdots = 0.02222\cdots$ remains.] In stage 2, C_2 is formed by removing from C_1 every number whose ternary decimal requires "1" as second digit. In general, C_{n+1} is formed by removing from C_n every number whose ternary decimal requires "1" as n+1st digit. What remains in the Cantor set are those numbers in $[0,1]$ whose ternary decimal can be written using only 0's and 2's.

7. (a) $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] = [0.0, 0.1] \cup [0.2, 1.0]$ in base-3, so $L_1 = \{0.0, 0.2\}$ in base-3.

$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, so $L_2 = \{0.00, 0.02, 0.20, 0.22\}$ in base-3.

In the general induction step, the typical interval comprising C_k is of the form

$$[0.d_1d_2\cdots d_k,\ 0.d_1'd_2'\cdots d_k'] \text{ or } [0.d_1d_2\cdots d_k,\ 1.00\cdots 0],$$

where $d_i \in \{0,2\}$. Thus, the two left endpoints contributed to L_{k+1} by removing the middle third from this interval are, in base-3, $0.d_1d_2\cdots d_k0$ and $0.d_1d_2\cdots d_k2$.

(b) By Thm. 3.4.10, every $x \in \mathbf{C}$ is a ternary decimal $x = \sum_{i=1}^{\infty} \frac{d_i}{3^i}$ where $d_i \in \{0,2\}$. That is, $x = \lim\limits_{n\to\infty} \sum_{i=1}^{n} \frac{d_i}{3^i}$. Since each $\sum_{i=1}^{n} \frac{d_i}{3^i}$ is a terminating base-3 decimal consisting of 0's and 2's, it is a member of L. Thus, every $x \in \mathbf{C}$ is the limit of a sequence of members of L.

9. ($\Rightarrow$) A perfect $\Rightarrow A' = A \Rightarrow A$ closed (Exercise 3.2.21) and A has no isolated points since {isolated points} $= A - A'$ by Exercise 3.2.24.

($\Leftarrow$) A closed with no isolated points $\Rightarrow \overline{A} = A$ and $A - A' = \varnothing \Rightarrow A' \subseteq A$ and $A - A' = \varnothing \Rightarrow A = A'$.

11. A nowhere dense $\Rightarrow \overline{A}$ contains no nonempty open intervals, $\Rightarrow \overline{A}^\circ = \varnothing \Rightarrow A^\circ = \varnothing \Rightarrow A^{c\,cl\,c} = \varnothing$ (Exercise 3.2.23) $\Rightarrow A^{c\,cl} = \mathbb{R} \Rightarrow A^c$ dense in $\mathbb{R}$. The converse is not true since $\mathbb{Q}^c$ is dense in $\mathbb{R}$, but $\mathbb{Q}$ is not nowhere dense.

13. $\mathbb{Q}$ is dense in $\mathbb{R}$ but $\mu(\mathbb{Q}) = 0$ since $\mathbb{Q}$ is countable. (Thm. 3.4.20)

15. (a) $A = (A - B) \cup (A \cap B)$, a union of disjoint sets, so by (μ4), $\mu(A) = \mu(A - B) + \mu(A \cap B)$.

(b) $A \cup B = (A - B) \cup B$, a union of disjoint sets, so by (μ4), $\mu(A \cup B) = \mu(A - B) + \mu(B) = \mu(A) - \mu(A \cap B) + \mu(B)$.

(c) If $B \subseteq A$, then $A \cap B = B$, and by ($\mu 1$) and ($\mu 5$),
$0 \le \mu(A - B) = \mu(A) - \mu(A \cap B) = \mu(A) - \mu(B)$, so $\mu(B) \le \mu(A)$.

17. By ($\mu 5$), $\mu([0,1] - \mathbf{C}) = \mu([0,1]) - \mu(\mathbf{C}) = length[0,1] - 0 = 1$.

Chapter 4

EXERCISE SET 4.1

1. (a) 0.002, 0.0001, $\varepsilon/5$ (b) $0.00\overline{3}$, $0.0001\overline{6}$, $\varepsilon/3$
(c) 0.001, 0.00008, $\min\{1, \varepsilon/7\}$, (d) 0.0008, 0.00004, $\min\{1, \varepsilon/19\}$

2. Choose ε as follows: (a) $\varepsilon/2$ (c) $\min\{1, \varepsilon/11\}$ (e) $\min\{1, \varepsilon/9\}$
(g) $\min\{1, \varepsilon/12\}$ (i) $\varepsilon/3$

3. (e) If $\{x_n\}$ is any sequence in $\mathcal{D}(f) - \{-3\} \ni x_n \to -3$, then by the algebra of limits of sequences, $2x_n^2 + 5x_n + 1 \to 2(-3)^2 + 5(-3) + 1 = 4$.

(i) If $\{x_n\}$ is any sequence in $\mathcal{D}(f) - \{-2\} \ni x_n \to -2$, then by the algebra of limits of sequences, $\frac{3x_n^2 - 12}{x_n + 2} = 3x_n - 6 \to 3(-2) - 6 = -12$.

5. $f(x) = ax + b$. If $a = 0$, choose $\delta = \varepsilon$. Then $0 < |x - x_0| < \delta \Rightarrow |(ax + b) - (ax_0 + b)| = 0 < \varepsilon$. If $a \neq 0$, choose $\delta = \varepsilon/|a|$. Then $0 < |x - x_0| < \delta \Rightarrow |(ax + b) - (ax_0 + b)| = |a(x - x_0)| = |a||x - x_0| < |a| \cdot \varepsilon/|a| = \varepsilon$.

7. First prove that $\lim\limits_{x \to 0} f(x) = 0$. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. Then $0 < |x - 0| < \delta \Rightarrow |f(x) - 0| = |f(x)| = |x| < \varepsilon$. Next, show that if $x_0 \neq 0$, $\lim\limits_{x \to x_0} f(x)$ does not exist.

By Thm. 2.3.6, $\exists$ sequences $\{r_n\}$ of rationals and $\{z_n\}$ of irrationals converging to x_0. Then $f(r_n) = r_n \to x_0$, whereas $f(z_n) = -z_n \to -x_0$. Since $x_0 \neq 0$, $\lim\limits_{x \to x_0} f(r_n) \neq \lim\limits_{x \to x_0} f(z_n)$, so by Cor. 4.1.11 $\lim\limits_{x \to x_0} f(x)$ does not exist.

9. 1 is not a cluster point of $\mathcal{D}(f) = (-\infty, 0] \cup \{1\} \cup [2, +\infty)$.

EXERCISE SET 4.2

1. (b)
$$\begin{aligned} \lim_{x \to x_0} f(x) = L &\Leftrightarrow \forall \varepsilon > 0 \ni 0 < |x - 0| < \delta \Rightarrow |f(x) - L| < \varepsilon \\ &\Leftrightarrow \forall \varepsilon > 0 \ni 0 < |x - 0| < \delta \Rightarrow ||f(x) - L| - 0| < \varepsilon \\ &\Leftrightarrow \lim_{x \to x_0} |f(x) - L| = 0. \end{aligned}$$

(c) Use the inequality in Thm. 1.2.15 (c). A counterexample for the converse is $f(x) = \left\{ \begin{array}{c} 1 \text{ if } x \ge 0 \\ -1 \text{ if } x < 0 \end{array} \right\}$ and $L = 1$.

3. $\forall \varepsilon > 0$, take $\delta = \varepsilon$. Then $0 < |x - x_0| < \delta \Rightarrow |f(x) - x_0| = |x - x_0| < \varepsilon$.

5. By Parts (a) and (b), $\lim_{x\to x_0}[f(x)-g(x)] = \lim_{x\to x_0}[f(x)+(-1)g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0}(-1)g(x) = \lim_{x\to x_0} f(x) + (-1)\lim_{x\to x_0} g(x) = \lim_{x\to x_0} f(x) - \lim_{x\to x_0} g(x)$.

7. (d) $$\lim_{x\to x_0}\frac{f(x)}{f(x)+2g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0}[f(x)+2g(x)]} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} f(x) + \lim_{x\to x_0}[2g(x)]}$$
$$= \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} f(x) + 2\lim_{x\to x_0} g(x)} = \frac{2}{2+2\cdot 3} = \frac{1}{4}.$$

8. (a) 4 (c) $\frac{13}{9}$ (e) $\frac{1}{2}$ (g) 3 (i) 3/2 (k) $\frac{1}{2\sqrt{a}}$ (m) $3a^2$ (o) n

9. For (a),(b),(c), use $f(x) = \left\{\begin{array}{l} 1 \text{ if } x \geq 0 \\ -1 \text{ if } x < 0 \end{array}\right\}$, $g(x) = \left\{\begin{array}{l} -1 \text{ if } x \geq 0 \\ 1 \text{ if } x < 0 \end{array}\right\}$.

11. Suppose $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. Let $\{x_n\}$ be any sequence in $\mathcal{D}(f)\cap\mathcal{D}(g) - \{x_0\}$ converging to x_0. By the sequential criterion for limits of functions, $f(x_n)\to L$ and $g(x_n)\to M$. By the algebra of limits of sequences, $f(x_n)+g(x_n)\to L+M$. Therefore, by the sequential criterion for limits of functions, $\lim_{x\to x_0}[f(x)+g(x)] = L+M$.

13. Done like Exercise 11.

15. $\forall x \neq x_0$, $f(x) = \frac{f(x)}{g(x)}g(x)$. Thus, $\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\frac{f(x)}{g(x)}\cdot\lim_{x\to x_0} g(x) = 0$.

17. Revise the proof of (a) by changing inequalities, etc.

19. Let $\varepsilon > 0$. Then $\exists\ \delta_1, B > 0 \ni 0 < |x-0| < \delta_1 \Rightarrow |g(x)| < B$ and $\exists\ \delta_2 > 0 \ni 0 < |x-0| < \delta_2 \Rightarrow |f(x)| < \varepsilon/B$. Let $\delta = \min\{\delta_1,\delta_2\}$. Then $0 < |x-0| < \delta \Rightarrow |f(x)g(x)| = |f(x)||g(x)| < (\varepsilon/B)B = \varepsilon$. $\therefore \lim_{x\to x_0} f(x)g(x) = 0$.

21. Suppose $\lim_{x\to x_0} f(x) < \lim_{x\to x_0} g(x)$. Let $h(x) = g(x)-f(x)$. Then $\lim_{x\to x_0} h(x) = \lim_{x\to x_0}(g(x)-f(x)) = \lim_{x\to x_0} g(x) - \lim_{x\to x_0} f(x) > 0$. So, by Ex. 20, $\exists\ \delta > 0 \ni 0 < |x-0| < \delta \Rightarrow h(x) > 0$; i.e., $g(x)-f(x) > 0$; i.e., $f(x) < g(x)$.

23. If $0 < |x-x_0| < \delta \Rightarrow |f(x)-L| < \varepsilon$ then $0 < |x-(x_0+p)| = |(x-p)-x_0| < \delta \Rightarrow |(f(x-p)-L| < \varepsilon \Rightarrow |(f(x)-L| < \varepsilon$. The argument is reversible.

EXERCISE SET 4.3

1. (a),(b) $f(x_0^-) = -1$ and $f(x_0^+) = 1$, so $\lim_{x\to x_0} f(x)$ does not exist.

(c),(f) $f(x_0^-) = f(x_0^+) = 0$, so $\lim_{x\to x_0} f(x) = 0$.

(d) $f(x_0^-)$ does not exist; $f(x_0^+) = 0$; $\lim_{x\to x_0} f(x) = 0$.

(e) $f(x_0^-) = 2$ and $f(x_0^+) = 3$, so $\lim\limits_{x \to x_0} f(x)$ does not exist.

(g) $f(x_0^-) = 0$ and $f(x_0^+) = 1$, so $\lim\limits_{x \to x_0} f(x)$ does not exist.

3. If $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) \cap (-\infty, x_0) \ni x_n \to x_0$ but $\{f(x_n)\} \not\to L$ then f does not have limit L from the left at x_0. If $\exists$ sequence $\{x_n\}$ in $\mathcal{D}(f) \cap (x_0, +\infty) \ni x_n \to x_0$ but $\{f(x_n)\} \not\to L$ then f does not have limit L from the right at x_0.

7. If $\lim\limits_{x \to x_0^-} f(x) = L \neq 0$, then f is bounded away from 0 on some interval of the form $(x_0 - \delta, x_0)$. If $\lim\limits_{x \to x_0^+} f(x) = L \neq 0$, then f is bounded away from 0 on some interval of the form $(x_0, x_0 + \delta)$.

15. Since $A \cap B = \varnothing$, the statement, $\forall x \in A \cup B, 0 < |x - x_0| < \delta \Rightarrow |h(x) - L| < \varepsilon$ is equivalent to $\left\{ \begin{array}{l} \forall x \in A, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon, \text{ and} \\ \forall x \in B, 0 < |x - x_0| < \delta \Rightarrow |g(x) - L| < \varepsilon \end{array} \right\}$.

EXERCISE SET 4.4-A

1. (a) Let $M > 1$. Choose $\delta = \frac{1}{M}$. Then $0 < |x| < \delta \Rightarrow 0 < x^2 < |x| < \frac{1}{M} \Rightarrow \frac{1}{x^2} > M$.

(c) Let $M > 0$. Choose $\delta = \frac{1}{\sqrt{M}}$. Then $0 < |x+1| < \delta \Rightarrow \frac{1}{(x+1)^2} > \frac{1}{\delta^2} = M$.

(e) Let $M > 0$. Choose $\delta = \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2M}}\right\}$. Then $0 < |x - 2| < \delta \Rightarrow$
$|x - 2| < \frac{1}{2}$ and $|x - 2| < \frac{1}{\sqrt{2M}} \Rightarrow x - 1 > \frac{1}{2}$ and $|x - 2| < \frac{1}{\sqrt{2M}}$
$\Rightarrow \frac{x-1}{(x-2)^2} > \frac{1}{2} \cdot 2M = M \Rightarrow \frac{1-x}{(x-2)^2} < -M$.

3. (a) $\lim\limits_{x \to x_0^-} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow f(x) > M$.

(b) $\lim\limits_{x \to x_0^-} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow f(x) < -M$.

(c) $\lim\limits_{x \to x_0^+} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni x_0 < x < x_0 + \delta \Rightarrow f(x) > M$.

(d) $\lim\limits_{x \to x_0^+} f(x) = -\infty \Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni x_0 < x < x_0 + \delta \Rightarrow f(x) < -M$.

5. (a) $\lim\limits_{x \to x_0^-} f(x) = +\infty \Leftrightarrow \forall \{x_n\}$ in $\mathcal{D}(f) \cap (-\infty, x_0) \ni x_n \to x_0, f(x_n) \to +\infty$.

(b) $\lim\limits_{x \to x_0^-} f(x) = -\infty \Leftrightarrow \forall \{x_n\}$ in $\mathcal{D}(f) \cap (x_0, +\infty) \ni x_n \to x_0, f(x_n) \to -\infty$.

7. (a) $\lim\limits_{x \to x_0} f(x) = +\infty \Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni 0 < |x - x_0| < \delta \Rightarrow f(x) > M$

$$\Leftrightarrow \forall M > 0, \exists\, \delta > 0 \ni \left\{ \begin{array}{l} x_0 - \delta < x < x_0 \Rightarrow f(x) > M, \text{ and} \\ x_0 < x < x_0 + \delta \Rightarrow f(x) > M. \end{array} \right\}$$

9. Let $M > 0$. Then $\exists\, \delta_1 > 0 \ni 0 < |x - x_0| < \delta_1 \Rightarrow f(x) > \sqrt{M}$,

and $\exists\ \delta_2 > 0 \ni 0 < |x - x_0| < \delta_2 \Rightarrow g(x) > \sqrt{M}$. Then
$0 < |x - x_0| < \min\{\delta_1, \delta_2\} \Rightarrow f(x)g(x) > \sqrt{M}\sqrt{M} = M$.

11. Suppose $f(x) \leq g(x)\ \forall x \in N_{\delta_1}(x_0)$, and suppose $\lim\limits_{x\to x_0} f(x) = +\infty$.
Let $M > 0$. Then $\exists\ \delta_2 > 0 \ni 0 < |x - x_0| < \delta_2 \Rightarrow f(x) > M$. Then
$0 < |x - x_0| < \min\{\delta_1, \delta_2\} \Rightarrow f(x) \leq g(x)$ and $f(x) > M \Rightarrow g(x) > M$.

13. (a) $f(x) = \frac{1}{x^2}$, $g(x) = -\frac{1}{x^2}$, $x_0 = 0$ (b) $f(x) = \frac{2}{x^2}$, $g(x) = -\frac{1}{x^2}$, $x_0 = 0$
(c) $f(x) = \frac{1}{x^2}$, $g(x) = -\frac{2}{x^2}$, $x_0 = 0$ (d) $f(x) = \frac{1}{x^2} + L$, $g(x) = -\frac{1}{x^2}$, $x_0 = 0$

15. (a) $\lim\limits_{x\to 1} \frac{(x-1)^2}{x} = 0$. $\frac{x}{(x-1)^2} > 0$ when $x > 1$ and when $0 < x < 1$, so by Thm. 4.4.3, $\lim\limits_{x\to 1^-} \frac{x}{(x-1)^2} = +\infty$ and $\lim\limits_{x\to 1^+} \frac{x}{(x-1)^2} = +\infty$. $\therefore \lim\limits_{x\to 1} \frac{x}{(x-1)^2} = +\infty$.

(b) $+\infty, +\infty, +\infty$,
(c), (f), (g) $-\infty, +\infty$, does not exist (d), (h) $+\infty, -\infty$, does not exist
(e) $\lim\limits_{x\to 3^-} \frac{x-3}{x^2-1} = 0$. $\frac{x^2-1}{x-3} > 0$ when $x > 3$ and < 0 when $1 < x < 3$, so $\lim\limits_{x\to 3^-} \frac{x^2-1}{x-3} = -\infty$, $\lim\limits_{x\to 3^+} \frac{x^2-1}{x-3} = +\infty$, and $\lim\limits_{x\to 3} \frac{x^2-1}{x-3}$ does not exist.

EXERCISE SET 4.4-B

1. $\lim\limits_{x\to +\infty} f(x) = +\infty \Leftrightarrow \mathcal{D}(f)$ is unbounded above and $\forall M > 0, \exists\ N > 0 \ni$
$x > N \Rightarrow f(x) > M$.

3. $\lim\limits_{x\to -\infty} f(x) = -\infty \Leftrightarrow \mathcal{D}(f)$ is unbounded below and $\forall M > 0, \exists\ N > 0 \ni$
$x < -N \Rightarrow f(x) < -M$.

4. (a) $\left|\frac{3x-1}{6x+5} - \frac{1}{2}\right| = \frac{7}{12x+10} < \frac{7}{12x}$. To make $\frac{7}{12x} < \varepsilon$, take $\frac{12x}{7} > \frac{1}{\varepsilon}$;
i.e., $x > \frac{7}{12\varepsilon}$. Thus, take $N = \frac{7}{12\varepsilon}$. (b) To make $\frac{3x^2+2x-1}{x+4} > M$, take $x > \max\{4, 2m/3\}$. (c) To make $\frac{1-x^2}{x+2} > M$, take $|x| > N$, where $N = \max\{2, M\}$. (d) Take $x < -(\frac{2}{\varepsilon} + 1)$. (e) Take $x > M + 1$. (f) Take $x < -M$.

5. We know $\lim\limits_{x\to -\infty} x^1 = -\infty$, so assume $n \geq 3$ and odd; say $n = 2k+1$. Then $\lim\limits_{x\to -\infty} x^n = \lim\limits_{x\to -\infty} x^{2k+1} = \lim\limits_{x\to -\infty} x^{2k} \cdot \lim\limits_{x\to -\infty} x^1 = +\infty \cdot -\infty = -\infty$ (in the sense of Table 4.1).

7. n even $\Rightarrow \lim\limits_{x\to -\infty} x^n = +\infty$ by Thm. 4.4.18 (b), so by Thm. 4.4.21(c), $\lim\limits_{x\to -\infty} \frac{1}{x^n} = 0$.
n odd $\Rightarrow \lim\limits_{x\to -\infty} x^n = -\infty$ by Thm. 4.4.18 (c), so by Thm. 4.4.21(d), $\lim\limits_{x\to -\infty} \frac{1}{x^n} = 0$.

9. (a) Let $x_n = \pi n$, $y_n = \frac{\pi}{2} + 2n\pi$. Then $x_n \to +\infty$, $y_n \to +\infty$, but $x_n \sin x_n =$
$0 \to 0$, while $y_n \sin y_n = y_n \to +\infty$. $\therefore \lim\limits_{x\to -\infty} x \sin x$ does not exist.

(b) $\forall x \neq 0, 0 \leq \left|\frac{\sin x}{x}\right| \leq \frac{1}{|x|} \to 0$, so by squeeze principle, $\lim\limits_{x\to+\infty} \frac{1}{x}\sin x = 0$.

(c) $\forall x \neq 0, 0 \leq \left|\frac{1}{x}\sin\frac{1}{x}\right| \leq \frac{1}{|x|} \to 0$, so by squeeze, $\lim\limits_{x\to+\infty} \frac{1}{x}\sin\frac{1}{x} = 0$.

11. (a) Suppose $L > 0$. ($\Rightarrow$) Suppose $\lim\limits_{x\to\infty} f(x) = +\infty$. Then $f > 0$ on a nbd of ∞, and $\lim\limits_{x\to\infty} \frac{1}{f(x)} = 0$. Using (4.4.23), $\lim\limits_{x\to\infty} \frac{1}{g(x)} = \lim\limits_{x\to\infty} \frac{1}{f(x)} \cdot \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{1}{f(x)} \cdot \lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0 \cdot L = 0$. So, by Thm. 4.4.21, $\lim\limits_{x\to\infty} g(x) = +\infty$.

($\Leftarrow$) $\lim\limits_{x\to\infty} g(x) = +\infty \Rightarrow \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\frac{f(x)}{g(x)} \cdot g(x)\right) = \lim\limits_{x\to\infty} \frac{f(x)}{g(x)} \cdot \lim\limits_{x\to\infty} g(x) = L \cdot +\infty = +\infty$ since $L > 0$ (Table 4.2 and Thm 4.4.23).

(b) For $L < 0$, modify the proof of (a) appropriately.

13. (a) $y = \frac{1}{3}$ (b) none (c), (d) $y = 0$ (e) none (f) $y = -6$

16. ($\Rightarrow$) Suppose $\lim\limits_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$. Then $\exists\ N \in \mathbb{R} \ni x > N \Rightarrow |f(x) - L| < \varepsilon/2$. Then $x, y > N \Rightarrow |f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

($\Leftarrow$) Let $\varepsilon > 0$. By hypothesis, $\exists\ N \in \mathbb{R} \ni x, y > N \Rightarrow |f(x) - f(y)| < \varepsilon$. Let $\{x_n\}$ be any sequence in $\mathcal{D}(f)$ converging to $+\infty$. Then $\{f(x_n)\}$ is Cauchy so it converges. Thus, $\forall$ sequences $\{x_n\}$ in $\mathcal{D}(f)$ converging to $+\infty$, $\{f(x_n)\}$ converges. We must prove that all these sequences $\{f(x_n)\}$ have the same limit.

Let $\{x_n\}, \{y_n\}$ be sequences in $\mathcal{D}(f)$ converging to $+\infty$. Then the sequence $\{x_1, y_1, x_2, y_2, x_3, y_3, \cdots\} \to +\infty$, so the sequence $\{f(x_1), f(y_1), f(x_2), f(y_2), \cdots\}$ converges. But then any two of its subsequences $\{f(x_n)\}$ and $\{f(y_n)\}$ must converge to the same limit.

Chapter 5
EXERCISE SET 5.1

1. Suppose x_0 is an isolated point of $\mathcal{D}(f)$. Then

(a) $\exists\delta > 0 \ni N'_\delta(x_0)$ contains no point of $\mathcal{D}(f)$. Let $\varepsilon > 0$. Then $\forall x \in \mathcal{D}(f)$, $|x - x_0| < \delta \Rightarrow x = x_0 \Rightarrow |f(x) - f(x_0| = 0 < \varepsilon$, so f is continuous at x_0.

(b) By Exercise 3.2.24, x_0 is not a cluster point of $\mathcal{D}(f)$, so by Defn. 4.1.1, $\lim\limits_{x\to x_0} f(x)$ does not exist.

For $f(x) = \sqrt{x^3 - x^2} = \sqrt{x^2(x-1)}$, $\mathcal{D}(f) = \{0\} \cup [1, \infty)$. Since 0 is an isolated point of $\mathcal{D}(f)$, f is continuous at 0 but $\lim\limits_{x\to 0} f(x)$ does not exist.

3. For $f(x) = \sqrt{x^3 + 2x^2 + x} = \sqrt{x(x-1)^2}$, $\mathcal{D}(f) = \{-1\} \cup [0, \infty)$. Since -1 is an isolated point of $\mathcal{D}(f)$, f is continuous at -1 but $\lim\limits_{x\to -1} f(x)$ does not exist.

5. Let $\varepsilon > 0$. Choose $\delta = \min\{1, \varepsilon/15\}$. Then

$$\begin{aligned}
|x-2| < \delta &\Rightarrow |x-2| < 1 \text{ and } |x-2| < \varepsilon/15 \\
&\Rightarrow 1 < x < 3 \text{ and } |x-2| < \varepsilon/15 \\
&\Rightarrow 7 < 4x + 3 < 15 \text{ and } |x-2| < \varepsilon/15
\end{aligned}$$

$$\Rightarrow |4x+3|\,|x-2| < 15\,\tfrac{\varepsilon}{15} \Rightarrow |(4x^2-5x-3)-3| < \varepsilon$$
$$\Rightarrow |f(x)-f(2)| < \varepsilon.$$

7. Let $p(x)$ be a polynomial. Every x_0 is a cluster point of $\mathcal{D}(f) = (-\infty, \infty)$, and $\lim\limits_{x \to x_0} p(x) = p(x_0)$, by Thm. 4.2.13. Thus, by note (2) following Defn. 5.1.1, $p(x)$ is continuous at x_0.

9. Let $f(x) = |x|$ and $x_0 \in \mathbb{R}$. For $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0| = \big|\,|x| - |x_0|\,\big| \leq |x - x_0| < \varepsilon$ by the triangle inequality, Thm. 1.2.15 (c).

11. (a) 0 is a cluster point of $\mathcal{D}(f) = (-\infty, \infty)$. $\lim\limits_{x \to 0} f(x)$ does not exist, by (4.1.12). So, by note (2) following Defn. 5.1.1, f is not continuous at 0.

(b) 0 is a cluster point of $\mathcal{D}(g) = (-\infty, \infty)$. $\lim\limits_{x \to 0} g(x) = 0 = g(0)$ as shown in (4.2.21). So, by note (2) following Defn. 5.1.1, g is continuous at 0.

13. (a) Continuous on $(-\infty, \infty)$.

(c) Continuous on $(-\infty, -3)$, $(-3, 1)$, and $(1, \infty)$.

(e) Continuous on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

(g) Continuous on $[0, \infty)$.

(h) Continuous on every interval $\left[-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right]$, $n \in \mathbb{Z}$.

(j) Continuous everywhere.

15. Modify the proof of (b) appropriately.

17. Let $\{x_n\}$ be a sequence in $\mathcal{D}(f) \cap \mathcal{D}(g) \ni x_n \to x_0$. By the sequential criterion for continuity (5.1.3), $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$. By Ex. 2.2.21, $\max\{f(x_n), g(x_n)\} \to \max\{f(x_0), g(x_0)\}$ and $\min\{f(x_n), g(x_n)\} \to \min\{f(x_0), g(x_0)\}$. $\therefore$ Therefore, $\max\{f, g\}$ and $\min\{f, g\}$ are continuous at x_0.

19. $f(x) = 1$ if x is rational, -1 if x is irrational.

21. Use $f(x) = \left\{\begin{array}{c} 1 \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{array}\right\}$, $g(x) = \left\{\begin{array}{c} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{array}\right\}$.

23. (a) $f^{-1}(-\infty, a) = \bigcup_{n=1}^{\infty} f^{-1}(a-n, a)$. Each $f^{-1}(a-n, a)$ is open by Ex. 22, and the union of open sets is open. Similarly, $f^{-1}(a, +\infty) = \bigcup_{n=1}^{\infty} f^{-1}(a, a+n)$ and each $f^{-1}(a, a+n)$ is open.

(b) $f^{-1}(-\infty, a] = \mathbb{R} - f^{-1}(a, +\infty)$; $\quad f^{-1}[a, +\infty) = \mathbb{R} - f^{-1}(-\infty, a)$.

(c) $f^{-1}(a) = \mathbb{R} - \left(f^{-1}(-\infty, a) \cup f^{-1}(a, +\infty)\right)$, so it is closed.

25. Suppose f is continuous at x_0. Then $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| \leq 1 \Rightarrow ||f(x)| - |f(x_0)|| \leq 1$ (Thm. 1.2.15 (c)) $\Rightarrow |f(x)| \leq |f(x_0)| + 1$.

27. ($\Rightarrow$) Suppose f is continuous on $\mathbb{R}$, let U be open, and $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$, which is open, so $\exists \varepsilon > 0 \ni f(x_0) \in N_\varepsilon(f(x_0)) \subseteq U$. By continuity, $\exists \delta > 0 \ni x \in N_\delta(x_0) \Rightarrow |f(x) - f(x_0)| < \varepsilon \Rightarrow f(x) \in N_\varepsilon(f(x_0)) \subseteq U \Rightarrow x \in f^{-1}(U)$. Thus, $\forall x_0 \in f^{-1}(U)$, $\exists \delta > 0 \ni N_\delta(x_0) \subseteq f^{-1}(U)$. $\therefore f^{-1}(U)$ is open.

($\Leftarrow$) Suppose that $\forall$ open set U, $f^{-1}(U)$ is open. Let $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Then $f^{-1}(N_\varepsilon(f(x_0))$ is open. Since $x_0 \in f^{-1}(N_\varepsilon(f(x_0))$, which is open, $\exists \delta > 0 \ni N_\delta(x_0) \subseteq f^{-1}(N_\varepsilon(f(x_0))$; i.e., $f(N_\delta(x_0)) \subseteq N_\varepsilon(f(x_0))$. This means $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Therefore, f is continuous at x_0.

29. Suppose $f, g : A \to \mathbb{R}$ are continuous and $f(x) = g(x)$ for all x in a dense subset B of A. Define $h(x) = f(x) - g(x)$. Then h is continuous on A and $\forall x \in B$, $h(x) = 0$. By Ex. 28, $\forall x \in A$, $h(x) = 0$.

31. $f =$ the Dirichlet function, $A = \mathbb{Q}$, $B = \mathbb{Q}^c$.

EXERCISE SET 5.2

1. ($\Rightarrow$) Suppose f is continuous from the left at x_0. Let $\{x_n\}$ be a sequence in $\mathcal{D}(f) \cap (-\infty, x_0)$ converging to x_0. Let $\varepsilon > 0$. Then $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $x_0 - \delta < x < x_0 \Rightarrow |f(x) - f(x_0)| < \varepsilon$ and $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow x_0 - \delta < x_n < x_0 \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$. $\therefore f(x_n) \to f(x_0)$.

($\Leftarrow$) Suppose f is not continuous from the left at x_0. Modify the procedure used in the proof of Thm. 4.1.9 to find a sequence $\{x_n\}$ in $\mathcal{D}(f) \cap (-\infty, x_0)$ converging to x_0 such that $f(x_n) \not\to f(x_0)$.

3. (a) Let $x_0 \notin \mathbb{Z}$. Then $\exists \delta > 0 \ni \lfloor x \rfloor$ is constant on $N_\delta(x_0)$, say $\lfloor x \rfloor = c$. Then, $\lim_{x \to x_0} \lfloor x \rfloor = c = \lfloor x_0 \rfloor$.

(b), (c) Let $x_0 \in \mathbb{Z}$. Then $\exists \delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow \lfloor x \rfloor = x_0 - 1$ and $x_0 < x < x_0 + \delta \Rightarrow \lfloor x \rfloor = x_0$. Thus,

$$\lim_{x \to x_0^-} \lfloor x \rfloor = x_0 - 1 \neq \lfloor x_0 \rfloor \quad \text{while} \quad \lim_{x \to x_0^+} \lfloor x \rfloor = x_0 = \lfloor x_0 \rfloor.$$

5.

	not cont. from left at	not cont. from right at
(a)	b	a
(b)	a	b
(c)	a, b	nowhere
(d)	nowhere	a, b
(e)	a	nowhere
(f)	nowhere	a
(g)	nowhere	a
(h)	a	nowhere

7. f has a removable discontinuity at $x_0 \Rightarrow \lim_{x\to x_0} f(x)$ exists, but either

(a) $f(x_0)$ does not exist, in which case f is not continuous from the left or from the right at x_0, or

(b) $\lim_{x\to x_0} f(x) = L \neq f(x_0)$. Then $\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L \neq f(x_0)$, in which case f is not continuous from the left or from the right at x_0.

9. f is monotone increasing and bounded on (a, b), so

$$a < x < c \Rightarrow f(x) \leq f(c);\ \sup\{f(x) : a < x < c\} \leq f(c);\ \lim_{x\to c^-} f(x) \leq f(c);$$
$$c < x < b \Rightarrow f(x) \geq f(c);\ \inf\{f(x) : c < x < b\} \geq f(c);\ \lim_{x\to c^+} f(x) \geq f(c).$$

11. Modify appropriately the proofs given in Thm. 5.2.17 and Exercises 8–10.

13. Modify the proof given for the monotone increasing case, using Thm. 5.2.18 instead of Thm. 5.2.17.

15. By Thm. 5.2.17, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist. Define $f(a) = \lim_{x\to a^+} f(x)$ and $f(b) = \lim_{x\to b^-} f(x)$. Then $f : [a, b] \to \mathbb{R}$ is continuous on (a, b) and, by Thm. 5.2.17, f is monotone increasing on $[a, b]$. Since $\lim_{x\to a^+} f(x) = f(a)$, $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $a < x < a + \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. Since $\mathcal{D}(f) = [a, b]$, this is equivalent to $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \forall x \in \mathcal{D}(f)$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. $\therefore f$ is continuous at a.

Continuity at b is proved similarly.

17. (a) Suppose f is monotone increasing and bounded on $I = (a, b)$, and $c \in (a, b]$. By Thm. 5.2.17(a), $\lim_{x\to c} f(x)$ exists and equals $\sup A$, where $A = \{f(x) : a < x < c\}$. Let $B = \{f(r) : r \in \mathbb{Q},\ a < r < c\}$.

Since $B \subseteq A$, $\sup B \leq \sup A$. On the other hand,
$a < x < c \Rightarrow \exists r \in \mathbb{Q} \ni x < r < c \Rightarrow \exists r \in \mathbb{Q} \ni f(x) \leq f(r) \Rightarrow f(x) \leq \sup B$.
$\therefore \sup A \leq \sup B$. We have proved that $\sup A = \sup B$.

The other parts are proved similarly.

19. As shown in Example 4.2.21, the graph of f "oscillates" between the lines $y = x$ and $y = -x$ infinitely often in any nbd of 0, and yet $\lim_{x\to 0} f(x) = 0$. Since $\lim_{x\to 0} f(x)$ exits but is unequal to $f(0)$, f has a removable discontinuity at 0.

EXERCISE SET 5.3

1. We must prove that Statement #2 $\Rightarrow$ Statement #1 $\Leftrightarrow$ A is open.

(a) Assume A open and Stmnt. #2 true; i.e., $f : A \to \mathbb{R}$ is continuous. Let $\varepsilon > 0$ and $x_0 \in A$. Then $\exists \delta_1 > 0 \ni \forall x \in A$, $|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Since A open, $\exists \delta_2 > 0 \ni N_{\delta_2}(x_0) \subseteq A$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x \in \mathcal{D}(f)$, $|x - x_0| < \delta \Rightarrow x \in A$ and $|f(x) - f(x_0)| < \varepsilon$. $\therefore f$ is continuous on A. (Stmnt. #1)

(b) Suppose A not open. Let $g : \mathbb{R} \to \mathbb{R}$ denote the characteristic function of A. Then

(1) for $x_0 \in A$ and $\varepsilon > 0$ take $\delta = 1$. Then $\forall x \in A$, $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| = |1 - 1| = 0 < \varepsilon$, so $g : A \to \mathbb{R}$ is continuous. (Stmnt. #2.)

(2) since A not open, $\exists x_0 \in A \ni \forall \delta > 0$, $N_\delta(x_0) \nsubseteq A$. Thus, $\forall n \in \mathbb{N}$, $\exists a_n \in N_{1/n}(x_0) \ni a_n \notin A$. Then, $\forall n \in \mathbb{N}$, $|a_n - x_0| < \frac{1}{n}$ but $a_n \notin A$. So, $a_n \to x_0$ but $g(a_n) = 0$ while $g(x_0) = 1$. $\therefore g : \mathbb{R} \to \mathbb{R}$ is not continuous at x_0. (Stmnt. #1 not true.)

Therefore, when A is not open, Statement #2 $\not\Rightarrow$ Statement #1.

3. Let A be a nonempty compact set; i.e., closed and bounded. Then $\exists u = \inf A$, $v = \sup A$. By Ex. 3.2.15, $u, v \in \overline{A}$. But A closed, so $u, v \in A$. $\therefore u = \min A$, $v = \max A$.

5. By Thm. 5.3.6, $f(A)$ is compact, hence closed and bounded. Thus, $\exists u = \inf f(A)$ and $v = \sup f(A)$. By Ex. 3.2.15, $u, v \in \overline{f(A)} = f(A)$ since $f(A)$ is closed. $\therefore u = \min f(A)$ and $v = \max f(A)$.

7. Let $B = \{x \in A : f(x) = c\}$ and $\{x_n\}$ be any convergent sequence in B, say $x_n \to L$. Since A is closed, $L \in A$. By continuity of f on A, $f(x_n) \to f(L)$. But $\{f(x_n)\}$ is the constant sequence $\{c\}$, so $f(x_n) \to c$. $\therefore f(L) = c$. $\therefore L \in B$. By the sequential criterion for closed sets, B is closed.

9. By Thm. 5.3.6, $f(A)$ is compact, so $\exists u = \inf f(A)$. By Cor. 5.3.7, $u \in f(A)$, so $u > 0$. $\therefore \forall x \in A$, $f(x) \geq u > 0$.

11. Define $\{x_n\}$ by $x_1 =$ arbitrary element of $[a, b]$ and $x_{n+1} \in [a, b] \ni |f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$. Then $|f(x_n)| \leq \frac{1}{2^n}|f(x_1)| \to 0$. $\therefore f(x_n) \to 0$.

Since $\{x_n\}$ is bounded, it has a convergent subsequence; say $x_{n_k} \to x_0$. Since $\{x_n\}$ is in the closed set $[a, b]$, $x_0 \in [a, b]$. Then f is continuous at x_0, so $\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$. $\therefore f(x_0) = 0$.

13. $x = 1.705$ **14.** $x_1 = 2.092$, $x_2 = -1.572$

15. Let $p(x)$ be a polynomial of odd degree. By Thm. 4.4.24, $\lim_{x \to \infty} p(x) = \pm\infty$ and $\lim_{x \to -\infty} p(x) = -\lim_{x \to \infty} p(x)$. Thus, $\exists\ a, b \ni p(a) < 0$ and $p(b) > 0$. By the intermediate value theorem, $\exists c$ between a and $b \ni p(c) = 0$.

17. Let $h(x) = \cos x - x$. Then h is continuous on $[0, \frac{\pi}{2}]$ and $h(0) = 1 > 0$, while $h(\frac{\pi}{2}) = -\frac{\pi}{2} < 0$. Apply the intermediate value theorem.

19. Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$ and $h(a) = f(a) - g(a) < 0$ while $h(b) = f(b) - g(b) > 0$. Thus 0 is between $h(a)$ and $h(b)$, so $\exists c \in (a, b) \ni h(c) = 0$; i.e., $f(c) = g(c)$.

21. Suppose $f : I \to \mathbb{R}$ is continuous, strictly monotone, and bounded on $I = (a, b)$. Let $c = \inf f(I)$ and $d = \sup f(I)$.

Case 1: (f strictly increasing on I) In this case,

(a) $\forall x \in (a, b)$, $f(x) \in f(I)$ so $c < f(x) < d$. Thus, $f(I) \subseteq (c, d)$.

(b) $\forall y \in (c, d)$, $c < y < d$, so by definition of inf and sup, $\exists\, y_1, y_2 \in f(I) \ni c < y_1 < y < y_2 < d$. Then, $\exists\, x_1, x_2 \in I \ni f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is strictly increasing on I, $x_1 < x_2$. Then, by the intermediate value theorem, $\exists x \in I \ni f(x) = y$; i.e., $(c, d) \subseteq f(I)$.

(c) Therefore, $f(I) = (c, d)$.

Extension to $[a, b]$: Define $f(a) = c$ and $f(b) = d$. By Thm. 5.2.17, $\lim_{x \to a^+} f(x) = \inf f(I) = c = f(a)$ and $\lim_{x \to b^-} f(x) = \sup f(I) = d = f(b)$. Then f is continuous on $[a, b]$ and $f[a, b] = [c, d]$.

To see that f is strictly increasing on $[a, b]$, let $a < x < b$. Then $\exists\, x_1, x_2 \in I \ni a < x_1 < x < x_2 < b$. So, $f(a) = c = \inf f(I) \leq f(x_1) < f(x) < f(x_2) \leq \sup f(I) = d = f(b)$; i.e., $f(a) < f(x) < f(b)$.

Case 2: (f strictly decreasing on I) Modify the proof of Case 1.

23. Let $y_0 \in f(A)$. We shall prove f^{-1} continuous at y_0. Let $\{y_n\}$ be a sequence in $f(A) \ni y_n \to y_0$. Then $\exists\, x_n, x_0 \in A \ni f(x_n) = y_n$ and $f(x_0) = y_0$ Since A compact, $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be any *convergent* subsequence of $\{x_n\}$, say $x_{n_k} \to L$. Since A is closed, $L \in A$. Since f continuous on A, $f(x_{n_k}) \to f(L)$; i.e., $y_{n_k} \to f(L)$. But $y_{n_k} \to y_0$, so by the uniqueness of limits, $f(L) = y_0 = f(x_0)$. Since f is 1-1 on A, $L = x_0$. That is, $x_{n_k} \to x_0$. Thus, all convergent subsequences of $\{x_n\}$ have the same limit, x_0. By Exercise 2.6.21, this means $x_n \to x_0$; i.e., $f^{-1}(y_n) \to f^{-1}(y_0)$. $\therefore f^{-1}$ is continuous at y_0.

25. (a) Define g:$\left[a, \frac{a+b}{2}\right] \to \mathbb{R}$ by $g(x) = f(x) - f\left(x + \frac{b-a}{2}\right)$. If $g(a) = 0$, take $x = a$, $y = \frac{a+b}{2}$. Suppose $g(a) \neq 0$. Then $g\left(\frac{a+b}{2}\right) = -g(a)$, so by the intermediate value theorem, $\exists\, c \in \left[a, \frac{a+b}{2}\right] \ni g(c) = 0$; i.e., $f(c) = f\left(c + \frac{b-a}{2}\right)$. Take $x = c$ and $y = c + \frac{b-a}{2}$.

(b) By (a), $\ni x_1, y_1 \in [a, b] \ni y_1 - x_1 = \frac{1}{2}(b - a)$ and $f(x_1) = f(y_1)$. Continuing by induction, $\forall n \in \mathbb{N}, \exists\, x_n, y_n \in [a, b] \ni y_n - x_n = \frac{1}{2}(y_{n-1} - x_{n-1}) = \frac{1}{2^n}(b - a)$ and $f(x_n) = f(y_n)$. Since $\frac{1}{2^n}(b - a) \to 0$, $\exists n_0 \in \mathbb{N} \ni \frac{1}{2^{n_0}}(b - a) < \varepsilon$.

27. ($\Leftarrow$) Suppose every continuous $f : A \to \mathbb{R}$ has a max and a min on A. Since the function $i : A \to \mathbb{R}$ given by $i(x) = x$ is continuous, A must have a max and a min, so A is bounded. We must prove A closed. Let $x_0 \in A^c$. Define $f(x) = \frac{1}{|x - x_0|}$. Then f is continuous on A since $x_0 \notin A$, so by hypothesis f has a maximum value on A. That is, $\exists M > 0 \ni \forall x \in A$, $f(x) \leq M$; i.e., $\frac{1}{|x - x_0|} \leq M$. Then $\forall x \in A$, $|x - x_0| \geq \frac{1}{M}$. Thus, $N_{\frac{1}{M}}(x_0)$ contains no point of A, so $N_{\frac{1}{M}}(x_0) \subseteq A^c$. $\therefore A^c$ is open, so A is closed.

EXERCISE SET 5.4

1. For $[-3,1]$, choose $\delta = \varepsilon/17$. For $(-2,2)$, choose $\delta = \varepsilon/13$.

3. Let $x_0 \in A$ and $\varepsilon > 0$. Since f is uniformly continuous on A, $\exists \delta > 0 \ni \forall x, y \in A$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Thus, $\forall x \in A$, $|x - x_0| < \delta \Rightarrow |f(x) - fx_0)| < \varepsilon$. That is, f is continuous at x_0.

5. $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| = |(7x-8) - (7y-8)| = 7|x-y|$. For $\varepsilon > 0$, choose $\delta = \varepsilon/7$. Then $|x - x_0| < \delta \Rightarrow |f(x) - f(y)| = 7|x-y| < 7 \cdot \varepsilon/7 = \varepsilon$.

7. $\forall x, y \in [1, +\infty)$, $|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} \leq |x-y|$.

9. In Section 5.1 we showed that $\forall x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$.

11. All but (e) are uniformly continuous on the given interval.

13. Suppose f, g are uniformly continuous on A. Let $\varepsilon > 0$. Then $\exists\ \delta_1, \delta_2 > 0 \ni |x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \varepsilon/2$ and $|x - y| < \delta_2 \Rightarrow |g(x) - g(y)| < \varepsilon/2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then $\forall x, y \in A$, $|x - x_0| < \delta \Rightarrow$

$$|f(x) + g(x) - (f(y) + g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

15. As shown in Section 5.1, $\sin x$ and $\cos x$ are continuous everywhere. Thus, by Thm. 5.1.13, $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ are continuous wherever $\cos x \neq 0$. Hence, they are continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. But they are not uniformly continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ since they are unbounded there.

17. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{R} \ni x \geq N \Rightarrow |f(x) - L| < \varepsilon/4$. Thus, $x, y \geq N \Rightarrow |f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \varepsilon/2$.

Since f is continuous on $[a, N]$, it is uniformly continuous there. So $\exists \delta > 0 \ni \forall x, y \in [a, N]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$.

Thus, for all $x, y \geq a$ and $|x - y| < \delta$,
(i) if $x, y \in [a, N]$, then $|f(x) - f(y)| < \varepsilon$;
(ii) if $x, y \in (N, \infty)$, then $|f(x) - f(y)| < \varepsilon$;
(iii) if $x \in [a, N]$ and $y \in (N, \infty)$, then

$$|f(x) - f(y)| \leq |f(x) - f(N)| + |f(N) - f(y)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

$\therefore f$ is uniformly continuous on $[a, \infty)$.

19. Let $\varepsilon > 0$. Then

$\exists \delta_1 > 0 \ni \forall u, v \in f(A)$, $|u - v| < \delta_1 \Rightarrow |g(u) - g(v)| < \varepsilon$, and
$\exists \delta_2 > 0 \ni \forall x, y \in A$, $|x - y| < \delta_2 \Rightarrow |f(x) - f(y)| < \delta_1$

$$\Rightarrow |g(f(x)) - g(f(y))| < \varepsilon \Rightarrow |(g \circ f)(x) - (g \circ f)(y)| < \varepsilon.$$

Therefore, $g \circ f$ is uniformly continuous on A.

21. By Cor. 5.1.15, f is continuous on $[0,1]$, so it is uniformly continuous there. But $\forall x, y \in [0,1]$, $\frac{|\sqrt{x} - \sqrt{y}|}{|x-y|} = \frac{1}{\sqrt{x} + \sqrt{y}}$, which is unbounded on $[0,1]$. Hence, $\nexists K > 0 \ni \frac{|\sqrt{x} - \sqrt{y}|}{|x-y|} \leq K$. $\therefore f$ cannot satisfy a Lipschitz condition on $[0,1]$.

23. (a) Suppose f is uniformly continuous on (a,b) and (c,d), where $a<b<c<d$. Let $\varepsilon>0$. Then $\exists\delta_1,\delta_2>0\ni$

$$\forall x,y\in(a,b),\ |x-y|<\delta_1\Rightarrow|f(x)-f(y)|<\varepsilon,\text{ and}$$
$$\forall x,y\in(c,d),\ |x-y|<\delta_2\Rightarrow|f(x)-f(y)|<\varepsilon.$$

Choose $\delta\le\min\{\delta_1,\delta_2,c-b\}$. Then $\forall x,y\in(a,b)\cup(c,d)$,

$$|x-y|<\delta\Rightarrow x,y\in(a,b)\text{ or }x,y\in(c,d),\text{ and }|x-y|<\delta_1\text{ and }\delta_2$$
$$\Rightarrow|f(x)-f(y)|<\varepsilon.$$

$\therefore f$ is uniformly continuous on $(a,b)\cup(c,d)$.

EXERCISE SET 5.5

1. Suppose f is monotone increasing on A and $x_1,x_2\in A$. Then $x_2\le x_1\Rightarrow f(x_2)\le f(x_1)$, so $f(x_1)<f(x_2)\Rightarrow f(x_2)\not\le f(x_1)\Rightarrow x_2\not\le x_1\Rightarrow x_1<x_2$.

3. (a) $x<y$ in $I\Rightarrow[f(x)\le f(y)$ and $g(x)\le g(y)]\Rightarrow f(x)+g(x)\le f(y)+g(y)$. $\therefore f+g$ is monotone increasing on I.

(b) Take $f(x)=g(x)=x$ on $[-1,1]$.

(c) Suppose f,g nonnegative on I. Then $\forall x,y\in I$,

$x<y\Rightarrow[f(x)\le f(y)$ and $g(x)\le g(y)]\Rightarrow f(x)g(x)\le f(y)g(x)\le f(y)g(y)$.

$\therefore fg$ is monotone increasing on I.

5. Take $f(x)=x$ if x is rational, $1-x$ if x is irrational.

7. Redo the proof of Thm. 5.5.2, changing "increasing" to "decreasing" and changing inequalities appropriately.

9. Suppose $a<b<c$ in I and $f(b)<\min\{f(a),f(c)\}$. Then $\exists r\in\mathbb{R}\ni f(b)<y<\min\{f(a),f(c)\}$. Then $f(b)<y<f(a)$, so by the intermediate value thm, $\exists x_1\in(a,b)\ni f(x_1)=y$. Similarly, $f(b)<y<f(c)$, so $\exists x_2\in(a,b)\ni f(x_2)=y$. Then we have $x_1\ne x_2$ in $I\ni f(x_1)=f(x_2)$, contradicting the hypothesis that f is 1-1 on I. $\therefore$ in this case, $f(b)$ is between $f(a)$ and $f(c)$.

11. Let $x\in\mathbb{R}$. Then $\exists$ unique $n\in\mathbb{N}\ni x\in[n,n+1)$, so $0\le x-n<1$. Define $\overline{\varphi}(x)=\varphi(x-n)+n$. Since $\varphi:[0,1)\to[0,1)$ is continuous and increasing, $\overline{\varphi}:[n,n+1)\to[n,n+1)$ is continuous and increasing. Also, $\forall n\in\mathbb{N}$, $\lim\limits_{x\to(n+1)^-}\overline{\varphi}(x)=\lim\limits_{x\to(n+1)}\varphi(x-n)+n=\varphi(1)+n=1+n=\overline{\varphi}(n+1)$. Thus, $\lim\limits_{x\to n^-}\overline{\varphi}(x)=\overline{\varphi}(n)$ and $\lim\limits_{x\to n^+}\overline{\varphi}(x)=\lim\limits_{x\to n+}\varphi(x-n)+n=\varphi(0)+n=n=\overline{\varphi}(n)$, so $\overline{\varphi}$ is continuous on $\mathbb{R}$. Moreover, $\overline{\varphi}$ is increasing on $\mathbb{R}$ since $\overline{\varphi}[n,n+1)\subseteq[n,n+1)$.

EXERCISE SET 5.6

1. Let $x\in\mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists$ rational $r_1\ni x-1<r_1<x$. Then, $\exists$ rational $r_2\ni\max\{x-\frac{1}{2},r_1\}<r_2<x$. Continuing inductively, $\exists$ rational

$r_{k+1} \ni \max\{x - \frac{1}{k+1}, r_k\} < r_{k+1} < x$. Then, $\forall n \in \mathbb{N}$, $x - \frac{1}{n} < r_n < r_{n+1} < x$, so $\{r_n\}$ is a strictly increasing sequence of rational numbers converging to x.

3. By (b), $a^{x-y}a^y = a^x$, so $a^{x-y} = a^x/a^y$ since $a^y \neq 0$ by Thm. 5.6.6.

5. By (d) and (e), $\left(\frac{a}{b}\right)^x = (ab^{-1})^x = a^x(b^{-1})^x = a^x(b^x)^{-1} = \frac{a^x}{b^x}$.

7. Let $f(x) = a^x$, $0 < a < 1$. Then $f(x) = 1/g(x)$ where $g(x) = \left(\frac{1}{a}\right)^x$. Then $\frac{1}{a} > 1$ so by Thm. 5.6.11 g is continuous and strictly increasing on $\mathbb{R}$ with range $(0, \infty)$. Thus, f is continuous and strictly decreasing on $\mathbb{R}$ with range $(0, \infty)$.

9. Let $t < 0$ and consider the power function $f(x) = x^t$. Then $-t > 0$. By Thm. 5.6.14, the function $g(x) = x^{-t}$ is strictly increasing, positive, and continuous on $(0, \infty)$ with $\lim\limits_{x\to\infty} g(x) = +\infty$. Since $f(x) = 1/g(x)$, f is positive, strictly decreasing, and continuous on $(0, \infty)$ with $\lim\limits_{x\to\infty} f(x) = +0$.

11. By Thm. 5.6.17, $\lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$. Thus,

(i) by Thm. 4.4.19, $\lim\limits_{x\to 0+}(1+x)^{1/x} = \lim\limits_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$;

(ii) by Ex. 10, $\lim\limits_{x\to -\infty}\left(1 + \frac{1}{x}\right)^x = e$, so by Thm. 4.4.19, $\lim\limits_{x\to 0-}(1+x)^{1/x} = e$.

$\therefore \lim\limits_{x\to 0}(1+x)^{1/x} = e$.

13. (a) $f : (0, \infty) \to \mathbb{R}$, and is 1-1 and onto, by definition of inverse.

(b) f is continuous on $(0, \infty)$ by Cor. 5.5.3.

(c) The function $g(x) = a^x$ is strictly increasing if $a > 1$ and strictly decreasing if $0 < a < 1$, by Thms. 5.6.11 and 5.6.12. Thus, $f = g^{-1}$ has the same properties by Cor. 5.5.3.

(d) $a^0 = 1 \Rightarrow g(0) = 1 \Rightarrow g^{-1}(1) = 0 \Rightarrow f(1) = 0$.

(e) Suppose $a > 1$. Since $\log_a x$ is strictly decreasing, $x > 1 \Rightarrow \log_a x > \log_a 1 = 0$ and $0 < x < 1 \Rightarrow \log_a x < \log_a 1 = 0$.

(g) Note that $a^x > 0$ and $\log_a(a^x) = f(g(x)) = f(f^{-1}(x)) = x$.

15. Let $a, b > 0$, $a, b \neq 1$, and $x > 0$. Let $y = \log_b x$. Then $b^y = x$, so $y \log_a b = \log_a x$.

Chapter 6

EXERCISE SET 6.1

1. (b) $\lim\limits_{x\to x_0} \dfrac{x^3 - x_0^3}{x - x_0} = \lim\limits_{x\to x_0}(x_0^2 + x_0 x + x^2) = 3x_0^2$.

(d) $\lim\limits_{x\to x_0} \dfrac{\frac{x+1}{x-1} - \frac{x_0+1}{x_0-1}}{x - x_0} = \lim\limits_{x\to x_0} \dfrac{2(x_0 - x)}{(x - x_0)(x-1)(x_0-1)} = \dfrac{-2}{(x_0-1)^2}$.

(f) $\displaystyle\lim_{x\to x_0}\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{x_0}}}{x-x_0}=\lim_{x\to x_0}\frac{\sqrt{x}-\sqrt{x_0}}{\sqrt{x}\sqrt{x_0}(x-x_0)}=\lim_{x\to x_0}\frac{-1}{\sqrt{x}\sqrt{x_0}(\sqrt{x}+\sqrt{x_0})}$
$\displaystyle=\frac{-1}{2x_0\sqrt{x_0}}.$

2. (e) $\displaystyle\lim_{h\to 0}\frac{\sqrt{4(x+h)-1}-\sqrt{4x-1}}{h}=\lim_{h\to 0}\frac{(4x+4h-1)-(4x-1)}{h\left[\sqrt{4(x+h)-1}+\sqrt{4x-1}\right]}$
$\displaystyle=\lim_{h\to 0}\frac{4h}{h\left[\sqrt{4(x+h)-1}+\sqrt{4x-1}\right]}=\frac{2}{\sqrt{4x-1}}.$

3. (b) $\displaystyle\lim_{x\to 0^-}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to 0^-}\frac{0}{x}=\lim_{x\to 0^-}0=0$, and

$\displaystyle\lim_{x\to 0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to 0^+}\frac{x^2}{x}=\lim_{x\to 0^+}x=0;\qquad \therefore \lim_{x\to 0}\frac{f(x)-f(x_0)}{x-x_0}=0.$

(c) yes (d) no

5. (a) $\displaystyle\lim_{x\to 0}\left|\frac{f(x)-f(x_0)}{x-x_0}\right|=\lim_{x\to 0}\left|\frac{f(x)}{x}\right|\le\lim_{x\to 0}\frac{x^2}{|x|}=\lim_{x\to 0}|x|=0.$ $\therefore$ f is differentiable at 0 and $f'(0)=0$.

(b) Suppose $x\neq 0$. Then $\exists$ sequence $\{r_n\}$ of rational numbers converging to x, and a sequence $\{z_n\}$ of irrationals converging to x. If x is rational, then $\displaystyle\lim_{n\to\infty}\left|\frac{f(z_n)-f(x)}{z_n-x}\right|=\lim_{n\to\infty}\left|\frac{0-x^2}{z_n-x}\right|=+\infty$, so f is not differentiable at x. If x is irrational, then $\displaystyle\lim_{n\to\infty}\left|\frac{f(r_n)-f(x)}{r_n-x}\right|=\lim_{n\to\infty}\left|\frac{r_n^2-0}{r_n-x}\right|=+\infty$, so f is not differentiable at x. $\therefore$ f is not differentiable at any $x\neq 0$.

7. (a) Continuous everywhere; differentiable $\Leftrightarrow x\neq 0$; $f'(x)=0$ if $x<0$, 2 if $x>0$; f' continuous $\Leftrightarrow x\neq 0$.

(c) Continuous everywhere; differentiable $\Leftrightarrow x\neq n\pi$ $(n\in\mathbb{Z})$; $f'(x)=(-1)^n\cos x$ if $x\in(n\pi,(n+1)\pi)$; f' continuous $\Leftrightarrow x\neq n\pi$.

(e) Continuous on $(-1,1)$ and on $(n,n+1)$, $n\in\mathbb{Z}$. $\forall n\in\mathbb{Z}$, f is differentiable on $(n,n+1)$, $f'(x)=n$ on $(n,n+1)$, and f' is continuous there.

9. (a) $\displaystyle\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x^r\sin\tfrac{1}{x}$.
(i) If $r>0$, $\displaystyle\lim_{x\to 0^+}x^r=0$ and $\sin\frac{1}{x}$ is bounded, so by Ex. 4.2.19, $\displaystyle\lim_{x\to 0^+}x^r\sin\tfrac{1}{x}=0=f(0)$. Thus, f is continuous from the right at 0.
(ii) If $r=0$, $\displaystyle\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\sin\tfrac{1}{x}$, which does not exist.
(iii) If $r<0$, let $x_n=\frac{1}{\frac{\pi}{2}+2n\pi}$. Then $x_n\to 0^+$ but $f(x_n)=\frac{1}{\left(\frac{\pi}{2}+2n\pi\right)^r}\to\infty$.

(b) $\lim\limits_{x\to 0^+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} x^{r-1}\sin\frac{1}{x}$. By the above argument,
(i) If $r > 1$, $\lim\limits_{x\to 0^+} x^{r-1}\sin\frac{1}{x} = 0$, so $f'_+(0) = 0$.
(ii) If $r \le 1$, $\lim\limits_{x\to 0^+} x^{r-1}\sin\frac{1}{x}$ does not exist, so $f'_+(0)$ does not exist.

11. $f'(x_0)$ exists $\Leftrightarrow \exists L \in \mathbb{R} \ni \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = L$

$\Leftrightarrow \exists L \in \mathbb{R} \ni \lim\limits_{x\to x_0^-} \dfrac{f(x)-f(x_0)}{x-x_0} = L$ and $\lim\limits_{x\to x_0^+} \dfrac{f(x)-f(x_0)}{x-x_0} = L$

$\Leftrightarrow f'_-(x_0)$ and $f'_+(x_0)$ exist and are equal.

13. Let $f(x) = \frac{|x|}{x}$ if $x \neq 0$, 0 if $x = 0$. Then $f'(x) = 0$ if $x \neq 0$, so $\lim\limits_{x\to 0^-} f'(x) = 0 = \lim\limits_{x\to 0^+} f'(x)$. But $\lim\limits_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^-} \frac{1}{x}$ and $\lim\limits_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0^+} \frac{1}{x}$, so $f'_-(x_0)$ and $f'_+(x_0)$ do not exist.

15. Assume the hypotheses. Then $\exists L \in \mathbb{R} \ni \lim\limits_{x\to 0^-} f'(x) = \lim\limits_{x\to 0^+} f'(x) = L$. By Thm. 6.1.14, $f'_-(x_0) = f'_+(x_0) = L$. By Thm. 6.1.13, $f'(x_0)$ exists and $= L$.

17. (a) Suppose $\exists \alpha > 1$ and $M > 0 \ni \forall x, y \in I$, $|f(x)-f(y)| \le M|x-y|^\alpha$. Let $x_0 \in I$. Then $\forall x \neq x_0$ in I, $|f(x)-f(x_0)| \le M|x-x_0|^\alpha$, so $\left|\dfrac{f(x)-f(x_0)}{x-x_0}\right| \le M|x-x_0|^{\alpha-1}$. Since $\alpha > 1$, $\lim\limits_{x\to x_0}(x-x_0)^{\alpha-1} = 0$ since the power function is continuous. $\therefore \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} = 0$.

(b) Take $f(x) = |x|$ on $I = (0, 1)$.

EXERCISE SET 6.2

1. By Parts (a) and (b), $(f-g)'(x_0) = (f+(-g))'(x_0) = f'(x_0) + (-g)'(x_0) = f'(x_0) - g'(x_0)$.

3. By Thm. 6.2.2, $\frac{d}{dx}\left[\frac{1}{f(x)}\right] = \dfrac{f(x)\frac{d}{dx}1 - 1\frac{d}{dx}f(x)}{f^2(x)} = \dfrac{-f'(x)}{f^2(x)}$.

5. (a) If f is odd and differentiable at x_0, then $f'(-x_0) = \lim\limits_{x\to -x_0} \dfrac{f(x)-f(-x_0)}{x-(-x_0)} = \lim\limits_{x\to -x_0} \dfrac{-f(-x)+f(x_0)}{-(-x)+x_0} = \lim\limits_{-x\to x_0} \dfrac{f(-x)-f(x_0)}{-x-x_0} = \lim\limits_{u\to x_0} \dfrac{f(u)-f(x_0)}{u-x_0} = f'(x_0)$.

(b) If f is even and differentiable at x_0, then $f'(-x_0) = \lim\limits_{x\to -x_0} \dfrac{f(x)-f(-x_0)}{x-(-x_0)} = \lim\limits_{x\to -x_0} \dfrac{f(-x)-f(x_0)}{x+x_0} = -\lim\limits_{-x\to x_0} \dfrac{f(-x)-f(x_0)}{-x-x_0} = -\lim\limits_{u\to x_0} \dfrac{f(u)-f(x_0)}{u-x_0} = -f'(x_0)$.

7. $\frac{d}{dx}[f\circ(g\circ h)](x) = f'[(g\circ h)(x)]\cdot(g\circ h)'(x) = f'[(g\circ h)(x)]\cdot g'(h(x))\cdot h'(x) = (f'\circ g\circ h)(x)\cdot(g'\circ h)(x)\cdot h'(x)$.

9. For $x > 0$,

$$\frac{d}{dx}\left[x+(x+\sqrt{x})^{\frac{1}{2}}\right]^{\frac{1}{2}} = \frac{1}{2}\left[x+(x+\sqrt{x})^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left[1+\frac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}}\frac{d}{dx}(x+\sqrt{x})\right]$$

$$= \frac{1+\dfrac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{2\sqrt{x+\sqrt{x}}+1+\frac{1}{2\sqrt{x}}}{4\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{1+2\sqrt{x}+4\sqrt{x^2+x\sqrt{x}}}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}.$$

11. Let $a > 0$, $a \neq 1$, and $f(x) = a^x$. By Exercise 5.6.16, $f(x) = e^{x\ln a}$.
(a) By Thm. 6.2.9 and the chain rule, $f'(x) = e^{x\ln a}\ln a = a^x \ln a$.
(b) By Thm. 5.6.25, $\log_a x = \frac{\ln x}{\ln a}$, so by Thm. 6.2.9 and the chain rule, $\frac{d}{dx}\log_a x = \frac{1}{x}\cdot\frac{1}{\ln a} = \frac{1}{x\ln a}$.

13. Use Ex. 12 and the chain rule.

15. Suppose f, g are differentiable everywhere and $f > 0$. Let $u = f(x)^{g(x)}$. Then $\ln u = g(x)\ln f(x)$ is differentiable, and taking the derivative of both sides,

$$\frac{1}{u}\frac{du}{dx} = g(x)\cdot\frac{f'(x)}{f(x)} + g'(x)\ln f(x), \text{ so } \frac{du}{dx} = u\left[g(x)\cdot\frac{f'(x)}{f(x)} + g'(x)\ln f(x)\right].$$

17. (a) For $x \neq 0$, $\frac{d}{dx}x^2\sin\frac{1}{x} = x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x\sin\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.
(b) $f'(x)$ is continuous whenever $x \neq 0$, but $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$, which does not exist for the following reason:

Let $x_n = \frac{1}{2\pi n}$. Then $x_n \to 0$ and $\lim\limits_{n\to\infty} f'(x_n) = \lim\limits_{n\to\infty}(0-1) = -1$.

Let $y_n = \frac{1}{\frac{\pi}{2}+2\pi n}$. Then $y_n \to 0$ but $\lim\limits_{n\to\infty} f'(y_n) = \lim\limits_{n\to\infty}(2y_n - 0) = 0$.

Since $\lim\limits_{n\to\infty} f'(x_n) \neq \lim\limits_{n\to\infty} f'(y_n)$, $\lim\limits_{x\to 0} f'(x)$ does not exist.

19. Suppose f is periodic with period $p > 0$ and differentiable on $[a, a+p)$.
(a) Let $x_0 \in \mathbb{R}$. Let k be the smallest integer $\ni x_0 + kp \geq a$. Then $x_0 + kp \in [a, a+p)$ and, letting $u = x + kp$, $u_0 = x_0 + kp$,

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{f(x+kp)-f(x_0+kp)}{(x+kp)-(x_0-kp)}$$

$$= \lim_{u\to u_0}\frac{f(u)-f(u_0)}{u-u_0} = f'(u), \text{ which exists since } u_0 \in [a, a+p).$$

(b) $f'(x+p) = \lim\limits_{h\to 0}\dfrac{f(x+p+h)-f(x+p)}{h} = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = f'(x)$.

EXERCISE SET 6.3

1. Revise the proof of Thm. 6.3.2, making appropriate changes.

3. Revise the proof of Part (a), making appropriate changes.

5. (a) Increasing on $(-2, \frac{1}{2})$ and $(3, +\infty)$; decreasing on $(-\infty, -2)$ and $(\frac{1}{2}, 3)$. Local minimum $y = 0$ at $x = -2, 3$; local maximum $y = \frac{25}{4}$ at $x = \frac{1}{2}$.

(c) Increasing on $(-\infty, -1)$ and $(-1, 0)$; decreasing on $(0, 1)$ and $(1, \infty)$. No local minimum; local maximum $y = -1$ at $x = 0$.

(e) Increasing on $(-\infty, -0)$; decreasing on $(0, \infty)$. No local minimum; local maximum $y = 1$ at $x = 0$.

(f) Increasing on $(-1, 1)$; decreasing on $(-\infty, -1)$ and $(1, \infty)$. Local minimum $y = -\frac{1}{2}$ at $x = -1$; local maximum $y = \frac{1}{2}$ at $x = 1$.

7. (a) f is differentiable everywhere, since

$$x \neq 0 \Rightarrow f'(x) = 1 + 4x \sin \tfrac{1}{x} - 2\cos \tfrac{1}{x}, \text{ and}$$

$$f'(0) = \lim_{h \to 0} \frac{x + 2x^2 \sin \frac{1}{x}}{x} = \lim_{h \to 0} (1 + 2x \sin \tfrac{1}{x}) = 1.$$

(b) Let I be any neighborhood of 0. Then I contains a tail of the sequence $\{\frac{1}{n\pi}\}$ since $\frac{1}{n\pi} \to 0$. Note that $f'\left(\frac{1}{n\pi}\right) = 1 + \frac{4}{n\pi} \sin n\pi - 2\cos n\pi = -1$ if n is even, 3 if n is odd. Thus, I contains points x where $f'(x) < 0$ and points x where $f'(x) > 0$. Apply Thm. 6.3.5.

9. Revise the proof of Case 1, changing inequalities appropriately.

11. Suppose f is differentiable on an open interval I and $f'(x) \neq 0$ on I. Suppose $\exists a, b \in I \ni f'(a) < 0$ and $f'(b) > 0$. Then, by Thm. 6.3.7, $\exists c$ between a and b such that $f'(c) = 0$. Contradiction.

EXERCISE SET 6.4

1. Given: $d \in [a, b]$ and $f(d) > f(a) = f(b)$. Then $d \neq a, b$, so $d \in (a, b)$ and $f(d) = \max f[a, b]$. That is, f has a local max at d. $\therefore$ by Thm. 6.3.4, $f'(d) = 0$.

3. Let $x_1 \neq x_2$ in I, say $x_1 < x_2$. If $f(x_1) = f(x_2)$ then by Rolle's Thm. applied to $[x_1, x_2]$, $\exists c \in (x_1, x_2) \ni f'(c) = 0$. But $\forall c \in I$, $f'(c) \neq 0$. $\therefore f(x_1) \neq f(x_2)$.

5. Let $f(x) = 3x^5 - 2x^3 + 12x - 8$. Then $f'(x) = 15x^4 - 6x^2 + 12$, a quadratic expression in x^2 with discriminant $D < 0$. Thus, $\nexists\, x \in \mathbb{R} \ni f'(x) = 0$. Hence, by Rolle's Thm., $\nexists\, x_1, x_2 \in \mathbb{R} \ni f(x_1) = f(x_2)$; that is, f is 1-1 on $\mathbb{R}$.

7. The function $f(x) = 7x^3 - 5x^2 + 4x - 10$ is continuous on $\mathbb{R}$, $f(0) < 0$, and $f(2) > 0$, so by the intermediate value theorem, $\exists c \in (0, 2) \ni f(c) = 0$. Thus, the equation $f(x) = 0$ has at least one root.

Suppose $\exists\, x_1 < x_2 \ni f(x_1) = 0$ and $f(x_2) = 0$. Then by Rolle's Thm., $\exists c \in (x_1, x_2) \ni f'(c) = 0$. But $f'(x) = 21x^2 - 10x + 4$, a quadratic with negative discriminant. So $\nexists\, c \in \mathbb{R} \ni f'(c) = 0$. Contradiction. $\therefore$ The equation $f(x) = 0$ cannot have more than one root.

9. Suppose $f''(x) = 0$ for all $x \in \mathbb{R}$. By Thm. 6.4.4, $f'(x)$ must be constant; say $f'(x) = c$ for all $x \in \mathbb{R}$. Let $g(x) = cx$. Since $\forall x \in \mathbb{R}$, $f'(x) = g'(x)$, Cor. 6.4.5 says that $\exists d \in \mathbb{R} \ni \forall x \in \mathbb{R}$, $f(x) = g(x) + d$. That is, $f(x) = cx + d$.

11. Suppose that $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|^2$. Then $\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x - y|$, so $\forall x_0 \in \mathbb{R}$, $\lim_{x \to x_0} \left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq \lim_{x \to x_0} |x - x_0| = 0$. That is, $\forall x_0 \in \mathbb{R}$, $f'(x) = 0$. $\therefore$ By Thm. 6.4.4, f is constant on $\mathbb{R}$.

13. Revise the proof of (a), making appropriate changes.

15. Let $f(x) = x^3$ and $g(x) = -x^3$ on $I = (-1, 1)$. Then f is strictly increasing on I, g is strictly decreasing on I, and $0 \in I$, yet $f'(0) \not> 0$ and $g'(0) \not< 0$.

17. Let $x \in \left(0, \frac{\pi}{2}\right)$. Then $\tan x$ is differentiable on $[0, x]$ so by the MVT, $\exists c \in (0, x) \ni \tan'(c) = \frac{\tan x - \tan 0}{x - 0}$; i.e., $\sec^2 c = \frac{\tan x}{x}$. Since $|\sec c| > 1$, this says that $\forall x \in \left(0, \frac{\pi}{2}\right)$, $\frac{\tan x}{x} > 1$; i.e., $\tan x > x$.

19. Let $x > 1$ and $f(x) = \ln x$. Applying the MVT to f on $[1, x]$, $\exists c \in (1, x) \ni \frac{1}{c} = \ln' c = \frac{\ln x - \ln 1}{x - 1} = \frac{\ln x}{x - 1}$. Since $1 < c < x$, $\frac{1}{x} < \frac{1}{c} < 1$, so $\frac{1}{x} < \frac{\ln x}{x-1} < 1$. Since $x - 1 > 0$, this implies $\frac{x-1}{x} < \ln x < x - 1$.

21. Suppose f' is continuous at some interior point x_0 of its domain, and $f'(x_0) > 0$. By the "nbd inequality property of continuous functions," (Exercise 5.1.26) $\exists \delta > 0 \ni \forall x \in N_\delta(x_0)$, $f'(x) > 0$. Then by Thm. 6.4.6 (c), f is strictly increasing on $N_\delta(x_0)$. Similarly for $f'(x_0) < 0$.

23. Define $h : [a, \infty) \to \mathbb{R}$ by $h(x) = g(x) - f(x)$. Then $\forall x > a$, $h'(x) = g'(x) - f'(x) > 0$, so by Thm. 6.4.6, h is strictly increasing on $[a, \infty)$. Then, $\forall x > a$, $h(x) > h(a)$; i.e., $g(x) - f(x) > g(a) - f(a) \geq 0$; $\therefore g(x) > f(x)$.

25. (a) Suppose $f'(x) \geq 0$ in $(x_0 - \delta, x_0)$ and $f'(x) \leq 0$ in $(x_0, x_0 + \delta)$. (i) Let $x \in (x_0 - \delta, x_0)$. By the MVT applied to f on $[x, x_0]$, $\exists c \in (x, x_0) \ni f'(c) = \frac{f(x_0)-f(x)}{x_0-x}$. Since $f'(c) \geq 0$, and $x_0 - x > 0$, we must have $f(x_0) - f(x) \geq 0$; i.e., $f(x_0) \geq f(x)$. (ii) Similarly, $x \in (x_0, x_0 + \delta) \Rightarrow$ by MVT, $\exists d \in (x_0, x) \ni f'(d) = \frac{f(x)-f(x_0)}{x-x_0} \leq 0 \Rightarrow f(x) \leq f(x_0)$.

Therefore, $f(x_0) = \max f(x_0 - \delta, x_0 + \delta)$, so f has a local maximum at x_0.

27. Assume the hypotheses. Then $\exists \delta > 0 \ni N_\delta(x_0) \subseteq I$ and $f(x_0) = \max f\left(N_\delta(x_0)\right)$. Suppose f is **not** the absolute max of f on I. Then $\exists x_1 \in I \ni f(x_1) > f(x_0)$.

Case 1: $x_1 > x_0$. By the MVT, $\exists c \in (x_0, x_1) \ni f'(c) = \frac{f(x_1)-f(x_0)}{x_1-x_0} > 0$. Since $f'(x) \neq 0$ in I, f is not constant on $(x_0, \min\{x_0 + \delta, c\})$, so $\exists x_2 \in (x_0, \min\{x_0 + \delta, c\}) \ni f(x_2) \neq f(x_0)$. Then $f(x_2) < f(x_0)$ since $f(x_0) = \max f(x_0 - \delta, x_0 + \delta)$. Then by the MVT, $\exists d \in (x_0, x_2) \ni f'(d) = \frac{f(x_2)-f(x_0)}{x_2-x_0} < 0$.

But then $f'(d) < 0 < f'(c)$, so by the intermediate value property of derivatives (Thm. 6.3.7) $\exists e$ between c and $d \ni f'(e) = 0$. Recall that $c, d > 0$, so e is a second point of I at which $f' = 0$. Contradiction. $\therefore f(x_0) = \max f(I)$.

Case 2: $x_1 < x_0$. Revise the above argument to cover this case.

29. (a) Suppose f is differentiable on $(x_0 - \delta, x_0)$, continuous from the left at x_0 and $\lim\limits_{x \to x_0^-} f'(x)$ exists. Let $x \in (x_0 - \delta, x_0)$. Applying the MVT to f on $[x, x_0]$, $\exists c_x \in (x, x_0) \ni f'(c_x) = \frac{f(x)-f(x_0)}{x-x_0}$. Thus, $f'_-(x_0) = \lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{x \to x_0^-} f'(c_x) = \lim\limits_{c_x \to x_0^-} f'(c_x) = \lim\limits_{x \to x_0^-} f'(x)$, which exists by hypothesis.

$\therefore f'_-(x_0)$ exists and equals $\lim\limits_{x \to x_0^-} f'(x)$.

31. For $x \neq 0$, $f'(x) = -\frac{2}{x} \cos \frac{1}{x^2} + 2x \sin \frac{1}{x^2}$. Also, $f'(0) = \lim\limits_{x \to 0} \frac{x^2 \sin \frac{1}{x^2} - f(0)}{x-0} = \lim\limits_{x \to 0} x \sin \frac{1}{x^2} = 0$ by Ex. 4.2.19. Thus, f is differentiable everywhere.

Let $x_n = 1/\sqrt{2\pi n}$. Then $x_n \to 0$ but $f'(x_n) = -2\sqrt{2\pi n} \to \infty$. Therefore, $\forall \varepsilon > 0$, f is differentiable on $[-\varepsilon, \varepsilon]$ but f' is unbounded there.

33. See solution to Exercise 6.1.17 and apply Thm. 6.4.4.

EXERCISE SET 6.5

1. $T_4(x) = 5 + 2(x-1) - (x-1)^2 + (x-1)^4 = 5 + 2x - 2 - (x^2 - 2x + 1) + (x^4 - 4x^3 + 6x^2 - 4x + 1) = 3 + 5x^2 - 4x^3 + x^4$.

3. $T_{2n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$;

$T_{2n}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = T_{2n-1}(x)$.

5. $T_{2n}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$; $T_{2n+1}(x) = T_{2n}(x)$.

7. $T_6(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{2^2 \cdot 2!}(x-1)^2 + \frac{3}{2^3 \cdot 3!}(x-1)^3 - \frac{3 \cdot 5}{2^4 \cdot 4!}(x-1)^4$
$+ \frac{3 \cdot 5 \cdot 7}{2^5 \cdot 5!}(x-1)^5 - \frac{3 \cdot 5 \cdot 7 \cdot 9}{2^6 \cdot 6!}(x-1)^6$.

Let $x \neq 1$. $R_6(x) = \frac{f^{(7)}(c)}{7!}(x-1)^7 = \frac{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{2^7 \cdot 7!} c^{-13/2}(x-1)^6$ for some c between 1 and x.

9. $f(x) = e^x$. For $|x| < 2$, Taylor's theorem says $\exists c \in (-2, 2) \ni |R_6(x)| = \left| \frac{f^{(7)}(c)}{7!} x^7 \right| = \frac{e^c}{7!}|x|^7 < \frac{e^2}{7!} 2^7 = \frac{128e^2}{7!} < 0.1877$. For $|x| \leq 1$, Taylor's theorem says $\exists c \in (-1, 1) \ni |R_6(x)| = \frac{e^c}{7!}|x|^7 < \frac{e^2}{7!} 2^7 = \frac{e}{7!} < 0.00054$.

11. (a) We want $|R_n(x)| < 0.005$ for all $x \in [-2, 2]$. Since $\frac{e^c}{(n+1)!}|x|^{n+1} < \frac{e^2 \cdot 2^{n+1}}{(n+1)!}$, we can accomplish this by making $\frac{e^2 \cdot 2^{n+1}}{(n+1)!} < 0.005$.

n	6	8	9	10
$\frac{e^2 2^{n+1}}{(n+1)!}$	.1877	.0104	.0021	.00038

Since the sequence $\left\{\frac{2^n}{n!}\right\}$ is decreasing when $n \geq 2$, $|R_n(x)| < 0.005$ when $n \geq 10$. Thus, $T_{10}(x)$ is the desired Taylor polynomial about 0.

(b) We want $|R_n(x)| < 0.005$ for all $x \in [-1,1]$. Since $\frac{e^c}{(n+1)!}|x|^{n+1} < \frac{e^1 \cdot 1^{n+1}}{(n+1)!}$, we can accomplish this by making $\frac{e}{(n+1)!} < 0.005$.

n	4	5
$\frac{e}{(n+1)!}$	.023	.0038

Since the sequence $\left\{\frac{e}{n!}\right\}$ is decreasing, $|R_n(x)| < 0.005$ when $n \geq 5$. Use $T_5(x)$.

13. For the Taylor polynomials $T_n(x)$ about 0 for the functions $\sin x$ or $\cos x$, $R_n(x) = \frac{\pm(\sin c \text{ or } \cos c)}{(n+1)!}x^{n+1}$, so $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$. By Cor. 2.3.11, $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for all $x \in \mathbb{R}$. $\therefore$ $R_n(x) \to 0$.

15. Let $x \in I$, $x \neq a$. By Taylor's thm., $\exists c$ between a and $x \ni R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, so $|R_n(x)| \leq \frac{M^{n+1}(x-a)^{n+1}}{(n+1)!}$. We shall apply Thm. 2.3.10. Let $y = \frac{M^{n+1}(x-a)^{n+1}}{(n+1)!}$. Then $\lim\limits_{n\to\infty}\left|\frac{y_{n+1}}{y_n}\right| = \lim\limits_{n\to\infty}\left|\frac{M^{n+2}(x-a)^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{M^{n+1}(x-a)^{n+1}}\right| = \lim\limits_{n\to\infty}\frac{M|x-a|}{n+2} = 0$. Thus, by Thm. 2.3.10, $\lim\limits_{n\to\infty} R_n(x) = 0$.

17. Calculating $f^{(k)}(x)$ by hand for $k = 1, 2, \cdots, 6$ requires some hard work, but a computer algebra system will make it easy. You will find that $f^{(k)}(0) = 0$ for $k = 1, 2, \cdots, 5$, but $f^{(6)}(0) = 720$. $\therefore$ By the n^{th} derivative test, f has a local minimum at 0.

19. See the footnote.

EXERCISE SET 6.6

1. Modify the proof of Case 2 by changing inequalities appropriately.

2. (a) $\lim\limits_{x\to x_0^-} f(x) = L \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \ni x_0 - \delta < x < x_0 \Rightarrow |f(x) - L| < \varepsilon$

$$\Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \ni -x_0 < -x < -x_0 + \delta \Rightarrow |f(x) - L| < \varepsilon$$
$$\Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0 \ni -x_0 < x < -x_0 + \delta \Rightarrow |f(-x) - L| < \varepsilon$$
$$\Leftrightarrow \lim_{x\to -x_0^+} f(-x) = L.$$

3. We prove Case 4; Cases 5 and 6 are proved similarly.

Given: (i) $f, g : I \to \mathbb{R}$, where I is an interval with right end-point $\alpha = x_0$.
(ii) f, g are differentiable on I; (iii) $\forall x \in I$, $g(x)g'(x) \neq 0$;
(iv) $\lim\limits_{x\to x_0^-} f(x) = \lim\limits_{x\to x_0^-} g(x) = 0$; (v) $\lim\limits_{x\to x_0^-} \frac{f'(x)}{g'(x)} = L$.

Let $-I = (-x_0, -a)$, for some $a < x_0$ in I. Define $F, G : -I \to \mathbb{R}$ by $F(x) = f(-x)$ and $G(x) = g(-x)$. Then $\forall x \in -I$, $F'(x) = -f'(-x)$ and

$G'(x) = -g'(-x)$. Consequently,

(a) $F, G : -I \to \mathbb{R}$ where $-I$ is an interval with left end-point $-x_0$;

(b) F, G are differentiable on $-I$;

(c) $\forall x \in -I$, $-x \in I$ so $g(-x)g'(-x) \neq 0$; i.e., $G(x)G'(x) \neq 0$;

(d) $\lim\limits_{x\to -x_0^+} F(x) = \lim\limits_{x\to -x_0^+} f(-x) = \lim\limits_{x\to x_0^-} f(x) = 0$ (using Ex. 2);

Similarly, $\lim\limits_{x\to -x_0^+} G(x) = 0$;

(e) $\lim\limits_{x\to -x_0^+} \frac{F'(x)}{G'(x)} = \lim\limits_{x\to -x_0^+} \frac{-f'(-x)}{-g'(-x)} = \lim\limits_{x\to -x_0^+} \frac{f'(-x)}{g'(-x)} = \lim\limits_{x\to x_0^-} \frac{f'(x)}{g'(x)} = L.$

$\therefore$ By Case 1, $\lim\limits_{x\to -x_0^+} \frac{F(x)}{G(x)} = L$, so by Ex. 2, $\lim\limits_{x\to x_0^-} \frac{F(-x)}{G(-x)} = L$; i.e., $\lim\limits_{x\to x_0^-} \frac{f(x)}{g(x)} = L.$

5. See how Exercise 3 was done.

7. Modify the proof of Case 2 by changing "$> M+1$" to "$< -M-1$" and "$> M$" to "$< -M$" etc.

9. To prove Cases 7–9, put together Cases 1 & 4, Cases 2 & 5, and Cases 3 & 6.

11. To prove Cases 13–15, modify the proofs of Cases 10–12 in obvious ways.

12. (a) $+\infty$ (c) 0 (e) $-\infty$ (g) 1

(b) $\lim\limits_{x\to -\infty} \frac{e^x}{x} = \lim\limits_{x\to -\infty} \frac{e^x}{1} = 0$; (L'Hôpital's rule is optional here.)

(d) $\lim\limits_{x\to 0+} \frac{\sec x}{\ln x} = \lim\limits_{x\to 0+} \frac{\sec x \tan x}{1/x} = \lim\limits_{x\to 0+} x \sec x \tan x = 0.$

(f) $\lim\limits_{x\to(\frac{\pi}{2})^-} \frac{3+4\sec x}{2+\tan x} = \lim\limits_{x\to(\frac{\pi}{2})^-} \frac{4\sec x \tan x}{\sec^2 x} = 4\cdot \lim\limits_{x\to(\frac{\pi}{2})^-} \frac{\tan x}{\sec x} = 4\cdot \lim\limits_{x\to(\frac{\pi}{2})^-} \sin x = 4.$

Without L'Hôpital's rule, $\lim\limits_{x\to(\frac{\pi}{2})^-} \frac{3\cos x+4}{2\cos x+\sin x} = \frac{4}{1} = 4.$

(h) $\lim\limits_{x\to 2+} \frac{1/(x-2)}{\ln(x-2)} = \lim\limits_{x\to 2+} \frac{-(x-2)^{-2}}{(x-2)^{-1}} = -\lim\limits_{x\to 2+} \frac{1}{x-2} = -\infty.$

13. (b) 1 (d) 1 (e) $1/e$ (g) e (h) 1/2

(a) $\lim\limits_{x\to(\frac{\pi}{2})+} (\sec x - \tan x) = \lim\limits_{x\to(\frac{\pi}{2})+} \frac{1-\sin x}{\cos x} = \lim\limits_{x\to(\frac{\pi}{2})+} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0.$

(c) $\lim\limits_{x\to 0+} x\ln(\sin x) = \lim\limits_{x\to 0+} \frac{\ln(\sin x)}{1/x} = \lim\limits_{x\to 0+} \frac{\cos x/\sin x}{-1/x^2} = -\lim\limits_{x\to 0+} \frac{x^2\cos x}{\sin x}$

$= -\lim\limits_{x\to 0+} \frac{-x^2\sin x+2x\cos x}{\cos x} = 0.$

(f) Take logarithm. $\ln\left[\lim\limits_{x\to\infty} \left(1-1/x^2\right)^x\right] = \lim\limits_{x\to\infty} \ln\left(1-1/x^2\right)^x =$

$\lim\limits_{x\to\infty} x\ln\left(1-1/x^2\right)$

$= \lim\limits_{x\to\infty} \frac{\ln\left(1-1/x^2\right)}{1/x} = \lim\limits_{x\to\infty} \frac{\ln\left(x^2-1\right)-\ln x^2}{1/x} = \lim\limits_{x\to\infty} \frac{\frac{2x}{x^2-1}-\frac{2}{x}}{-1/x^2} = \lim\limits_{x\to\infty} \frac{2x^3-2x(x^2-1)}{-(x^2-1)}$

$= \lim\limits_{x\to\infty} \frac{2x}{-(x^2-1)} = \lim\limits_{x\to\infty} \frac{2}{-2x} = 0.$ $\therefore \lim\limits_{x\to\infty} \left(1-1/x^2\right)^x = 1.$

15. (a) We use mathematical induction to prove that $\forall k \in \mathbb{N}$, $\exists$ polynomial $p_k(x)$ with constant term 0 such that $\forall x > 0$, $f^{(k)}(x) = p_k\left(\frac{1}{x}\right)e^{-1/x}$.

(i) For $x > 0$, $f'(x) = \frac{1}{x^2} e^{-1/x}$. Take $p_1(x) = x^2$. Thus, the statement is true when $n = 1$.

(ii) Suppose the statement is true when $n = k$. That is, $f^{(k)}(x) = p_k\left(\frac{1}{x}\right) e^{-1/x}$, where $p_k(x)$ is a polynomial with constant term 0. Then, $\forall x > 0$, $f^{(k+1)}(x) = p_k\left(\frac{1}{x}\right) \cdot \frac{1}{x^2} e^{-1/x} + p_k'\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2} e^{-1/x} = \frac{1}{x^2} e^{-1/x}\left[p_k\left(\frac{1}{x}\right) + p_k'\left(\frac{1}{x}\right)\right] = e^{-1/x} p_{k+1}\left(\frac{1}{x}\right)$, where $p_{k+1}(x)$ is a polynomial with constant term 0.

(b) We show that $\forall k \in \mathbb{N}$, $f^{(k)}(0) = 0$. First, note that $f_-^{(k)}(0) = 0$, since f is constant on $(-\infty, 0]$. We shall use mathematical induction to show that $\forall k \in \mathbb{N}$, $f_+^{(k)}(0) = 0$.

(i) $f_+^{(1)}(0) = \lim\limits_{x \to x_0^+} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to x_0^+} \frac{e^{-1/x}}{x} = 0$ by Ex. 14 (a).

(ii) Suppose true for $n = k$. Then $f_+^{(k+1)}(0) = \lim\limits_{x \to x_0^+} \frac{f^{(k)}(x) - f^{(k)}(0)}{x}$ $= \lim\limits_{x \to x_0^+} \frac{1}{x} p_k\left(\frac{1}{x}\right) e^{-1/x}$. Now $\frac{1}{x} p_k\left(\frac{1}{x}\right)$ is a polynomial in $\frac{1}{x}$ with constant term 0, so by Ex. 14 (c), $\lim\limits_{x \to x_0^+} \frac{1}{x} p_k\left(\frac{1}{x}\right) e^{-1/x} = 0$. $\therefore$ $f_+^{(k+1)}(0) = 0$.

(c) Since $f^{(k)}(0) = 0$ for every k, the n^{th} Taylor polynomial for f about 0 is $T_n(x) = 0$. But, $\forall x > 0$, $f(x) \neq 0$. Thus, $\forall x > 0$, $T_n(x) \not\to f(x)$.

17. (a) 1 (b) $-e/2$ (c) $-1/2$

Chapter 7

EXERCISE SET 7.1

1. By hypotheses, $\forall a \in A$, $a \in B$, so $a \leq \sup B$. Thus, $\sup B$ is an upper bound for A. $\therefore$ $\sup A \leq \sup B$. Similarly, $\forall a \in A$, $a \in B$, so $a \geq \inf B$. Thus, $\inf B$ is a lower bound for A. $\therefore$ $\inf A \geq \inf B$.

3. (a) $\forall a \in A$, $b \in B$, $a + b \in A + B$ and $a \geq \inf A$ and $b \geq \inf B$, so $a + b \geq \inf A + \inf B$. Thus, $\inf A + \inf B$ is a lower bound for $A + B$. Suppose w is another lower bound for $A + B$. Then $\forall a \in A$, $b \in B$, $a + b \geq w$; i.e., $a \geq w - b$. Thus, $\forall b \in B$, $w - b$ is a lower bound for A. Then,

$$\forall b \in B, \ w - b \leq \inf A; \text{ i.e., } w - \inf A \leq b.$$

Thus, $w - \inf A$ is a lower bound for B, so $w - \inf A \leq \inf B$. $\therefore$ $w \leq \inf A + \inf B$. Putting these results together, $\inf(A + B) = \inf A + \inf B$.

5. (a) Suppose that $\exists K > 0 \ni \forall \varepsilon > 0$, $x \leq a + K\varepsilon$. Let $\varepsilon > 0$. Then $\varepsilon/K > 0$, so $x \leq a + K(\varepsilon/K) = a + \varepsilon$. By the forcing principle (Thm. 1.5.9) $x \leq a$.

(b) Suppose that $\exists K > 0 \ni \forall \varepsilon > 0$, $x \geq a - K\varepsilon$. Let $\varepsilon > 0$. Then $\varepsilon/K > 0$, so $x \geq a - K(\varepsilon/K) = a - \varepsilon$, so $-x \leq -a + \varepsilon$. By the forcing principle, $-x \leq -a$; i.e., $x \geq a$.

EXERCISE SET 7.2

1. Since $m_i \le M_i$, $\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \le \sum_{i=1}^{n} M_i = \overline{S}(f,\mathcal{P})$.

3. Note that $\mathcal{P} = \{a,b\}$ is a partition of $[a,b]$. For this partition, $m_1 = \inf f[a,b]$ and $M_1 = \sup f[a,b]$, so $m \le m_1 \le M_1 \le M$ and $m(b-a) \le m_1(b-a) = \underline{S}(f,\mathcal{P}) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le \overline{S}(f,\mathcal{P}) = M_1(b-a) \le M(b-a)$.

5. For $\mathcal{P}_1 = \{2,9\}$, $\underline{S}(f,\mathcal{P}) = m_1\triangle_1 = 0\cdot 7 = 0$. $\therefore \underline{\int_a^b} f \ge 0$.

Let $\varepsilon > 0$. For $\mathcal{P}_2 = \{2, 5-\frac{\varepsilon}{2}, 5+\frac{\varepsilon}{2}, 9\}$, $\overline{S}(f,\mathcal{P}) = m_1\triangle_1 + m_2\triangle_2 + m_3\triangle_3 = 0\left(3-\frac{\varepsilon}{2}\right) + 1\cdot\varepsilon + 0\left(4-\frac{\varepsilon}{2}\right) = \varepsilon$. $\therefore \forall\varepsilon > 0$, $\overline{\int_a^b} f \le \varepsilon$. By the forcing principle, $\overline{\int_a^b} f \le 0$.

Thus we have $0 \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le 0$.

7. (a) For $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}$, $\underline{S}(f,\mathcal{P}) = 0\cdot\frac{1}{2} + \left[\frac{1}{4}+\frac{3}{2}\right]\frac{1}{2} + [1+3]\frac{1}{2} + \left[\frac{9}{4}+\frac{9}{2}\right]\frac{1}{2} = \frac{25}{4}$ and $\overline{S}(f,\mathcal{P}) = \left[\frac{1}{4}+\frac{3}{2}\right]\frac{1}{2} + [1+3]\frac{1}{2} + \left[\frac{9}{4}+\frac{9}{2}\right]\frac{1}{2} + [4+6]\frac{1}{2} = \frac{45}{4}$.

(b) For $\mathcal{P} = \left\{0, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, 2\right\}$, $\underline{S}(f,\mathcal{P}) = \frac{1}{4}\left[0 + \frac{7}{4} + \frac{45}{16} + 4 + \frac{85}{16}\right] + \frac{27}{4}\cdot\frac{1}{2} = \frac{438}{64}$, and $\overline{S}(f,\mathcal{P}) = \frac{1}{4}\left[\frac{7}{2} + \frac{45}{16} + 4 + \frac{85}{16} + \frac{27}{4}\right] + 10\cdot\frac{1}{2} = \frac{294}{64}$.

(c) For $\mathcal{P} = \left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{5}{3}, 2\right\}$, $\underline{S}(f,\mathcal{P}) = \frac{1}{3}\left[0 + \frac{10}{9} + \frac{22}{9} + 8 + \frac{79}{9}\right] = \frac{61}{9}$, and $\overline{S}(f,\mathcal{P}) = \frac{1}{3}\left[\frac{10}{9} + \frac{22}{9} + 4 + \frac{158}{9} + 10\right] = \frac{316}{27}$.

(d) For $\mathcal{P} = \left\{0, \frac{2}{n}, \frac{4}{n}, \cdots, \frac{2i}{n}, \cdots 2\right\}$, $\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2 + 3\left(\frac{2i}{n}\right)\right]\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i^2}{n^2} + \frac{6i}{n}\right] = \frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{6}{n}\frac{n(n+1)}{2}\right] = \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 6\left(1+\frac{1}{n}\right)$ and $\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n}\left[\left(\frac{2(i-1)}{n}\right)^2 + 3\left(\frac{2(i-1)}{n}\right)\right]\frac{2}{n} = \frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 6\left(1-\frac{1}{n}\right)$.

9. If $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ are sequences of partitions of $[a,b]$ such that $\underline{S}(f,\mathcal{P}_n) \to L$ and $\overline{S}(f,\mathcal{Q}_n) \to L$, then by (a), (b) and Thm. 7.2.7, $L \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le L$.

11. (a) Let $\mathcal{P}_n = \left\{2, 2+\frac{3}{n}, 2+\frac{6}{n}, \cdots, 2+\frac{3n}{n}\right\}$. For each i, $x_i = 2 + \frac{3i}{n}$ and $\triangle_i = \frac{3}{n}$. Since f is increasing on $[2,5]$, $m_i = f(x_{i-1})$ and $M_i = f(x_i)$. Thus,

$$\overline{S}(f,\mathcal{P}_n) = \sum_{i=1}^{n} M_i\triangle_i = \sum_{i=1}^{n}\left(x_i^2 - 2x_i\right)\cdot\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)^2 - 2\left(2+\frac{3i}{n}\right)\right]$$
$$= \frac{3}{n}\sum_{i=1}^{n}\left[\frac{9i^2}{n^2} + \frac{6i}{n}\right] = \frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2 + \frac{6}{n}\sum_{i=1}^{n} i\right] = \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{18}{n^2}\frac{n(n+1)}{2}$$
$$= \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 9\left(1+\frac{1}{n}\right) \to 18.$$

$$\underline{S}(f,\mathcal{P}_n)=\sum_{i=1}^{n} m_i\triangle_i=\sum_{i=1}^{n}\left(x_{i-1}^2-2x_{i-1}\right)\cdot\tfrac{3}{n}=\tfrac{3}{n}\sum_{i=1}^{n}\left[\left(2+\tfrac{3(i-1)}{n}\right)^2-2\left(2+\tfrac{3(i-1)}{n}\right)\right]$$
$$=\tfrac{3}{n}\sum_{i=1}^{n-1}\left[\tfrac{9i^2}{n^2}+\tfrac{6i}{n}\right]=\tfrac{3}{n}\left[\tfrac{9}{n^2}\sum_{i=1}^{n-1} i^2+\tfrac{6}{n}\sum_{i=1}^{n-1} i\right]=\tfrac{27}{n^3}\tfrac{(n-1)n(2(n-1)+1)}{6}+\tfrac{18}{n^2}\tfrac{(n-1)n}{2}$$
$$=\tfrac{9}{2}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)+9\left(1-\tfrac{1}{n}\right)\to 18.$$

By Thm. 7.2.12 (c), f is integrable on $[2,5]$ and $\int_2^5 f=18$.

(b) Let $\mathcal{P}_n=\left\{1,1+\frac{5}{n},1+\frac{10}{n},\cdots,1+\frac{5n}{n}\right\}$. For each i, $x_i=1+\frac{5i}{n}$ and $\triangle_i=\frac{5}{n}$. Since f is increasing on $[1,6]$, $m_i=f(x_{i-1})$ and $M_i=f(x_i)$. Thus,

$$\overline{S}(f,\mathcal{P}_n)=\sum_{i=1}^{n} M_i\triangle_i=\tfrac{5}{n}\sum_{i=1}^{n}\left[\left(1+\tfrac{5i}{n}\right)^2+\left(1+\tfrac{5i}{n}\right)+3\right]$$
$$=\tfrac{5}{n}\left[\sum_{i=1}^{n}5+\tfrac{15}{n}\sum_{i=1}^{n} i+\tfrac{25}{n^2}\sum_{i=1}^{n} i^2\right]$$
$$=25+\tfrac{75}{2}\left(1+\tfrac{1}{n}\right)+\tfrac{125}{6}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\to\tfrac{625}{6}.$$

$$\underline{S}(f,\mathcal{P}_n)=\sum_{i=1}^{n} m_i\triangle_i=\tfrac{5}{n}\sum_{i=1}^{n}\left[\left(1+\tfrac{5(i-1)}{n}\right)^2+\left(1+\tfrac{5(i-1)}{n}\right)+3\right]$$
$$=\tfrac{5}{n}\left[\sum_{i=1}^{n}5+\tfrac{15}{n}\sum_{i=1}^{n-1} i+\tfrac{25}{n^2}\sum_{i=1}^{n-1} i^2\right]$$
$$=25+\tfrac{75}{2}\left(1-\tfrac{1}{n}\right)+\tfrac{125}{6}\left(1-\tfrac{1}{n}\right)\left(2-\tfrac{1}{n}\right)\to\tfrac{625}{6}.$$

By Thm. 7.2.12 (c), f is integrable on $[1,6]$ and $\int_1^6 f=\frac{625}{6}=104.1\overline{6}$.

(c) Let $\mathcal{P}_n=\left\{0,\frac{2}{n},\frac{4}{n},\cdots,\frac{2n}{n}\right\}$. For each i, $x_i=\frac{2i}{n}$ and $\triangle_i=\frac{2}{n}$. Since f is increasing on $[0,2]$, $m_i=f(x_{i-1})$ and $M_i=f(x_i)$. Thus,

$$\overline{S}(f,\mathcal{P}_n)=\sum_{i=1}^{n} M_i\triangle_i=\tfrac{2}{n}\sum_{i=1}^{n}\left(\tfrac{2i}{n}\right)^3=\tfrac{16}{n^4}\sum_{i=1}^{n} i^3=\tfrac{16}{n^4}\tfrac{n^2(n+1)^2}{4}=4\left(1+\tfrac{1}{n}\right)^2\to 4;$$

$$\underline{S}(f,\mathcal{P}_n)=\sum_{i=1}^{n} m_i\triangle_i=\tfrac{2}{n}\sum_{i=1}^{n}\left(\tfrac{2(i-1)}{n}\right)^3=\tfrac{16}{n^4}\sum_{i=1}^{n-1} i^3=\tfrac{16}{n^4}\tfrac{(n-1)^2n^2}{4}\to 4.$$

By Thm. 7.2.12 (c), f is integrable on $[0,2]$ and $\int_0^2 f=4$.

(d) Let $\mathcal{P}_n=\left\{-1,-1+\frac{3}{n},-1+\frac{6}{n},\cdots,-1+\frac{3n}{n}\right\}$. For each i, $x_i=-1+\frac{3i}{n}$ and $\triangle_i=\frac{3}{n}$. Since f is decreasing on $[-1,2]$, $m_i=f(x_i)$ and $M_i=f(x_{i-1})$.

Thus, $\underline{S}(f,\mathcal{P}_n) = \sum\limits_{i=1}^{n} f(x_i)\triangle_i = \sum\limits_{i=1}^{n}\left(1-\left(-1+\frac{3i}{n}\right)^3\right)\cdot\frac{3}{n}$

$= \frac{3}{n}\sum\limits_{i=1}^{n}\left[2-\frac{9i}{n}+\frac{27i^2}{n^2}-\frac{27i^3}{n^3}\right] = 6-\frac{27}{n^2}\frac{n(n+1)}{2}+\frac{81}{n^3}\frac{n(n+1)(2n+1)}{6}-\frac{81}{n^4}\frac{n^2(n+1)^2}{4}$

$= 6-\frac{27}{2}\left(1+\frac{1}{n}\right)+\frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{81}{4}\left(1+\frac{1}{n}\right)^2 \to -\frac{3}{4}$. Similarly,

$\overline{S}(f,\mathcal{P}_n) = \sum\limits_{i=1}^{n} f(x_{i-1})\triangle_i = \frac{3}{n}\sum\limits_{i=1}^{n}\left[1-\left(-1+\frac{3(i-1)}{n}\right)^3\right] = \frac{3}{n}\sum\limits_{i=1}^{n-1}\left[1-\left(-1+\frac{3i}{n}\right)^3\right]$

$= 6\left(1-\frac{1}{n}\right)-\frac{27}{2}\left(1-\frac{1}{n}\right)+\frac{27}{2}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)-\frac{81}{4}\left(1-\frac{1}{n}\right)^2 \to -\frac{3}{4}$.

13. For f, g satisfying the hypotheses, and $\forall$ partitions $\mathcal{P}$ of $[a,b]$, $\underline{S}(f,\mathcal{P}) \leq \overline{S}(f,\mathcal{P})$, so $\int_a^b f = \underline{\int_a^b} f \leq \overline{\int_a^b} g = \int_a^b g$.

15. (a) For f and $\{\mathcal{P}_n\}$ satisfying the hypotheses, $\{\underline{S}(f,\mathcal{P}_n)\}$ and $\{\overline{S}(f,\mathcal{P}_n)\}$ are monotone increasing and monotone decreasing sequences, respectively, by Thm. 7.2.4. By Thm. 7.2.5, $\{\underline{S}(f,\mathcal{P}_n)\}$ is bounded above by $\overline{S}(f,\mathcal{P}_1)$ and $\{\overline{S}(f,\mathcal{P}_n)\}$ is bounded below by $\underline{S}(f,\mathcal{P}_1)$. Thus, by the monotone convergence theorem (2.5.3) both of these sequences converge. Since $\forall n \in \mathbb{N}$, $\underline{S}(f,\mathcal{P}_n) \leq \overline{S}(f,\mathcal{P}_n)$, and since limits preserve inequalities, $\lim\limits_{n\to\infty}\underline{S}(f,\mathcal{P}_n) \leq \lim\limits_{n\to\infty}\overline{S}(f,\mathcal{P}_n)$.

(b) By the monotone convergence thm, $\lim\limits_{n\to\infty}\underline{S}(f,\mathcal{P}_n) = \sup\{\underline{S}(f,\mathcal{P}_n) : n \in \mathbb{N}\}$ $\leq \underline{\int_a^b} f$ and $\lim\limits_{n\to\infty}\overline{S}(f,\mathcal{P}_n) = \inf\{\overline{S}(f,\mathcal{P}_n) : n\in\mathbb{N}\} \geq \overline{\int_a^b} f$. Thus, $\lim\limits_{n\to\infty}\underline{S}(f,\mathcal{P}_n) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \lim\limits_{n\to\infty}\overline{S}(f,\mathcal{P}_n)$. So, if $\lim\limits_{n\to\infty}\underline{S}(f,\mathcal{P}_n) = \lim\limits_{n\to\infty}\overline{S}(f,\mathcal{P}_n) = L$, then $\underline{\int_a^b} f = \overline{\int_a^b} f = L$.

(c) Let $f(x) = x$ on $[0,2]$. Then f is integrable on $[0,2]$. For the partition $\mathcal{P}_n = \left\{0, \frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \cdots, \frac{2^n}{2^n}, 2\right\}$, take $K = 2^n$. Then $\underline{S}(f,\mathcal{P}_n) = 1+\frac{1}{2}(1-\frac{1}{K}) \to \frac{3}{2}$, while $\overline{S}(f,\mathcal{P}_n) = 2+\frac{1}{2}(1+\frac{1}{K}) \to \frac{5}{2}$.

17. Trivial. Go through the proof, replacing "$<$" by "$\leq$", and see that it works.

19. The proof of Case 2 is just like that of Case 1 except that this time $m_i = f(x_i)$ and $M_i = f(x_{i-1})$. Then $\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{Q}) = \frac{b-a}{n}\sum\limits_{i=1}^{n}[f(x_i)-f(x_{i-1})] = \frac{b-a}{n}[f(a)-f(b)] < (b-a)[f(a)-f(b)]\,\varepsilon$. Apply Riemann's criterion.

21. $\underline{S}(f,\mathcal{Q}) - \underline{S}(f,\mathcal{P}) = \left[\sum\limits_{i=1}^{k-1} m_i(x_i-x_{i-1}) + m_{k,1}(x_k^*-x_{k-1}) + m_{k,2}(x_k-x_k^*)\right.$

$\left.+\sum\limits_{i=k+1}^{n} m_i(x_i-x_{i-1})\right] - \left[\sum\limits_{i=1}^{k-1} m_i(x_i-x_{i-1}) + m_k(x_k-x_{k-1})\right.$

$\left.+\sum\limits_{i=k+1}^{n} m_i(x_i-x_{i-1})\right] = m_{k,1}(x_k^*-x_{k-1})+m_{k,2}(x_k-x_k^*)-m_k(x_k-x_{k-1})$.

Similarly, $\overline{S}(f,\mathcal{Q}) - \overline{S}(f,\mathcal{P}) = M_{k,1}(x_k^*-x_{k-1})+M_{k,2}(x_k-x_k^*)-M_k(x_k-x_{k-1})$.

EXERCISE SET 7.3

1. Using the notation of Exercise 7.2.21,
$$\underline{S}(f,\mathcal{Q})-\underline{S}(f,\mathcal{P})=m_{k,1}(x_k^*-x_{k-1})+m_{k,2}(x_k-x_k^*)-m_k(x_k-x_{k-1})$$
$$\le M(x_k^*-x_{k-1})+M(x_k-x_k^*)+M(x_k^*-x_{k-1})+M(x_k-x_k^*)\le 4M\,\|\mathcal{Q}\|.$$

3. Replace the appropriate section of the proof by the following:

By the ε-criterion for infimum, we can select tags $x_i^*\in[x_{i-1},x_i]\ni f(x_i^*)>M_i-\frac{\varepsilon}{b-a}$. Then $R(f,\mathcal{P}^*)=\sum\limits_{i=1}^{n}f(x_i^*)\Delta_i>\sum\limits_{i=1}^{n}(M_i-\frac{\varepsilon}{b-a})\Delta_i=\overline{S}(f,\mathcal{P})-\varepsilon$. Thus, $\overline{S}(f,\mathcal{P})<R(f,\mathcal{P}^*)+\varepsilon<I+2\varepsilon$. $\therefore\ \forall\varepsilon>0$, $\overline{S}(f,\mathcal{P})<I+2\varepsilon$, so $\overline{\int_a^b}f<I+2\varepsilon$. Therefore, by the forcing principle, $I\ge\overline{\int_a^b}f$.

4. Each given function is continuous on the given interval, so it is integrable there. Let $\mathcal{P}_n=\{x_0,x_1,\cdots,x_n\}$ be the partition of $[a,b]$ into n subintervals of equal length, $\triangle=\frac{b-a}{n}$. Then $x_i=a+i\triangle$; choose $x_i^*=x_i$. Then $\|\mathcal{P}_n\|=\frac{b-a}{n}\to0$, so by Thm. 7.3.6, $R(f,\mathcal{P}_n)\to\int_a^b f$.

(d) $\int_{-1}^{3}(x^3+2x)dx$; $\triangle=\frac{4}{n}$, $x_i=-1+\frac{4i}{n}$. Then $R(f,\mathcal{P}_n)=\sum\limits_{i=1}^{n}f(x_i^*)\triangle=\sum\limits_{i=1}^{n}\left[\left(-1+\frac{4i}{n}\right)^3+2\left(-1+\frac{4i}{n}\right)\right]\cdot\frac{4}{n}=\frac{4}{n}\left[-3n+\frac{20}{n}\sum\limits_{i=1}^{n}i-\frac{48}{n^2}\sum\limits_{i=1}^{n}i^2+\frac{64}{n^3}\sum\limits_{i=1}^{n}i^3\right]=$
$-12+40\left(1+\frac{1}{n}\right)-32\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+64\left(1+\frac{1}{n}\right)^2\to28.$

5. (a) Using the notation used in solving Ex. 4, $R(f,\mathcal{P}_n)=\sum\limits_{i=1}^{n}f(x_i^*)\triangle=$
$$\sum_{i=1}^{n}(a+i\triangle)\triangle=\frac{b-a}{n}\sum_{i=1}^{n}\left(a+i\left(\frac{b-a}{n}\right)\right)=\frac{b-a}{n}\left[an+\frac{b-a}{n}\frac{n(n+1)}{2}\right]$$
$$=(b-a)a+\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)\to ab-a^2+\frac{b^2}{2}-ab+\frac{a^2}{2}=\frac{b^2-a^2}{2}.$$

7. Let $\mathcal{P}_n=\left\{0,\frac{1^2}{n^2},\frac{2^2}{n^2},\frac{3^2}{n^2},\cdots,\frac{n^2}{n^2}\right\}$. Then $\triangle_i=\frac{i^2}{n^2}-\frac{(i-1)^2}{n^2}=\frac{2i-1}{n^2}<\frac{2}{n}$, so $\|\mathcal{P}_n\|<\frac{2}{n}\to0$. Let $x_i^*=x_i$. Then $R(f,\mathcal{P}_n)=\sum\limits_{i=1}^{n}f(x_i^*)\triangle_i=\sum\limits_{i=1}^{n}\sqrt{\frac{i^2}{n^2}}\cdot\frac{2i-1}{n^2}$
$=\sum\limits_{i=1}^{n}\frac{2i^2-i}{n^3}=\frac{2}{n^3}\frac{n(n+1)(2n+1)}{6}-\frac{1}{n^3}\frac{n(n+1)}{2}\to\frac{2}{3}$. $\therefore\ \int_0^1 f=\frac{2}{3}$.

9. Suppose f is integrable on $[a,b]$, and let $\mathcal{P}_n=\{0,\frac{1}{n},\frac{2}{n},\cdots,\frac{n}{n}\}$. Then $x_i=\frac{i}{n}$ and $\triangle_i=\frac{1}{n}$. To make $\mathcal{P}$ a tagged partition, choose $x_i^*=x_i$. Then $R(f,\mathcal{P}_n^*)=\sum\limits_{i=1}^{n}f(x_i^*)\triangle_i=\sum\limits_{i=1}^{n}f(\frac{i}{n})\frac{1}{n}=\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i}{n})$. By Thm. 7.3.6, $\lim\limits_{n\to\infty}R(f,\mathcal{P}_n^*)=\int_0^1 f$ since $\|\mathcal{P}_n\|=\frac{1}{n}\to0$. Thus, $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{i=1}^{n}f(\frac{i}{n})=\int_0^1 f$.

10. (a) $1/2$ (b) Let $f(x) = x^2$. Then
$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{k^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{k}{n}\right)^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f = \int_0^1 x^2 dx = \frac{1}{3}$.
(c) $1/6$ (d) $\ln 2$ (e) $\frac{1}{2}\ln 2$ (f) $\pi/4$

11. (a) Think of $f(x) = \cos x$ over $\left[0, \frac{\pi}{2}\right]$, with $\mathcal{P}_n$ dividing the interval into n subintervals of equal length. Then $\forall i$, $x_i = \frac{i\pi}{2n}$, and $\|\mathcal{P}_n\| = \frac{\pi}{2n} \to 0$. Choose $x_i^* = x_i$. Then $R(f, \mathcal{P}_n^*) = \sum_{i=1}^{n} f(x_i^*)\triangle_i = \sum_{i=1}^{n} \cos\left(\frac{i\pi}{2n}\right)\frac{\pi}{2n}$. Thus, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \cos\left(\frac{i\pi}{2n}\right) = \frac{2}{\pi} \lim_{n\to\infty} R(f, \mathcal{P}_n^*) = \frac{2}{\pi}\int_0^{\pi/2} \cos x dx = \frac{2}{\pi}$.
(b) $3/(2\pi)$ (c) 4 (d) $\sqrt{8}$

13. Let $\mathcal{P}_n = \{x_0, x_1, \cdots, x_n\}$ be the partition of $[a, b]$ into n subintervals of equal length, $\triangle = \frac{b-a}{n}$; let $\mathcal{P}_n^*$ be the tagged partition using the left endpoints of each subinterval $[x_{i-1}, x_i]$, and let $\mathcal{P}_n^{**}$ be the tagged partition using the right endpoints. By Thm. 7.3.6, $\int_a^b f = \lim_{n\to\infty} R(f, \mathcal{P}_n^*) = \lim_{n\to\infty} R(f, \mathcal{P}_n^{**})$. Thus,
$\int_a^b f = \frac{1}{2}\left[\lim_{n\to\infty} R(f, \mathcal{P}_n^*) + \lim_{n\to\infty} R(f, \mathcal{P}_n^{**})\right] = \frac{1}{2}\lim_{n\to\infty}\left[R(f, \mathcal{P}_n^*) + R(f, \mathcal{P}_n^{**})\right] =$
$\frac{1}{2}\lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f(x_{i-1}) + \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\right]$
$= \lim_{n\to\infty} \frac{b-a}{2n}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$.

15. Revise the portion of the proof beginning in line 7, as follows: change m_i, $\overline{m}_k$, inf, and $\le$ to M_i, $\overline{M}_k$, sup, and $\ge$. Also, change "$+B$" and "$+(B - \overline{m}_{k_1})$" to "$-B$" and "$-(B + \overline{M}_{k_1})$", and so on.

17. ($\Rightarrow$) Suppose f integrable on $[a, b]$. By Cor. 7.3.12, $\exists\, L \in \mathbb{R} \ni L = \sup\{\underline{S}(f, \mathcal{Q}) : \mathcal{Q}$ is a regular partition of $[a, b]\} = \inf\{\overline{S}(f, \mathcal{Q}) : \mathcal{Q}$ is a regular partition of $[a, b]\}$. Let $\varepsilon > 0$. By the ε-criterion for sup and inf, $\exists$ regular partitions $\mathcal{Q}_m$, $\mathcal{Q}_n$ of $[a, b] \ni \underline{S}(f, \mathcal{Q}_m) > L - \frac{\varepsilon}{2}$ and $\overline{S}(f, \mathcal{Q}_n) < L + \frac{\varepsilon}{2}$. For $k \ge \max\{m, n\}$, $\overline{S}(f, \mathcal{Q}_k) - \underline{S}(f, \mathcal{Q}_k) \le \overline{S}(f, \mathcal{Q}_n) - \underline{S}(f, \mathcal{Q}_m) < \varepsilon$.

($\Leftarrow$) Apply Thm. 7.2.14.

19. ($\Rightarrow$) Suppose f integrable on $[a, b]$. Let $\varepsilon > 0$. By Thm. 7.3.2, $\exists\, \delta > 0 \ni \forall$ partitions $\mathcal{P}$ of $[a, b]$, $\|\mathcal{P}\| < \delta \Rightarrow \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$. Choose $n_0 \in \mathbb{N} \ni \frac{b-a}{n_0} < \delta$. Then $n \ge n_0 \Rightarrow \|\mathcal{Q}_n\| = \frac{b-a}{n} < \delta \Rightarrow \overline{S}(f, \mathcal{Q}_n) - \underline{S}(f, \mathcal{Q}_n) < \varepsilon$.

($\Leftarrow$) Apply Riemann's criterion, Thm. 7.2.14.

21. Let f be the Dirichlet function on $[0, 1]$, and $\mathcal{Q}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\right\}$. Note that x_i, x_{i-1}, and $\frac{x_i + x_{i-1}}{2}$ are all rational. Thus,

(a) For $x_i^* = x_{i-1}$, $R(f, \mathcal{Q}_n^*) = \frac{1}{n} \sum_{i=1}^{n} f(x_{i-1}) = \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{n}{n} = 1$.

(b) For $x_i^* = x_i$, $R(f, \mathcal{Q}_n^*) = \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{n}{n} = 1$.

(c) For $x_i^* = \frac{x_{i-1}+x_i}{2}$, $R(f, \mathcal{Q}_n^*) = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right) = \frac{1}{n} \sum_{i=1}^{n} 1 = \frac{n}{n} = 1$.

(d) $\frac{1}{2n}[f(0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)] = \frac{2n}{2n} = 1$.

(e) $\frac{1}{3(2n)}[f(0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{2n-1}) + f(x_{2n})]$

$= \frac{1}{6n}\left[2(1) + \frac{2n}{2}(4) + \left(\frac{2n}{2} - 1\right)(2)\right] = \frac{1}{6n}[2 + 4n + 2n - 2] = 1$.

EXERCISE SET 7.4

1. Modify the proof of (a) as appropriate.

3. Suppose f is integrable on $[a, b]$, and $[c, d] \subsetneq [a, b]$. If $a = c < d < b$, or if $a < c < d = b$, then f is integrable on $[c, d]$ by Thm. 7.4.2. If $a < c < d < b$, the desired conclusion follows from Cor. 7.4.3.

5. Use mathematical induction. The case $n = 1$ is trivial; the case $n = 2$ is Thm. 7.4.5. Suppose the theorem is true when $n = k$. Suppose f is integrable on each subinterval created by $\mathcal{P} = \{x_1, x_2, \cdots, x_{k+1}\}$. By Thm. 7.4.5, f is integrable on $[x_0, x_k]$ and $[x_k, x_{k+1}]$, and

$$\int_a^b f = \int_a^{x_k} f + \int_{x_k}^b f. \qquad (1)$$

Since $\mathcal{P}' = \{x_1, x_2, \cdots, x_k\}$ is a partition of $[x_0, x_k]$, our induction hypothesis says that f is integrable over $[x_0, x_1], [x_0, x_2], \cdots, [x_{k-1}, x_k]$ and

$$\int_a^{x_k} f = \sum_{i=1}^{k} \left(\int_{x_{i-1}}^{x_i} f\right). \qquad (2)$$

Putting (1) and (2) together, $\int_a^b f = \sum_{i=1}^{k+1} \left(\int_{x_{i-1}}^{x_i} f\right)$.

7. (i) By Thm. 7.4.2, f is integrable on $[a, b-h]$ and $[b-h, b]$, and $\int_a^b f = \int_a^{b-h} f + \int_{b-h}^b f$. Thus, by Exercise 6, $\left|\int_a^b f - \int_a^{b-h} f\right| = \left|\int_{b-h}^b f\right| \leq Mh$.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$. Then $0 < h < \delta \Rightarrow \left|\int_a^b f - \int_a^{b-h} f\right| < M\delta = \varepsilon$, so $\lim_{h \to 0^+} \int_a^{b-h} f = \int_a^b f$.

(ii) For any $c \in (a, b)$, $\lim_{h \to 0^+} \int_{a+h}^{b-h} f = \lim_{h \to 0^+} \left[\int_{a+h}^c f + \int_c^{b-h} f\right] = \int_a^c f + \int_c^b f = \int_a^b f$ by earlier parts of the proof.

9. (a) f is bounded and continuous, hence integrable, on every $[c, d] \subseteq (0, 1)$, so by Thm. 7.4.7, f is integrable on $[0, 1]$. The same is true on $[-1, 0]$. Thus, by Thm. 7.4.5, f is integrable on $[-1, 1]$.

(b) The sequence $\left\{ g\left(\frac{1}{\frac{\pi}{2}+2n\pi} \right) \right\}$ shows that g is unbounded on $[-1, 1]$, hence is not integrable there.

11. (a) 10 (b) 5/2 (c) 10

13. (a) For $x \in (x_{i-1}, x_i)$, $\sigma(x) = m_i$ and $\tau(x) = M_i$, so σ and τ are step functions on $[a, b]$.

(b) For $x \in [a, b]$, either $x = b$ or there is a unique $i \ni x \in [x_{i-1}, x_i)$. In the first case, $\sigma(x) = f(x) = \tau(x)$. In the second case, $\sigma(x) = m_i \leq f(x) \leq M_i = \tau(x)$. Thus, $\forall x \in [a, b]$, $\sigma(x) \leq f(x) \leq \tau(x)$.

(c) By (7.4.3), (7.4.8), and (7.2.9), $\int_a^b \sigma = \sum\limits_{i=1}^{k} \left(\int_{x_{i-1}}^{x_i} \sigma \right) = \sum\limits_{i=1}^{k} m_i(x_i - x_{i-1}) = \underline{S}(f, \mathcal{P})$ and $\int_a^b \tau = \sum\limits_{i=1}^{k} \left(\int_{x_{i-1}}^{x_i} \tau \right) = \sum\limits_{i=1}^{k} M_i(x_i - x_{i-1}) = \overline{S}(f, \mathcal{P})$.

15. The function $f(x) = 1/x$ if $x \neq 0$, 0 if $x = 0$, is continuous (hence integrable) on every proper closed subinterval $[c, d]$ of $[0, 1]$, but is not bounded on $[0, 1]$ so is not integrable on $[0, 1]$ This does not contradict Thm. 7.4.7 because that theorem pertains only to *bounded* functions.

17. The function $f(x) = \sin \frac{1}{x}$ if $x \neq 0$, 0 if $x = 0$, is continuous (hence integrable) on every $[c, d] \subseteq (0, 1)$, so by Thm. 7.4.7 it is integrable on $[0, 1]$. As shown in (4.1.12), $\lim\limits_{x \to 0+} f(x)$ does not exist, so f is not regulated on $[0, 1]$.

19. Let $a < b$. By Thm. 5.7.3, $\exists$ bounded, monotone increasing $f : [a, b] \to \mathbb{R}$ having $\mathbb{Q} \cap [a, b]$ as its set of discontinuities. Since f is monotone, it is integrable on $[a, b]$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\mathbb{Q} \cap [a, b]$ is dense in $[a, b]$. By Thm. 5.2.17, f is regulated on $[a, b]$.

EXERCISE SET 7.5

1. (a) Given $\mathcal{P} = \{x_1, x_2, \cdots, x_n\}$, define $m_i(f) = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$. Define $m_i(g)$ and $m_i(f + g)$ similarly. Then $\forall x \in [x_{i-1}, x_i]$, $m_i(f) + m_i(g) \leq f(x) + g(x)$, and so $m_i(f) + m_i(g) \leq m_i(f + g)$. Thus, $\sum\limits_{i=1}^{n} m_i(f)\Delta_i + \sum\limits_{i=1}^{n} m_i(g)\Delta_i \leq \sum\limits_{i=1}^{n} m_i(f + g)\Delta_i$. That is,

$$\underline{S}(f, \mathcal{P}) + \underline{S}(g, \mathcal{P}) \leq \underline{S}(f + g, \mathcal{P}) \leq \underline{\int_a^b}(f + g). \qquad (3)$$

Let $\varepsilon > 0$. Then $\exists\, \mathcal{P}_1 \ni \underline{S}(f, \mathcal{P}_1) \geq \underline{\int_a^b} f - \frac{\varepsilon}{2}$ and $\exists\, \mathcal{P}_2 \ni \underline{S}(g, \mathcal{P}_2) \geq \underline{\int_a^b} g - \frac{\varepsilon}{2}$. Taking $\mathcal{Q} = \mathcal{P}_1 \cup \mathcal{P}_2$,

$$\underline{S}(f, \mathcal{Q}) + \underline{S}(g, \mathcal{Q}) \geq \underline{\int_a^b} f + \underline{\int_a^b} g - \varepsilon. \tag{4}$$

By (3) and (4) together, $\underline{\int_a^b}(f+g) \geq \underline{\int_a^b} f + \underline{\int_a^b} g - \varepsilon$. Apply the forcing principle.

(b) Use $f(x) = 1$ if x is rational, -1 if x is irrational, and $g = -f$.

(c) Modify the proof of (a), using $\overline{M_i(f)}, \overline{M_i(g)}, \overline{M_i(f+g)}$, upper sums, and other appropriate changes to prove that $\overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g$.

(d) If f, g are integrable on $[a, b]$, then from (a), (c), and Thm. 7.2.7,
$\int_a^b f + \int_a^b g = \underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b}(f+g) \leq \overline{\int_a^b}(f+g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g = \int_a^b f + \int_a^b g$.

3. Suppose f is integrable on $[-a, a]$. Let $\{\mathcal{P}_n^*\}$ be a sequence of tagged partitions of $[0, a] \ni \|\mathcal{P}_n\| \to 0$. By Thm. 7.3.6, $R(f, \mathcal{P}_n^*) \to \int_0^a f$.

For each $\mathcal{P}_n = \{0, x_1, x_2, \cdots x_{m_n}\}$ with tags $\{x_i^* : i = 1, 2, \cdots, m_n\}$, let $\mathcal{Q}_n = \{-x_{m_n}, \cdots, -x_2, -x_1, 0, x_1, x_2, \cdots x_{m_n}\}$ with tags $\{-x_i^* : i = 1, 2, \cdots, m_n\} \cup \{x_i^* : i = 1, 2, \cdots, m_n\}$. Then $\{\mathcal{Q}_n^*\}$ is a sequence of tagged partitions of $[0, a] \ni \|\mathcal{Q}_n\| \to 0$. By Thm. 7.3.6, $R(f, \mathcal{Q}_n^*) \to \int_{-a}^a f$.

But $R(f, \mathcal{Q}_n^*) = \sum_{i=1}^{m_n} f(-x_i^*)\triangle_i + \sum_{i=1}^{m_n} f(x_i^*)\triangle_i = \sum_{i=1}^{m_n}[f(-x_i^*) + f(x_i^*)]\triangle_i$. If f is even, then $f(-x_i^*) + f(x_i^*) = 2f(x_i^*)$, while if f is odd, $f(-x_i^*) + f(x_i^*) = 0$.

5. (a) In this case, $\underline{\int_a^b} f \geq \underline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i \geq 0$.

(b) For $\mathcal{P} = \{0, x_1, x_2, \cdots x_n,\}$, $m \leq m_i \leq M_i \leq M$, so $\underline{\int_a^b} f \geq \underline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i \geq \sum_{i=1}^{n} m \triangle_i = m(b-a)$ and $\overline{\int_a^b} f \leq \overline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i \leq \sum_{i=1}^{n} M \triangle_i = M(b-a)$.

(c) By (b), $-M(b-a) \leq \int_a^b f \leq M(b-a)$; i.e., $\left|\int_a^b f\right| \leq M(b-a)$.

(d) $\forall x \in [a, b]$, $g(x) - f(x) \geq 0$, so by (a), $\int_a^b (g - f) \geq 0$; i.e., $\int_a^b g - \int_a^b f \geq 0$.

7. T is regulated and g is piecewise continuous, so T, g are integrable on $[0, 1]$. However, $g \circ T$ is the Dirichlet function, which is not integrable on $[0, 1]$.

9. Let $f(x) = x$ if $0 < x \leq 1$, 1 if $x = 0$. Note that $1/f$ is unbounded on $[0, 1]$.

11. Let f be the constant $f(x) = 1$, and g denote the Dirichlet function.

13. By Ex. 7.2.13, $\underline{\int_a^b} f \leq \underline{\int_a^b} h \leq \underline{\int_a^b} g$ and $\overline{\int_a^b} f \leq \overline{\int_a^b} h \leq \overline{\int_a^b} g$. Since f, g are integrable on $[a, b]$, this means $\int_a^b f \leq \underline{\int_a^b} h \leq \int_a^b g$ and $\int_a^b f \leq \overline{\int_a^b} h \leq \int_a^b g$. But $\int_a^b f = \int_a^b g$, so h is integrable on $[a, b]$ and $\int_a^b h = \int_a^b f = \int_a^b g$.

EXERCISE SET 7.6

1. If $a = b$, then $\int_a^c f + \int_c^b = \int_a^c f + \int_c^a f = \int_a^c f - \int_a^c = 0 = \int_a^a f = \int_a^b f$.
If $b = c$, then $\int_a^c f + \int_c^b = \int_a^c f + \int_b^b f = \int_a^c f + 0 = \int_a^c f = \int_a^b f$.
If $a = c$, then $\int_a^c f + \int_c^b = \int_a^a f + \int_c^b f = 0 + \int_c^b = \int_a^b f$.

3. $Sgn(x)$ is piecewise continuous on $[-1, 1]$, so it is integrable there. $Sgn(x)$ does not have the intermediate value property, so by Thm. 6.3.7, it cannot be a derivative on $[-1, 1]$; that is, it does not have an antiderivative there.

5. In Exercise 6.2.17 we showed that $f'(0) = 0$. For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Note that f' is not continuous at 0 since $\lim\limits_{x \to 0} f'(x)$ does not exist.

For $0 < \varepsilon < \frac{2}{\pi}$, f' is continuous on $[\varepsilon, \frac{2}{\pi}]$ so by FTC-I, $\int_\varepsilon^{\frac{\pi}{2}} f' = [f(x)]_\varepsilon^{\frac{2}{\pi}} = \left[x^2 \sin \frac{1}{x}\right]_\varepsilon^{\frac{2}{\pi}} = \frac{4}{\pi^2} - \varepsilon^2 \sin \frac{1}{\varepsilon}$. By Thm. 7.4.7, f' is integrable on $[0, \frac{2}{\pi}]$ and $\int_0^{\frac{\pi}{2}} f' = \lim\limits_{\varepsilon \to 0} \int_\varepsilon^{\frac{\pi}{2}} f' = \lim\limits_{\varepsilon \to 0} \left[\frac{4}{\pi^2} - \varepsilon^2 \sin \frac{1}{\varepsilon}\right] = \frac{4}{\pi^2}$.

7. (a) Suppose $x < 0$.
If $x < -1$, $\int_{-1}^x sgn = -\int_x^{-1}(-1) = \int_x^{-1} 1 = -1 - x = |x| - 1$;
If $x = -1$, $\int_{-1}^x sgn = \int_{-1}^{-1} sgn = 0 = |x| - 1$;
If $-1 < x < 0$, $\int_{-1}^x sgn = \int_{-1}^x(-1) = -(x+1) = -x - 1 = |x| - 1$.

(b) For $x = 0$, $\int_{-1}^0 sgn = \lim\limits_{\varepsilon \to 0^-} \int_{-1}^\varepsilon(-1) = -\left(\lim\limits_{\varepsilon \to 0^-} \varepsilon + 1\right) = -1 = |x| - 1$.

(c) Suppose $x > 0$. Then $\int_{-1}^x sgn = \int_{-1}^0 sgn + \int_0^x sgn = -1 + \lim\limits_{\varepsilon \to 0^+} \int_\varepsilon^x 1 = -1 + \lim\limits_{\varepsilon \to 0^+} (x - \varepsilon) = x - 1 = |x| - 1$.

9. Suppose f is integrable on $[a, b]$ and define $F(x) = \int_a^x f$ on $[a, b]$.
(a) Suppose $f \geq 0$ on $[a, b]$. If $x_1 < x_2$ in $[a, b]$, then by Exercise 7.2.12 (a), $\int_{x_1}^{x_2} f \geq 0$, so $F(x_2) = \int_a^{x_2} f = \int_a^{x_1} f + \int_{x_1}^{x_2} f \geq \int_a^{x_1} f = F(x_1)$.
(b) Redo (a), changing inequalities appropriately.

11. (a) $-f(x)$ (b) $(f \circ g)(x)g'(x)$ (c) $-(f \circ g)(x)g'(x)$
(d) $(f \circ h)(x)h'(x) - (f \circ g)(x)g'(x)$

13. (a) 3/4 (b) 8/15

14. (a) $\frac{1}{3}(x^2 + 2)\sqrt{x^2 - 1} + C$ (c) $6 \ln 2 - 2$ (e) $4e^3 + 2$
(g) $\sin(\ln 2) - \cos(\ln 2) + 1/2$ (i) $x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$

15. (a) $\left[(x - t)f'(t) - \int f'(t)(-dt)\right]_a^x = [0 - (x - a)f'(a)] + \int_a^x f'(t)dt$
$= -(x - a)f'(a) + f(x) - f(a) = f(x) - [f(a) + f'(a)(x - a)] = f(x) - T_1(x)$.

(b) $\left[(x-t)^{k+1}f^{(k+1)}(t) - \int f^{(k+1)}(t)(k+1)(x-t)^k(-dt)\right]_a^x$
$= \left[(x-t)^{k+1}f^{(k+1)}(t)\right]_a^x + (k+1)\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$
$= 0\cdot f^{(k+1)}(x) - f^{(k+1)}(a)(x-a)^{k+1} + \frac{(k+1)!}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$
$= -f^{(k+1)}(a)(x-a)^{k+1} + (k+1)!R_k(x)$
$= -f^{(k+1)}(a)(x-a)^{k+1} + (k+1)![f(x) - T_k(x)]$
$= (k+1)!\left\{f(x) - \left[T_k(x) + \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1}\right]\right\}$
$= (k+1)!\left\{f(x) - T_{k+1}(x)\right\} = (k+1)!R_{k+1}(x)$.

17. $\forall x \in [a,b]$, $f(a) \le f(x) \le f(b)$, so by Thm. 7.5.2 (b), $f(a)(b-a) \le \int_a^b f \le f(b)(b-a)$. Define g on $[a,b]$ by $g(x) = f(a)(x-a) + f(b)(b-x)$. Then g is continuous and $g(a) = f(b)(b-a)$, $g(b) = f(a)(b-a)$. Thus, $g(a) \le \int_a^b f \le g(b)$. By the intermediate value theorem (5.3.9), $\exists c \in [a,b] \ni \int_a^b f = g(c) = f(c)(c-a) + f(b)(b-c)$.

19. By Thm. 5.7.3, there is a bounded, monotone increasing (hence integrable) function $f : \mathbb{R} \to \mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$. By Cor. 5.2.19, all of the discontinuities of f are jump discontinuities. Thus, f has a jump discontinuity at every rational number.

If $a < c < d < b$, then f has jump discontinuities on a dense subset of $[c,d]$. So, by Exercise 6.3.12, f cannot be the derivative of any function on $[c,d]$.

21. (a) Let $x_0 \in (a,b)$. By Thm. 5.2.17, $f(x_0^-)$ and $f(x_0^+)$ exist, and by Ex. 20, f is differentiable from the left and right at x_0, and $F'_-(x_0) = f(x_0^-)$, $F'_+(x_0) = f(x_0^+)$. Suppose f is not continuous at x_0. Then, by Thm. 5.2.17, $f(x_0^-) < f(x_0^+)$, so $F'_-(x_0) < F'_+(x_0)$. $\therefore$ F is not differentiable at x_0.

(b) Let $a < b$ and $A = \mathbb{Q} \cap [a,b]$. By Thm. 5.7.3, there is a bounded, nonnegative, monotone increasing $f : [a,b] \to \mathbb{R}$ having A as its set of discontinuities. Define $F(x) = \int_a^x f$ on $[a,b]$. Then,

1. F is continuous on $[a,b]$, by Thm. 7.6.6;
2. F is monotone increasing, by Ex. 20.
3. F is differentiable at every irrational number in $[a,b]$ since f is continuous there. (See FTC-II)
4. $\forall x \in A$, f is discontinuous at x, so by Part (a), F is not differentiable at x. $\therefore$ $\forall$ rational numbers x in $[a,b]$, F is not differentiable at x.

23. $F_n'(x) = \displaystyle\sum_{k=0}^{2n}(-1)^k p^{2n-k}\psi_n^{(k+1)}(x) = \sum_{m=1}^{2n+1}(-1)^{m-1}p^{2n-m+1}\psi_n^{(m)}(x) =$

$$\left[\sum_{m=0}^{2n}(-1)^{m-1}p^{2n-m+1}\psi_n^{(m)}(x)\right] - (-1)p^{2n+1}\psi_n^{(0)}(x) + (-1)^{2n-1}p\,\psi_n^{(2n+1)}(x)$$

$$= \left[-p\sum_{m=0}^{2n}(-1)^m p^{2n-m}\psi_n^{(m)}(x)\right] + p^{2n+1}\psi_n(x) = -pF_n(x) + p^{2n+1}\psi_n(x).$$

EXERCISE SET 7.8-A

1. Suppose f integrable on every $[c, b]$ such that $a < c < b$. Suppose for contradiction that $\forall \varepsilon > 0$, f is bounded on $[a, a+\varepsilon)$. Then, since f is also bounded on $[a+\varepsilon, b]$, it is bounded on $[a, b]$. By Thm. 7.4.7, f is integrable on $[a, b]$. Contradiction.

3. (a) Improper; diverges to $+\infty$. (b) Improper; converges to 2.
(c) Not improper. (d) Improper; diverges to $+\infty$.
(e) Improper; converges to -1. (f) Improper; converges to $2\sqrt{8}$.
(g) Improper; diverges. $\int_0^1 f$ diverges to $+\infty$, $\int_{-1}^0 f$ div. to $-\infty$.
(h) Improper; diverges to $+\infty$. (i) Improper; diverges to $+\infty$.
(j) Not improper. (k) Improper; converges to $2(e-1)$. (l) Not improper.
(m) Not improper. (n) Improper; diverges to $-\infty$.

5. $\displaystyle\int_0^{1/2} \frac{dx}{x(\ln x)^2} = \lim_{c\to 0+} \left[\frac{-1}{\ln x}\right]_c^{1/2} = 1/\ln 2$, converges. $\displaystyle\int_{1/2}^1 \frac{dx}{x(\ln x)^2} =$
$\displaystyle\lim_{c\to 1-} \left[\tfrac{-1}{\ln x}\right]_{1/2}^c = +\infty$, diverges. Therefore $\displaystyle\int_0^1 \frac{dx}{x(\ln x)^2}$ diverges.

EXERCISE SET 7.8-B

1. (a) Converges to $\frac{1}{2}$. (b) Conv. to 1. (c) Diverges to $+\infty$.
(d) Conv. to 2. (e) Div. to $+\infty$. (f) Diverges. (g) Conv. to $\frac{\pi}{2}$.
(h) Conv. to $1/e$. (i) Converges to π. (j) Conv. to $2-1/e$.
(k) Conv. to 1. (l),(m),(n) Div. to $+\infty$.

2. (a) Converges $\Leftrightarrow r > 1$. (b) Converges $\Leftrightarrow r < 1$. (c) Diverges for all r.

3. (a) Conv. by comp. with $\int_1^\infty x^{-3/2}dx$. (b) Div. by comp. with $\int_1^\infty x^{-1/2}dx$.
(c) Diverges, since $\int_1^\infty f$ diverges by comparison with $\int_1^\infty x^{-1/2}dx$.
(d) Conv. by comp. with $\int_1^\infty x^{-3/2}dx$. (e) Conv. by comp. with $\int_{-1}^\infty \frac{dx}{(x+2)^2}$.
(f) Converges, since $\int_{-\infty}^{-3} f$ conv. by comp. with $\int_{-\infty}^{-3} \frac{dx}{(x+2)^2}$, $\int_{-3}^{-1} f$ exists since f is continuous on $[-3,-1]$, and $\int_{-1}^\infty f$ conv. by (e).

5. (a) For $x \geq \pi$, $0 \leq \frac{|\cos x|}{x^2} \leq \frac{1}{x^2}$, and $\int_1^\infty \frac{1}{x^2}dx$ converges.

(b) $\int_\pi^b \frac{\sin x}{x}dx = \left[-\frac{1}{x}\cos x\right]_\pi^b - \int_\pi^b \frac{\cos x}{x^2}dx$, so $\int_\pi^\infty \frac{\sin x}{x}dx = -1 - \int_\pi^\infty \frac{\cos x}{x^2}dx$.

(c) $\{\pi, 2\pi, 3\pi, \cdots, \pi + n\pi\}$ is a partition of $[\pi, (n+1)\pi\}$. Apply Cor. 7.4.3.

(d) For even n, $x \in [n\pi, (n+1)\pi] \Rightarrow \sin x \geq \frac{1}{2} \Rightarrow \frac{\sin x}{x} \geq \frac{1}{2(n+1)\pi}$;
For odd n, $x \in [n\pi, (n+1)\pi] \Rightarrow \sin x \leq \frac{-1}{2} \Rightarrow \frac{\sin x}{x} \leq \frac{-1}{2(n+1)\pi}$;

(e) $\int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x} dx = \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x} dx \geq \sum_{k=1}^{n} \int_{k\pi}^{(k+1)\pi} \frac{1}{2(k+1)\pi} dx$
$= \frac{1}{2} \sum_{k=1}^{n} \frac{k}{k+1} > \sum_{k=1}^{n} \frac{1}{k+1} \to +\infty.$

7. $\forall a < c < d < b$, $\int_a^c f + \int_c^b f = \left(\int_a^d f + \int_d^c f\right) + \left(\int_c^d f + \int_d^b f\right) = \int_a^d f + \int_d^b f$.

9. (a) Assume the hypotheses and define $F(x) = \int_x^b f$, $G(x) = \int_x^b g$ on $(-\infty, b]$. Then F, G are monotone decreasing and $G(x) \geq F(x)$ on $(-\infty, b]$. $\lim_{x \to -\infty} G(x)$ exists, so by Exercise 4.4-B.15, G is bounded above on $(-\infty, b]$ and $\int_{-\infty}^b g = \lim_{x \to -\infty} G(x) = \sup\{G(x) : x \leq b\}$. Thus, $\lim_{x \to -\infty} F(x) \leq \lim_{x \to -\infty} G(x) = \int_{-\infty}^b g$.

(b) Suppose that $\forall x \leq b$, $0 \geq f(x) \geq g(x)$. If $\int_{-\infty}^b g$ converges, so does $\int_{-\infty}^b f$ and $\int_{-\infty}^b f \geq \int_{-\infty}^b g$.

EXERCISE SET 7.9

1. (a) Obvious from Defn. 7.9.2.

(b) Suppose A, B have measure 0 and $\varepsilon > 0$. Then $\exists$ collections of open intervals $\{I_n : n \in \mathbb{N}\}$ and $\{J_n : n \in \mathbb{N}\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $B \subseteq \bigcup_{n=1}^{\infty} J_n$ and $\sum l(I_n) < \varepsilon/2$ and $\sum l(J_n) < \varepsilon/2$. Define the open intervals $K_n = I_{n/2}$ if n is even and $K_n = I_{(n+1)/2}$ if n is odd. Then $A \cup B \subseteq \bigcup_{n=1}^{\infty} K_n$ and $\sum l(K_n) < \varepsilon$.

(c) Use (b) and mathematical induction.

(d) Suppose $\{A_k : k \in \mathbb{N}\}$ is a countable collection of sets of measure 0, and let $\varepsilon > 0$. Then $\forall k \in \mathbb{N}$, $\exists$ collection $\{I_{kn} : n \in \mathbb{N}\}$ of open intervals $\ni A_k \subseteq \bigcup_{n=1}^{\infty} I_{kn}$ and $\sum_{n=1}^{\infty} l(I_{kn}) < \varepsilon/2^k$. Use the diagonal scheme shown in Thm 2.8.5 to arrange this collection of intervals into a sequence $\{I_n\}$ of open intervals. Then $\bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} \left(\bigcup_{n=1}^{\infty} I_{kn}\right) = \bigcup_{n=1}^{\infty} I_n$. Since $\sum_{k=1}^{\infty} l(I_k) \leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} l(I_{kn})\right) < \sum_{k=1}^{\infty} \varepsilon/2^k = \varepsilon$, $\bigcup_{k=1}^{\infty} A_k$ has measure 0.

3. Let $A = \{x \in [a, b] : f(x) \neq g(x)\}$, and $\forall n \in \mathbb{N}$, let $A_n = \{x \in [a, b] : |f(x) - g(x)| \geq 1/n\}$. Then $x \in A \Leftrightarrow \exists n \in \mathbb{N} \ni x \in A_n$. Thus, $A = \bigcup_{n=1}^{\infty} A_n$. By Exercises 7.9.1 and 7.9.2, this set has measure 0.

5. Suppose f is integrable on $[a, b]$, and define $F(x) = \int_a^x f$. By FTC-II, $F'(x) = f(x)$ at every point of $[a, b]$ except possibly at a, b and points where f is

discontinuous. By Lebesgue's criterion (7.9.5), the set of points of discontinuity of f in $[a, b]$ must have measure 0.

6. Let $0 < \varepsilon < 1/4$, $\mathbb{Q} \cap [0,1] = \{r_n : n \in \mathbb{N}\}$, and $\forall n \in \mathbb{N}$, $J_n = (r_n - \varepsilon/2^n, r_n + \varepsilon/2^n)$. Then $A = \bigcup_{n=1}^{\infty} J_n$ is open and bounded (why?). Let $a = \inf A$, $b = \sup A$, and let $\mathcal{P} = \{x_0, x_1, x_2, \cdots, x_n\}$ be any partition of $[a, b]$. Since each $[x_{i-1}, x_i]$ contains a rational number, $\overline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i = \sum_{i=1}^{n} \triangle_i = b - a \geq 1$, while $\underline{S}(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i = \sum \{\triangle_i : [x_{i-1}, x_i] \subseteq A\} \leq \sum_{n=1}^{\infty} l(J_n) = \sum_{n=1}^{\infty} \varepsilon/2^{n-1} = 2\varepsilon < 1/2$. $\therefore \overline{\int_a^b} f \geq 1$ and $\underline{\int_a^b} f \leq 1/2$, so f is not integrable on any interval containing A.

Chapter 8

EXERCISE SET 8.1

2. (a) 1/99 (b) 9876/9999 (c) 2/3 (d) 1/4 (e) 15/2

(f) $+\infty$ (g),(h),(j) diverge by general term test (i) 0

3. By Exercise 6.2.19, $\cos nx \nrightarrow 0$, and $\sin x \nrightarrow 0$ unless x is an integral multiple of π, in which case $\sin nx = 0$. Apply general term test.

5. Let $\sum_{n=1}^{\infty} b_n$ be the result of altering or deleting a finite number of terms from $\sum_{n=1}^{\infty} a_n$, and let a_{m-1} be the last term altered or deleted. Then $a_m = b_{m'}$ for some $m' \in \mathbb{N}$, and $\sum_{n=m}^{\infty} a_n = \sum_{n=m'}^{\infty} b_n$. Apply Ex. 4.

7. The n^{th} partial sum of $\sum_{k=1}^{\infty} (b_n - b_{n+1})$ is $S_n = b_1 - b_{n+1}$. $\therefore \{S_n\}$ converges $\Leftrightarrow \{b_{n+1}\}$ converges, and $\lim_{n \to \infty} S_n = b_1 - \lim_{n \to \infty} b_n$.

9. $\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1}{k+2} - \frac{1}{k+3}\right] = \lim_{n \to \infty} \left[\frac{1}{3} - \frac{1}{n+3}\right] = \frac{1}{3}$.

11. $\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1/2}{k} - \frac{1/2}{k+2}\right] = \frac{1}{2} \lim_{n \to \infty} \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2}\right] = \frac{5}{12}$.

13. Let $S_0 = 0$ and $S_n = \sum_{k=1}^{n} x_k$. Then $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} (S_k - S_{k-1})$.

15. (a) Suppose $\sum a_n$ converges. Let $\sum b_n$ be formed by inserting parentheses in $\sum a_n$. Then each partial sum of $\sum b_n$ is, after removing parentheses, a partial sum of $\sum a_n$, so the sequence of partial sums of $\sum b_n$ is a subsequence of the sequence of partial sums of $\sum a_n$. Thus, $\sum b_n = \sum a_n$.

(b) The contrapositive of (a).
(c) $\sum(-1)^n$ diverges, but $(1-1)+(1-1)+(1-1)+\cdots$ converges.

17. Suppose $A_n = \sum_{k=1}^{n} a_k \to A$ and $B_n = \sum_{k=1}^{n} k_k \to B$. Then, by the algebra of limits of sequences, $\sum_{k=1}^{n}(a_k + b_k) \to A + B$ and $\sum_{k=1}^{n} ca_k \to cA$.

EXERCISE SET 8.2

1. The sequence of partial sums of a nonnegative series is monotone increasing. Apply Cor. 2.5.4.

3. Converges. Comparison test, with $\sum\left(\frac{1}{2}\right)^k$.

5. Diverges. Comparison test, with $\sum\frac{1}{3\sqrt{n}}$.

7. Diverges. Comparison test, with $\sum\frac{1}{n+1}$.

9. Converges. Ratio test; $L = 1/2$.

11. Diverges. Root test; $R = +\infty$.

13. Diverges. Comparison test, with $\sum\frac{3}{n}$.

15. Converges by the integral test.

17. Converges by comparison with Exercise 8.2.15.

19. Diverges by the general term test.

21. Converges. Ratio test; $L = 1/e$.

23. Diverges. Ratio test; $L = +\infty$.

25. Converges. Comparison test, with geometric series $\sum(\sin 1)^k$.

27. Converges. Root test, $R = 1/9$.

29. Diverges, since it has a regrouping that diverges: $1 + \left(\frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{2}}\right) + \left(\frac{1}{\sqrt[3]{5}} + \frac{1}{\sqrt[3]{4}}\right) + \cdots$. The sequence of partial sums of this series is a subsequence of the sequence of partial sums of $\sum\frac{1}{\sqrt[3]{k}}$, a divergent p-series.

31. If $p > 1$, the integral test shows convergence. If $p = 1$, the integral test shows divergence. If $p < 1$, use the comparison test with the $p = 1$ case.

33. (a) Suppose $p > 1$. Show that $\frac{d}{dx}\frac{\ln x}{x^p} < 0$ if $\ln x > p$; hence $\left\{\frac{\ln n}{n^p}\right\}$ is eventually decreasing. By L'Hôpital's rule, $\frac{\ln x}{x^p} \to 0$, so the integral test applies, to show that the series converges.

(b) For $p = 1$ the series diverges (see Exercise 8.2.16).
(c) For $p < 1$, the comparison test using (b) shows divergence.

35. Both $L = 1$ and $R = 1$.

37. (a) Suppose $\overline{L} < 1$; choose any $\overline{L} < r < 1$. By ε-criterion for upper limit (2.9.7), $\frac{a_{n+1}}{a_n} < r$ for all but finitely many n, so by Thm. 8.2.10, $\sum a_n$ converges.

(b) Suppose $\underline{L} > 1$. If $\underline{L}$ is finite, then by the ε-criterion for lower limit (2.9.8), $\frac{a_{n+1}}{a_n} > 1$ for all but finitely many n, so by Thm. 8.2.10, $\sum a_n$ diverges. If $\underline{L} = +\infty$, then $\frac{a_{n+1}}{a_n} > 1$ for all but finitely many n, so the series diverges.

39. Modify the proof given in Exercise 8.2.37.

41. By formula (5) in the proof of the integral test (8.2.3), $\int_1^n \frac{1}{x}dx \le \sum\limits_{k=1}^{n} \frac{1}{k} - \frac{1}{n}$. Thus, $\frac{1}{n} \le \gamma_n$. So, $\{\gamma_n\}$ is bounded below. To show $\{\gamma_n\}$ is monotone decreasing, first show that $\gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \frac{\ln(n+1)-\ln n}{(n+1)-n}$. Then apply the MVT to the last fraction to see that $\ln(n+1) - \ln n = \frac{1}{c}$ for some $c \in (n, n+1)$. $\therefore \gamma_{n+1} - \gamma_n = \frac{1}{n+1} - \frac{1}{c} < 0$. Apply the monotone convergence theorem (2.5.3).

43. Show $\frac{a_{k+1}}{a_k} = \frac{2k+1}{2k+2}$, so $\lim\limits_{k\to\infty} k\left(1 - \frac{a_{k+1}}{a_k}\right) = \lim\limits_{k\to\infty} k\left(1 - \frac{2k+1}{2k+2}\right) = \frac{1}{2}$. Then use Raabe's test.

45. First show $\frac{a_{k+1}}{a_k} = \left(\frac{2k+2}{2k+3}\right)^p$, so $\lim\limits_{k\to\infty} k\left(1 - \frac{a_{k+1}}{a_k}\right) = \lim\limits_{k\to\infty} k\frac{(2k+3)^p-(2k+2)^p}{(2k+3)^p} = \lim\limits_{k\to\infty} \frac{2^{p-1}pk^p + \text{ terms of degree } \le p-1 \text{ in } k}{2^p k^p + \text{ terms of degree } \le p-1 \text{ in } k} = \frac{p}{2}$. By Raabe's test, the series converges if $\frac{p}{2} > 1$ and diverges if $\frac{p}{2} < 1$.

47. Suppose that $\forall n \in \mathbb{N}$, $0 < a_n < 1$.
(a) By continuity of $\ln x$, $a_n \to 0 \Rightarrow \ln(1 - a_n) \to 0$. By continuity of e^x, the converse is true.
(b) $(\Rightarrow)$ Suppose $\sum a_n$ converges. Then $a_n \to 0$ so $1 - a_n \to 1$. By L'Hôpital's rule, $\lim\limits_{x\to 0} \frac{x}{\ln(1-x)} = -\lim\limits_{x\to 0}(1 - x) = -1$. By the sequential criterion for continuity of $\frac{x}{\ln(1-x)}$ at 0, $\lim\limits_{n\to\infty} \frac{a_n}{-\ln(1-a_n)} = 1$. Note that $0 < a_n < 1 \Rightarrow 0 < 1 - a_n < 1 \Rightarrow \ln(1 - a_n) < 0 \Rightarrow -\ln(1 - a_n) > 0$. So, by the limit comparison test, $\sum \ln(1 - a_n)$ converges.

$(\Leftarrow)$ If $\sum \ln(1 - a_n)$ converges, then $\ln(1 - a_n) \to 0$, so $1 - a_n \to 1$, so $a_n \to 0$. Apply L'Hôpital's rule and the limit comparison test as in $(\Rightarrow)$.
(c) Let $b_n = (1 - a_1)(1 - a_2)\cdots(1 - a_n)$. Then $0 < b_n < 1$ and $\ln b_n = \sum \ln(1 - a_n)$. By (b), $\sum a_n$ diverges $\Leftrightarrow \{\ln b_n\}$ diverges. Since $\{\ln b_n\}$ is monotone decreasing, the only way it can diverge is to $-\infty$. Thus, $\sum a_n$ diverges $\Leftrightarrow \ln b_n \to -\infty \Leftrightarrow b_n \to 0$.

49. Let $a_k = \frac{1\cdot3\cdot5\cdot7\cdot\dots\cdot(2k-1)}{2\cdot4\cdot6\cdot8\cdot\dots\cdot(2k)}$. Let $b_k = 1 - \frac{a_{k+1}}{a_k} = 1 - \frac{2k+1}{2k+2} = \frac{1}{2k+2}$. Since $\sum b_k$ diverges, $a_k \to 0$ by Ex. 48.

EXERCISE SET 8.3

1. This series has a grouping that diverges: $(1-2) + \left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{2}{3}\right) + \left(\frac{1}{4} - \frac{2}{4}\right) + \cdots = -1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots$, so it diverges by Thm. 8.1.10.

3. Prove the contrapositive: Suppose $\sum a_n$ converges.
(a) If $\sum a_n$ converges absolutely, both $\sum a_n^+$ and $\sum a_n^-$ converge by (a).
(b) If $\sum a_n$ converges conditionally, both $\sum a_n^+$ and $\sum a_n^-$ diverge by (b).

5. By grouping pairs of successive terms we see that the even-numbered partial sums are $S_{2n} = \sum_{k=1}^{n} \left[\left(\frac{1}{2}\right)^k - \left(\frac{1}{3}\right)^k\right] = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k - \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$, so $S_{2n} \to 1 - \frac{1}{2} = \frac{1}{2}$. For the odd-numbered partial sums, $S_{2n+1} = S_{2n} + \frac{1}{2^{n+1}} \to \frac{1}{2}$. Thus, by Ex. 2.6.6, the series converges to $\frac{1}{2}$. The series converges absolutely, since $\sum |a_n| = \sum \left[\left(\frac{1}{2}\right)^k + \left(\frac{1}{3}\right)^k\right] = \sum \left(\frac{1}{2}\right)^k + \sum \left(\frac{1}{3}\right)^k$.

6. (a) Conv. but not abs. $n > 9{,}998$. (b) Conv. but not abs. $n > 31$.
(c) Diverges, by the general term test. (d) Converges absolutely. $n \geq 5$.
(e) Conv. absolutely. $n > 10$. (f) Conv. but not abs. $n > 646$.

8. $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = S_{2n}$ from Ex. 7.

10. (a),(d) converge conditionally (b),(e),(f) converge absolutely (c) diverges

12. Suppose $\sum |a_k|$ converges and $\forall n \in \mathbb{N}$, $|b_k| \leq B$. Then the partial sums $S_n = \sum_{k=1}^{n} |a_k|$ are bounded above, say by B'. Then $\forall n \in \mathbb{N}$, $\sum_{k=1}^{n} |a_k b_k| \leq B \sum_{k=1}^{n} |a_k| \leq BB'$. $\therefore \sum_{k=1}^{\infty} |a_k b_k|$ converges, by Thm. 8.2.2.

14. Consider $\{a_n\} = \left\{\left(1 + \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) - \left(\frac{1}{6} + \frac{1}{8}\right) + \cdots\right\}$. Then,
(a) $a_n = \frac{4n}{4n^2-1}$ if n is odd, $\frac{4n-2}{4n^2-4n}$ if n is even.
(b) $a_n \to 0$ (easy to show).
(c) For odd n, $\frac{a_{n+1}}{a_n} = \frac{8n^3+4n^2-2n-1}{8n^3+8n^2} < 1$. For even n, $\frac{a_{n+1}}{a_n} = \frac{8n^3-8n^2-2n-1}{8n^3+12n^2-2n-3} < 1$. Thus, $\forall n \in \mathbb{N}$, $\frac{a_{n+1}}{a_n} < 1$, so $\{a_n\}$ is monotone decreasing.
(d) By (b), (c), and the alternating series test, $\sum a_k$ converges.
(e) If S_n denotes the partial sum, then $S_{4n} =$ the sum of the first $4n$ terms of the alternating harmonic series. Thus, by Ex. 7, $S_{4n} \to \ln 2$, so $S_n \to \ln 2$.

16. The series $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$ diverges since its sequence $\{S_n\}$ of partial sums has the subsequence $S_{3n} = \sum_{k=1}^{n}\left[\frac{1}{3k-2}+\frac{1}{3k-1}-\frac{1}{3k}\right]$. $S_{3n} \to \infty$ since $\frac{1}{3k-2}+\frac{1}{3k-1}-\frac{1}{3k} = \frac{9k^2-2}{3k(3k-1)(3k-2)} > \frac{8k^2}{(3k)(9k^2)} = \frac{8}{27k}$. Apply Thm. 8.1.10.

18. Using a computer to check the inequalities, the first 24 terms are $1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\frac{1}{18}-\frac{1}{20}-\frac{1}{22}-\frac{1}{24}+\frac{1}{7}-\frac{1}{26}-\frac{1}{28}-\frac{1}{30}-\frac{1}{32}+\frac{1}{9}-\frac{1}{34}-\frac{1}{36}-\frac{1}{38}$.

EXERCISE SET 8.4

1. $\sum c_k = 1+0+0+0+0+\cdots = 1$.

3. For $|r|<1$, the series $\sum_{k=0}^{\infty} r^k$ converges absolutely, so the Cauchy product $\sum c_k$ of this series with itself must converge to $\left(\sum_{k=0}^{\infty} r^k\right)^2$. Now $c_k = \sum_{i=0}^{k} r^i\, r^{k-i} = (k+1)\,r^k$. Thus, $\sum_{k=0}^{\infty}(k+1)\,r^k = \left(\sum_{k=0}^{\infty} r^k\right)^2 = \left(\frac{1}{1-r}\right)^2$. So, $\sum_{k=0}^{\infty} k\,r^k + \sum_{k=0}^{\infty} r^k = \frac{1}{(1-r)^2}$, from which we get $\sum_{k=0}^{\infty} k\,r^k = \frac{1}{(1-r)^2} - \frac{1}{1-r} = \frac{r}{(1-r)^2}$.

5. Let $|r|<1$. We know that $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ and from Ex. 3, $\frac{1}{(1-r)^2} = \sum_{k=0}^{\infty}(k+1)\,r^k$. Thus, $\frac{1}{(1-r)^3} = \left(\sum_{k=0}^{\infty} r^k\right)\left(\sum_{k=0}^{\infty}(k+1)\,r^k\right)$. Since both of these series converge absolutely, their Cauchy product series $\sum c_k$ will have the same sum. Now, $c_k = \sum_{i=0}^{k} r^i\left[(k+1-i)\,r^{k-i}\right] = r^k \sum_{i=1}^{k+1} i = r^k\frac{(k+1)(k+2)}{2}$. Thus, $\frac{1}{(1-r)^3} = \sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}\,r^k$.

7. The series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ converges to $\ln 2$ (Exercise 8.3.7), and the geometric series $\sum_{k=1}^{\infty}\frac{1}{3^k}$ converges absolutely to $\frac{1/3}{1-1/3} = \frac{1}{2}$. Thus, by Mertens' theorem, their Cauchy product series $\sum_{k=1}^{\infty} c_k$ converges to the product of their sums, $\frac{1}{2}\ln 2$. Note that $c_k = \sum_{i=1}^{k}\frac{(-1)^i}{i}\,\frac{1}{3^{k+1-i}}$.

9. Let $a_k = \frac{(-1)^{k+1}}{k^{3/2}}$ and $b_k = \frac{(-1)^{k+1}}{k^{1/2}}$. Then

(a) $\sum a_k$ converges absolutely, since $\sum |a_k| = \sum \frac{1}{k^{3/2}}$, a convergent p-series.

(b) $\sum b_k$ converges by the alternating series test, but not absolutely, since $\sum |b_k| = \sum \frac{1}{\sqrt{k}}$, a divergent p-series.

(c) By Mertens' Thm., their Cauchy product $\sum c_k$ converges, but the convergence is not absolute, since $|c_k| = \left|\sum_{i=1}^{k} \frac{(-1)^{k+1}}{i^{3/2}} \frac{(-1)^{k+1}}{(k+1-i)^{1/2}}\right| = \sum_{i=1}^{k} \frac{1}{i^{3/2}(k+1-i)^{1/2}} \geq \frac{1}{1^{3/2}k^{1/2}} = \frac{1}{\sqrt{k}} \geq \frac{1}{k}$.

11. The k^{th} term of the Cauchy product $(\sum a_k)(\sum b_k + \sum c_k)$ is the k^{th} term of $(\sum a_k)(\sum(b_k + c_k))$. By definition, this is $\sum_{i=1}^{k} a_i(b_{k+1-i} + c_{k+1-i}) = \left(\sum_{i=1}^{k} a_i b_{k+1-i}\right) + \left(\sum_{i=1}^{k} a_i c_{k+1-i}\right)$, which is the sum of the k^{th} term of $(\sum a_k)(\sum b_k)$ and the k^{th} term of $(\sum a_k)(\sum c_k)$, so it must be the k^{th} term of $(\sum a_k)(\sum b_k) + (\sum a_k)(\sum c_k)$. Thus, the two series $(\sum a_k)(\sum b_k + \sum c_k)$ and $(\sum a_k)(\sum b_k) + (\sum a_k)(\sum c_k)$ have the same terms, so they must be equal.

13. (a) $\sum_{k=1}^{n}\left(\sum_{i=1}^{k} a_i b_{k+1-i}\right) = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots$
$= a_1(b_1+b_2+\cdots+b_n)+a_2(b_1+b_2+\cdots+b_{n-1})+a_3(b_1+b_2+\cdots+b_{n-2})+\cdots+a_nb_1$
$= a_1B_n + a_2B_{n-1} + \cdots + a_nB_1 = \sum_{k=1}^{n} a_kB_{n-k+1} = \sum_{k=1}^{n} a_k(B + \overline{B}_{n-k+1})$.

(b) $\sum_{k=1}^{n_0} |a_{n+1-k}| = |a_n| + |a_{n-1}| + |a_{n-2}| + \cdots + |a_{n+1-n_0}| = \sum_{k=n+1-n_0}^{n} |a_k|$ and $\sum_{k=n_0+1}^{n} |a_{n+1-k}| < \sum_{k=1}^{\infty} |a_k| = A'$.

EXERCISE SET 8.5

1. $\sum a_k = \sum b_k = \sum \frac{(-1)^k}{\sqrt{k}}$.

3. See the solution to Exercise 8.3.12.

5. Let $\{a_k\}$, $\{b_k\}$ be sequences. By Ex. 8.1.5, $\{a_k\}$ is the sequence of partial sums of some $\sum x_k$. Then $\forall k \in \mathbb{N}$, $a_k = \sum_{i=1}^{k} x_i$. Applying Abel's summation by parts (8.5.2) with $\{x_k\}$ in place of $\{a_k\}$, and with $m = 1$, we get $\sum_{k=1}^{n} x_kb_k = \sum_{k=1}^{n} a_k(b_k - b_{k+1}) + a_nb_{n+1} - (0)b_1$. That is, $\sum_{k=1}^{n} b_k(a_k - a_{k+1}) = a_nb_{n+1} - \sum_{k=1}^{n} a_k(b_{k+1} - b_k)$.

7. Assume $t \neq 2n\pi$ is fixed. By Lemma 8.5.4(b), $\sum_{k=1}^{n} \cos kt = \frac{\sin(n+1/2)t - \sin t/2}{2\sin t/2}$ $\leq \frac{2}{|2\sin t/2|}$.

9. (a) When $t = n\pi$, convergence is absolute since every term is 0. Suppose $t \neq n\pi$. Then $|\sin kt| \geq \sin^2 kt = \frac{1}{2}(1 - \cos 2kt)$, so $\sum_{k=1}^{n} \frac{1}{k}|\sin kt| \geq \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\cos k(2t)$. Recall that $\sum_{k=1}^{n} \frac{1}{k} = +\infty$ and $\sum_{k=1}^{n} \frac{1}{k}\cos k(2t)$ converges by Example 8.5.7(b). $\therefore \sum_{k=1}^{n} \frac{1}{k}|\sin kt|$ diverges.

11. Suppose $\{b_k\}$ is monotone increasing. Then $b_k \to b = \sup\{b_k\}$, so $\{b - b_k\}$ is a monotone decreasing sequence converging to 0. Thus, by Dirichlet's test, $\sum a_k(b - b_k)$ converges. Then $\sum a_k b_k = \sum a_k b - \sum a_k(b - b_k)$ converges. The proof is similar if $\{b_k\}$ is monotone decreasing.

13. (a) If $\sum |a_k|$ converges, then by Ex. 4, $\sum (a_k)^2$ converges. (b) $\{1/k\}$

15. $\{1/k\}$ is square summable. Apply Thm. 8.5.16.

17. Both $\{\sqrt{a_k}\}$ and $\{1/k\}$ are square summable. Apply Thm. 8.5.16.

EXERCISE SET 8.6

1. Suppose $\sum a_k(x_1 - c)^k$ does not converge absolutely, and $|x_2 - c| > |x_1 - c|$. If $\sum a_k(x_2 - c)^k$ converges, then by Thm. 8.6.2, $\sum a_k(x_1 - c)^k$ converges absolutely, a contradiction.

3. By Thm. 8.2.16, $\sum a_k(x_1 - c)^k$ converges absolutely when $\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k||x - c|^k} < 1$; that is, when $|x - c|\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} < 1$, and diverges when $|x - c|\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} > 1$. Let $R = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}$. Let $\rho = 1/R$ (0 if $R = \infty$, and ∞ if $R = 0$). Then the series converges absolutely when $|x - c| < \rho$ and diverges when $|x - c| > \rho$. Therefore, the radius of convergence is ρ.

4. (a) $(-2, 0)$, 1 (c) $\mathbb{R}$, ∞ (e) $(1, 5)$, 2 (g) $(-\frac{4}{3}, -\frac{2}{3}]$, $\frac{1}{3}$
(i) $(2, 4)$, 1 (k) $[-3, -1]$, 1 (m) $[-1, 1)$, 1

5. For (a) and (b), apply Thm. 8.1.12. For (c), observe that by Thm. 8.4.3,

$$f(x)g(x) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i(x-c)^i b_{k-i}(x-c)^{k-i}\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)(x-c)^k.$$

7. Suppose $f(x) = \sum_{k=0}^{\infty} a_k(x - c)^k$ on the interval $|x - c| < \rho$ and $c \neq 0$. Then $f(x) = g(u)$ where $g(u) = \sum_{k=0}^{\infty} a_k u^k$ and $u = x - c$. By Part (a), $\frac{d}{du}g(u) = \sum_{k=0}^{\infty} k a_k u^{k-1}$. By the chain rule, $f'(x) = \frac{d}{du}g(u)u'(x) = \sum_{k=0}^{\infty} k a_k(x - c)^{k-1}$.

9. (a) $\sum\limits_{k=0}^{\infty}(-x)^{3k}$ on $(-1,1)$ (c) $\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+3}$ on $(-1,1]$ (e) $\sum\limits_{k=0}^{\infty}(-1)^{k+1}kx^{3k-1}$ on $(-1,1)$

11. These functions are not differentiable at 0, so their Maclaurin coefficients do not exist.

13. (a) Suppose $f(x)=\sum\limits_{k=0}^{\infty}a_kx^k$ is even. Then $f(x)=f(-x)=\sum\limits_{k=0}^{\infty}a_k(-1)^kx^k$. Thus, $\sum\limits_{k=0}^{\infty}a_kx^k=\sum\limits_{k=0}^{\infty}a_k(-1)^kx^k$, so $\sum\limits_{k=0}^{\infty}a_k[1-(-1)^k]x^k=0=\sum\limits_{k=0}^{\infty}0\,x^k$. Then $\forall k\in\mathbb{N}$, $a_k[1-(-1)^k]=0$. $\therefore a_k=0$ when k is odd.

15. By 8.6.18 (b), $\tan^{-1}x=\sum\limits_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$ on $(-1,1)$. Testing endpoints, this series converges at ±1 by the alternating series test. $\therefore$ By Abel's Thm. (8.6.19), $\pi/4=\tan^{-1}1=\lim\limits_{x\to1^-}\tan^{-1}x=\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{k+1}$.

17. (a) Converges absolutely on $(-\infty,2)\cup(4,+\infty)$; diverges everywhere else.

(b) Converges absolutely in every interval $(2n\pi,2(n+1)\pi)$, $n\in\mathbb{Z}$, but diverges whenever $x=2n\pi$, $n\in\mathbb{Z}$.

EXERCISE SET 8.7

1. Assume (b). First, suppose $c<x$. Since $f^{(n+1)}$ is continuous on $[c,x]$, the first mean value theorem for integrals applies, so $\exists z\in(c,x)\ni\int_x^c(x-t)^nf^{(n+1)}(t)dt=(x-z)^nf^{(n+1)}(z)(x-c)$. For $c>x$, revise this proof slightly. $\therefore$ (c).

3. Let $p(x)$ be a polynomial of degree n. Then $p(x)$ is analytic at every $c\in\mathbb{R}$. The Maclaurin series for $p(x)$ is identical with $p(x)$, while the Taylor series of $p(x)$ about $c\neq0$ is a polynomial of degree n in $(x-c)$ which, when simplified, is identical with $p(x)$.

5. See the solutions of Exercises 6.5.5 and 6.5.13.

7. (a) $\sum\limits_{k=0}^{\infty}\frac{x^{k+2}}{k!}$ on $(-\infty,\infty)$ (b) $\sum\limits_{k=0}^{\infty}\frac{(-1)^kx^{2k+4}}{(2k+1)!}$ on $(-\infty,\infty)$

(c) $1+x-\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}-\frac{x^6}{6!}-\frac{x^7}{7!}+\cdots$ on $(-\infty,\infty)$

(d) $\sum\limits_{k=0}^{\infty}\frac{(-1)^kx^{k+2}}{k+1}$ on $(-1,1]$

(e) $1-\frac{2x^2}{2!}+\frac{2^3x^4}{4!}-\frac{2^5x^6}{6!}+\cdots+(-1)^k\frac{2^{2k-1}x^{2k}}{(2k)!}+\cdots$ on $(-\infty,\infty)$

(f) $\frac{2x^2}{2!}-\frac{2^3x^4}{4!}+\frac{2^5x^6}{6!}-\cdots+(-1)^{k+1}\frac{2^{2k-1}x^{2k}}{(2k)!}+\cdots$ on $(-\infty,\infty)$

9. $\cos x = \cos[c + (x - c)] = \cos c \cos(x - c) - \sin c \sin(x - c)$
$= \cos c \sum\limits_{k=0}^{\infty} \frac{(-1)^k (x-c)^{2k}}{(2k)!} - \sin c \sum\limits_{k=0}^{\infty} \frac{(-1)^k (x-c)^{2k+4}}{(2k+1)!} = \sum\limits_{k=0}^{\infty} a_k \frac{(x-c)^k}{k!}$
where $a_k = (-1)^{k/2} \cos c$ if k is even; $a_k = (-1)^{(k+1)/2} \sin c$ if k is odd.

11. (a) $x + \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} + \cdots + C$

(b) $\ln|x| - x + \frac{x^2}{2 \cdot 2!} - \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} - \frac{x^5}{5 \cdot 5!} + \cdots + C$

13. By Exercise 6.6.16, the Maclaurin series of f is $\sum\limits_{k=0}^{\infty} 0 \cdot x^k$.

15. $f(x) = (1 + x^2)^{-1}$, $f'(x) = \frac{-2x}{(1+x^2)^2}$, $f''(x) = \frac{6x-2}{(1+x^2)^3}$. In general, $f^{(n)}(x)$ is a rational function whose denominator is an integral power of $1 + x^2$ and so is never 0. Thus, f is infinitely differentiable everywhere. But its Maclaurin series is $\sum\limits_{k=0}^{\infty} (-1)^k x^{2k}$, which converges only in $(-1, 1)$.

17. $R_1 = -1$, but for $i \geq 2$, $R_i = 0$. $\therefore \sum\limits_{i=1}^{\infty} R_i = -1$;

$C_1 = 1$, but for $j \geq 2$, $C_j = 0$. $\therefore \sum\limits_{J=1}^{\infty} C_j = 1$.

19. (a) $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_{ij}$ is the sum of the entries in a rectangle in the upper left corner.

Suppose the sum by rows of $\sum\limits_{i,j=1}^{\infty} |a_{ij}|$ converges. Then, by Thm. 8.7.17, all the row sums $R_i = \sum\limits_{i,j=1}^{\infty} a_{ij}$ converge, and the sum by rows converges, say $\sum\limits_{i=1}^{\infty} R_i = S$. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow \left| \sum\limits_{i=1}^{n} R_i - S \right| < \varepsilon/2$, and $\forall i \in \mathbb{N}$, $\exists m_i \in \mathbb{N} \ni m \geq m_i \Rightarrow \left| \sum\limits_{j=1}^{m} a_{ij} - R_i \right| < \varepsilon/2^{i+1}$. Then $m, n \geq \max\{n_0, m_1, \cdots, m_{n_0}\} \Rightarrow \left| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{m} a_{ij} \right) - S \right| = \left| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{m} a_{ij} - R_i \right) + \sum\limits_{i=1}^{n} R_i - S \right|$
$\leq \left| \sum\limits_{i=1}^{n} \left(\sum\limits_{j=1}^{m} a_{ij} - R_i \right) \right| + \left| \sum\limits_{i=1}^{n} R_i - S \right| \leq \sum\limits_{i=1}^{n} \left| \sum\limits_{j=1}^{m} a_{ij} - R_i \right| + \frac{\varepsilon}{2} < \sum\limits_{i=1}^{n} \frac{\varepsilon}{2^{i+1}} + \frac{\varepsilon}{2} = \varepsilon.$

(b) For the double series of Ex. 17, $\forall i \in \mathbb{N}$, $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij} = 0$, $\sum\limits_{i=1}^{n+1} \sum\limits_{j=1}^{n} a_{ij} = 1$, and $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n+1} a_{ij} = -1$. Thus, $\nexists S, n_0 \ni m, n \geq n_0 \Rightarrow \left| \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} a_{ij} - S \right| < 2$.

Chapter 9

EXERCISE SET 9.1

1. (a) and (c), are subspaces; the others are not.

3. (a) $f_n(x) \to 0$ on $\mathbb{R}$. (c) $f_n(x) \to 0$ on $[0, \infty)$.
(b) $\lim_{n\to\infty} f_n(x) = -\pi/2$ if $x < 0$, 0 if $x = 0$, $\pi/2$ if $x > 0$.
(d) $f_n(x) \to 0$ if $x \in (-2, 0) \cup (0, 2)$, 1 if $x = 0$.
(e) $f_n(x) \to 0$ if $x \neq \frac{(2k+1)\pi}{2}$, 1 if $x = \frac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$.
(f) f_n converges pointwise on $\mathbb{R} - \{(2k+1)\pi : k \in \mathbb{Z}\}$: to 1 if $x = 2k\pi$, to 0 if $x \neq k\pi$, $k \in \mathbb{Z}$.
(g) $f_n(x)$ converges pointwise on $\mathbb{R}$ to 0 if $x = 0$, 1 if $x \neq 0$.
(h) $f_n(x) \to 0$ on $\mathbb{R}$.
(i) Converges pointwise on $\mathbb{R} - \{-1\}$ to 0 if $|x| < 1$, 1 if $|x| > 1$, $1/2$ if $x = 1$.
(j) $f_n(x) \to x/3$ on $\mathbb{R}$. (l) $f_n(x) \to e^x$ on $\mathbb{R}$.
(k) Converges pointwise on $[0, \infty)$ to 0 if $x > 0$, -1 if $x = 0$.

5. Yes. Each function $f_n(x) = |x|$ if $|x| \geq 1/n$, $\frac{1}{n} - |x|$ if $|x| < 1/n$, has a local max at 0, but the limit function, $f(x) = |x|$ does not.

7. $\forall n \in \mathbb{N}$, $\int_0^1 f_n = 3 + 1/n$, so $\lim_{n\to\infty} \int_0^1 f_n = 3$. But, $\int_0^1 \lim_{n\to\infty} f_n = \int_0^1 0 = 0$.

9. (a) $\forall 0 < x \leq 1$, $\sqrt[n]{x} \to 1$ by Thm. 2.3.9. $\therefore$ $\sqrt[2n-1]{x} \to 1$, since it is a subsequence.

(b) $\forall -1 \leq x < 0$, $x^{1/(2n-1)} = -(-x)^{1/(2n-1)} \to -1$ by (a).

(c) $$x^{\frac{2n}{2n-1}} = x^{1+\frac{1}{2n-1}} = x \cdot x^{\frac{1}{2n-1}} \to \left\{ \begin{array}{c} x \text{ if } 0 < x \leq 1, \\ -x \text{ if } -1 \leq x < 0 \end{array} \right\} \text{ on } [-1, 1].$$

EXERCISE SET 9.2

1. (a) $\sup\{|f(x)| : x \in S\} \geq 0$, and $= 0 \Leftrightarrow \forall x \in S$, $f(x) = 0$.
(b) $\sup\{|f(x) + g(x)| : x \in S\} \leq \sup\{|f(x)| : x \in S\} + \sup\{|g(x)| : x \in S\}$.
(c) $\sup\{|rf(x)| : x \in S\} = |r| \sup\{|f(x)| : x \in S\}$.

4. They are all equivalent.

5. (b) $\forall n \in \mathbb{N}$, $\|f_n - f\| = 2n \not\to 0$ (d) $\forall n \in \mathbb{N}$, $\|f_n - f\| = 1 \not\to 0$.

7. $\forall x \in \mathbb{R}$, $\left|\sin\left(x + \frac{1}{n}\right) - \sin x\right| = |\cos c|/n \leq 1/n$ for some $x < c < x + 1/n$. $\therefore \|fn - f\| \leq 1/n \to 0$.

8. (a) and (d) converge uniformly on $[0, +\infty)$.

9. (a) $\forall n \in \mathbb{N}$, $\|f_n\| = +\infty$, so convergence is not uniform on $\mathbb{R}$. But on any compact $[a, b]$, the convergence is uniform since $\|f_n\| = \max\left\{\frac{|a|}{n}, \frac{|b|}{n}\right\} \to 0$.

(b) $\forall n \in \mathbb{N}$, $\|f_n - f\| = \sup\{|f_n(x) - \frac{\pi}{2}| : x \in \mathbb{R}\} = \frac{\pi}{2}$, so convergence is not uniform on $\mathbb{R}$. But $\forall a > 0$, convergence is uniform on $[a, \infty)$ since $\|f_n - f\| = \sup\{|\tan^{-1} nx - \frac{\pi}{2}| : x \geq a\} = \tan^{-1} na - \frac{\pi}{2} \to 0$. Similarly on $(-\infty, -a]$.

(c) $f_n(x) \to 0$ on $[0, \infty)$. From $f'(x) = \frac{n(1-nx)}{e^{nx}}$, we find that f_n has its max when $x = 1/n$. Thus, $\|f_n\| = f_n(1/n) = 1/e \not\to 0$, so convergence is not uniform on $[0, \infty)$. But $\forall a > 0$, convergence is uniform on $[a, \infty)$ since $\|f_n\| = f_n(a) = \frac{na}{e^{na}} \to 0$ by L'Hôpital's rule.

(d) Convergence is uniform on any $[a, b]$, where $0 < a < b < 2$ or $-2 < a < b < 0$, but is not uniform on any interval containing -2, 0, or 2, or on any unbounded interval.

(e) Convergence is uniform on any $[a, b]$, where $\frac{(2k-1)\pi}{2} < a < b < \frac{2(k+1)\pi}{2}$, but is not uniform on any interval containing any $\frac{2(k+1)\pi}{2}$, $k \in \mathbb{Z}$.

(f) Convergence is uniform on any $[a, b]$, where $(k-1)\pi < a < b < k\pi$, but is not uniform on any interval containing $k\pi$, $k \in \mathbb{Z}$.

(g) Convergence is not uniform on any interval containing more than one point.

(h) Convergence is uniform on any $(-\infty, a]$ or $[b, +\infty)$, where $a < 0 < b$, but is not uniform on any interval containing 0.

(i) Convergence is uniform on any interval not containing -1 or 1.

(j), (l) Convergence is uniform on any compact $[a, b]$.

(k) Convergence is uniform on any $[a, b]$, where $0 < a < b$.

11. $\|f_n\| = \|f_n - f + f\| \leq \|f_n - f\| + \|f\|$, so $\|f_n\| - \|f\| \leq \|f_n - f\|$. Also, $\|f\| = \|f - f_n + f_n\| \leq \|f - f_n\| + \|f_n\|$, so $\|f\| - \|f_n\| \leq \|f - f_n\|$. Therefore, $|\,\|f_n\| - \|f\|\,| \leq \|f_n - f\|$. So, $f_n \to f$ uniformly $\Rightarrow \|f_n - f\| \to 0 \Rightarrow \|f_n\| \to \|f\|$.

13. ($\Rightarrow$) Suppose $\sum\limits_{k=0}^{\infty} f_k = f$ uniformly on $\mathcal{S}$, and let $S_n = \sum\limits_{k=0}^{n} f_k$. Then $S_n \to f$ uniformly on $\mathcal{S}$. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N} \ni n \geq n_0 \Rightarrow S_n - f$ is bounded and $\|S_n - f\| < \varepsilon/2$. Then $m, n \geq n_0 \Rightarrow \|S_m - S_n\| \leq \|S_n - f\| + \|f - S_m\| < \varepsilon$. That is, $m, n \geq n_0 \Rightarrow \left\| \sum\limits_{k=m+1}^{n} f_k \right\| < \varepsilon$.

($\Leftarrow$) Suppose the hypotheses. Let $\varepsilon > 0$. Then $\exists n_0 \in \mathbb{N} \ni m, n \geq n_0 \Rightarrow \sum_{k=m+1}^{n} f_k$ is bounded and $\left\| \sum_{k=m+1}^{n} f_k \right\| < \varepsilon$, so $\|S_m - S_n\| < \varepsilon$. By Thm 9.2.7, $\{S_n\}$ converges uniformly on $\mathcal{S}$.

15. Suppose $\sum_{k=0}^{\infty} a_k(x-c)^k$ has radius of convergence $\rho > 0$. Let $0 < r < \rho$. By Thm 8.6.2, $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely in $[c-r, c+r]$, so $\sum_{k=0}^{\infty} a_k r^k$ converges absolutely. On $[c-r, c+r]$, $\|a_k(x-c)^k\| \leq |a_k| r^k$, and $\sum_{k=0}^{\infty} |a_k| r^k$ converges. $\therefore$ By Weierstrass M-test, $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges uniformly on $[c-r, c+r]$.

17. (a) Let $f_k(x) = (-1)^k x^k$ and $g_k(x) = 1/k$. First we prove that $\forall\ 0 \leq x \leq 1$, $0 < \sum_{k=1}^{n} f_k < 1$, using math induction. The case $n = 1$ is trivial. To prove the general induction step, assume $0 < \sum_{k=1}^{n} f_k < 1$. Then $0 < 1 - \sum_{k=1}^{n} f_k < 1$, i.e., $0 < 1 - \sum_{k=1}^{n} (-1)^k x^k < 1$, so $0 < x \left[1 - \sum_{k=1}^{n} (-1)^k x^k \right] < 1$, so $0 < x + \sum_{k=1}^{n} (-1)^{k+2} x^{k+1} < 1$; i.e., $0 < \sum_{k=1}^{n+1} (-1)^{k+1} x^{k+1} < 1$; i.e., $0 < \sum_{k=1}^{n+1} f_k(x) < 1$.

By Dirichlet's test (9.2.16), $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on $[0, 1]$. But the convergence is not absolute at $x = 1$ since $\sum \frac{1}{k}$ diverges.

19. (a), (d), and (e) converge uniformly on the indicated sets.

EXERCISE SET 9.3

1. Let $0 < a < 1$. Let $S_n = \sum_{k=0}^{n} x^k$. Since the radius of convergence of $\sum_{k=0}^{\infty} x^k$ is 1, $S_n \to f(x) = \frac{1}{1-x}$ uniformly on $[0, a]$, by Thm. 9.2.15. For contradiction, suppose $S_n \to f$ uniformly on $[0, 1)$. Now, $\forall n \in \mathbb{N}$, $\lim_{x \to 1^-} S_n(x) = \lim_{x \to 1^-} \sum_{k=0}^{n} x^k = \sum_{k=0}^{n} \lim_{x \to 1^-} x^k = n + 1$. Thus, each S_n has a finite limit as $x \to 1^-$, but the limit function f does not. This would contradict Thm. 9.3.5.

3. Let $f_n(x) = 1/n$ if x is rational, 0 if x is irrational. Then, just like the Dirichlet function, each f_n is discontinuous everywhere, but $\|f_n\| = 1/n \to 0$, so $f_n \to 0$ uniformly on $(-\infty, \infty)$.

5. On $[0,2]$ the limit function is $f(x) = 0$ if $0 < x \le 2$, 1 if $x = 0$. Then $\forall n$, $\|f_n - f\| = 1$, so the convergence is not uniform.

Let $0 < \varepsilon < 2$. Then on $[\varepsilon, 2]$, $\|f_n\| = e^{-n\varepsilon^2} \to 0$, so $f_n \to 0$ uniformly on $[\varepsilon, 2]$. Thus, by Thm. 9.3.8, $\lim\limits_{n\to\infty} \int_\varepsilon^2 f_n = \int_\varepsilon^2 f = 0$. Then $\lim\limits_{n\to\infty} \int_0^2 f_n = \lim\limits_{n\to\infty} \left[\int_0^\varepsilon f_n + \int_\varepsilon^2 f_n\right] \le \lim\limits_{n\to\infty} \left[\int_0^\varepsilon 1 + \int_\varepsilon^2 f_n\right] = \varepsilon$. By the forcing principle, $\lim\limits_{n\to\infty} \int_0^2 f_n = 0 = \int_0^2 f$.

7. On $[0,\infty)$, $\|f_n\| = 1/n \to 0$, so $f_n \to 0$ uniformly. But $\int_0^\infty f_n = 1/2$, so $\lim\limits_{n\to\infty} \int_0^\infty f_n \ne \int_0^\infty f = 0$.

9. Let $f(x) = \sum\limits_{k=1}^{n} \frac{x^k}{k(k+1)}$. The radius of convergence is $\rho = 1$. Thus, by Cor. 9.3.12, f is differentiable on $(-1,1)$, and $f'(x) = \sum\limits_{k=1}^{n} \frac{kx^{k-1}}{k(k+1)} = \sum\limits_{k=1}^{n} \frac{x^{k-1}}{k+1}$.

11. (a) This series is the term-by-term differentiation of the series $g(x) = \sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ on $(-1,1)$. Thus, by Cor. 9.3.12, $f(x) = g'(x) = \frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$ for all x in $(-1,1)$.

(b) The radius of convergence is 1. Thus, by Cor. 9.3.12, f is differentiable term-by-term on $(-1,1)$, so $f'(x) = \frac{x^2}{1} + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{4} + \cdots = x\left[\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x4}{4} + \cdots\right] = -x\ln(1-x)$. Thus, on $(-1,1)$, $f(x) = -\int x\ln(1-x)dx$. Integrating by parts, $f(x) = \frac{1}{2}(1-x^2)\ln(1-x) + \frac{1}{4}x^2 + \frac{1}{2}x$.

13. (a) Each f_n is differentiable on $\mathbb{R}$, and $f'(x) = -\sin nx$.

(b) $\|f_n\| = 1/n \to 0$, so $f_n \to 0$ uniformly on $\mathbb{R}$.

(c) $\forall x \ne n\pi$ $(n \in \mathbb{Z})$, $\{f_n'(x)\}$ diverges, by Exercise 2.6.20.

(d) This does not contradict Thm. 9.3.11 because one of the hypotheses of that theorem (uniform convergence of $\{f_n'\}$) is not satisfied.

15. Let $a > 1$.

(a) $\forall x \in [a,\infty)$, $\forall k \in \mathbb{N}$, $k^x \ge k^a > 0$, so $0 < \frac{1}{k^x} \le \frac{1}{k^a}$. Thus, $\forall k \in \mathbb{N}$, $\left\|\frac{1}{k^x}\right\| \le \frac{1}{k^a}$ and since $\sum \frac{1}{k^a}$ converges, the Weierstrass M-test guarantees that $\sum \frac{1}{k^x}$ converges uniformly on $[a,\infty)$.

(b) $\forall k \in \mathbb{N}$, let $f_k(x) = 1/k^x = k^{-x}$. Consider the series $\sum\limits_{k=1}^{n} f_k'(x) = -\sum\limits_{k=1}^{n} \frac{\ln k}{k^x}$. Now, $\forall x \in [a,\infty)$, $0 < \frac{\ln k}{k^x} \le \frac{\ln k}{k^a}$ and the series $\sum\limits_{k=1}^{n} \frac{\ln k}{k^a}$ converges by Ex. 8.2.33. Thus, by the Weierstrass M-test, $\sum\limits_{k=1}^{n} \frac{\ln k}{k^x}$ converges uniformly on $[a,\infty)$. Therefore, by Thm. 9.3.11, $\forall x \ge a$, $\zeta'(x) = -\sum\limits_{k=1}^{n} \frac{\ln k}{k^x}$.

(c) Since this is true $\forall x \geq a > 0$, it is true for all $x > 0$.

Appendix

EXERCISE SET A.1-A

1. $\sim (E \vee P)$ or $\sim E \wedge \sim P$

3. $(H \wedge B) \wedge \sim (H \wedge B \wedge R)$

5. $Y \wedge (E \Rightarrow A)$

7. $(\sim S \vee F) \Rightarrow \sim P$

9. $P \Rightarrow [B \wedge L \wedge \sim (D \vee I)]$

11. $(D_4 \wedge D_3) \Rightarrow D_{12}$

13.

P	Q	R	S	$P \wedge \sim Q$	$Q \vee R$	$R \Rightarrow S$
T	F	T	T	T	T	T

15. (a) $A \Rightarrow U$ (b) $W \Rightarrow E$ (c) $(\sim W \vee \sim U) \Rightarrow \sim A$

(d) $(E \Rightarrow W) \wedge \sim (E \Rightarrow U)$

(e) If I understand the basic ideas and I work hard, I enjoy mathematics.

(f) If the assignment is not easy, then I don't understand the basic ideas or I don't work hard.

(g) It is not true that if I enjoy mathematics then if I work hard the assignment will be easy.

EXERCISE SET A.1-B

1.

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

3.

P	Q	$P \vee Q$	$P \Rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

5.

P	$\sim P$	$P \wedge \sim P$	$\sim (P \wedge \sim P)$
T	F	F	T
F	T	F	T

7.

P	Q	R	$P \vee Q$	$P \vee R$	$(P \vee Q) \Rightarrow (P \vee R)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	T
F	F	F	F	F	T

(not a tautology)

9.

P	Q	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow P)$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

(not a tautology)

11.

P	Q	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(4th and 7th columns identical)

13.

P	Q	$P \vee Q$	$\sim P$	$\sim P \Rightarrow Q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

(3rd and 5th columns identical)

15.

P	Q	$P \wedge Q$	$\sim (P \wedge Q)$	$\sim P$	$Q \Rightarrow \sim P$
T	T	T	F	F	F
T	F	F	T	F	T
F	T	F	T	T	T
F	F	F	T	T	T

(4th and 6th columns identical)

17.

P	Q	$P \Leftrightarrow Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

(3rd and 6th columns identical)

19.

P	Q	$P \Leftrightarrow Q$	$\sim (P \Leftrightarrow Q)$	$\sim Q$	$P \Leftrightarrow \sim Q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	F	T	F	T
F	F	T	F	T	F

(4th and 6th columns identical)

21. $\sim (P \vee \sim Q) \equiv \sim P \wedge \sim\sim Q \equiv \sim P \wedge Q$

23. $\sim [(P \vee Q) \Rightarrow R] \equiv (P \vee Q) \wedge \sim R$

25. $\sim [P \Rightarrow (Q \Rightarrow R)] \equiv P \wedge \sim (Q \Rightarrow R) \equiv P \wedge (Q \wedge \sim R)$

27. $\sim [(Q \wedge P) \Rightarrow (R \vee S] \equiv (Q \wedge P) \wedge \sim (R \vee S) \equiv (Q \wedge P) \wedge (\sim R \wedge \sim S)$

29. $\sim (J \wedge M) \equiv \sim J \vee \sim M$. John is not innocent or Mary's charge is not a lie.

31. $\sim (C \Rightarrow O) \equiv C \wedge \sim O$. You come tonight but I don't order pizza.

33. $\sim (C \vee F) \equiv \sim C \wedge \sim F$. I am not going to the concert and I am not going to the football game.

35. $\sim (T \Rightarrow P) \equiv T \wedge \sim P$. This statement is true but I can't prove it.

37. $\sim (\sim P \wedge \sim C) \equiv P \vee C$. 1 is a prime number or 1 is a composite number.

39. $\sim [H \Rightarrow (G \vee S)] \equiv H \wedge (\sim G \wedge \sim S)$. You come to my house tonight but I don't grill a steak and I don't make stir fry.

EXERCISE SET A.2

Part A:

1. $\forall x,\ [L(x) \Rightarrow W(x)]$; domain = {people}

3. $\sim \exists x \ni [A(x) \wedge M(x)]$ or $\forall x,\ [\sim A(x) \vee \sim M(x)]$ or $\forall x,\ [A(x) \Rightarrow \sim M(x)]$; domain = {people}

5. $[\exists x \ni G(x)] \wedge \sim [\exists x \ni C(x)]$ or $[\exists x \ni G(x)] \wedge [\forall x, \sim C(x)]$; domain = {people in this room}

7. $\exists x \ni [L(x) \wedge \sim G(x)]$; domain = {people}

9. $\forall x,\ [G(x) \Rightarrow C(x)]$; domain = {people in this room}

11. $\sim \exists x \ni x^2 + 3x - 1 = 0$, or $\forall x,\ x^2 + 3x - 1 \neq 0$; domain = {real numbers}

13. $\exists x \ni [\sim A(x)]$ or $\sim \forall x, A(x)$; domain = {your ideas}

15. $[\forall x,\ E(x)] \wedge \sim [\forall x,\ S(x)]$ or $[\forall x,\ E(x)] \wedge [\exists x \ni \sim S(x)]$; domain = {people}

17. $\forall x,\ [E(x) \Rightarrow \exists y \ni \sim H(y)]$; domain of x and y = {people}

19. $\forall x,\ \forall y,\ B(x,y)$, where $B(x,y) \equiv x$ is brother of y; domain of x and y = {people}

21. $\sim \exists x,\ \sim \exists y \ni [x \neq y \wedge L(x,y)]$ or $\forall x,\ \forall y,\ [x \neq y \Rightarrow \sim L(x,y)]$, where $L(x,y) \equiv x$ looks exactly like y; domain of x and y = {people}

23. $\exists x, \exists y \ni [S(x,y) \land \sim C(x,y)]$; domain of x and $y = \{\text{triangles}\}$

25. $\exists x \ni \forall y, [(y \neq x) \Rightarrow x < y]$; domain of x and $y = \{\text{numbers in this set}\}$

27. $\sim \exists x \ni \forall y, x \geq y$ or $\forall x, \exists y \ni y > x$; domain of x and $y = \{\text{real numbers}\}$

Part B:

1. $\exists x \ni \sim [L(x) \Rightarrow W(x)]$ or $\exists x \ni [L(x) \land \sim W(x)]$. Not all lawyers are wealthy. Or, some lawyers are not wealthy.

3. $\exists x \ni [A(x) \land M(x)]$. There is someone who wants an A in this course and can afford to miss an assignment.

5. $\sim [\exists x \ni G(x)] \lor \sim\sim [\exists x \ni C(x)]$ or $[\forall x, \sim G(x)] \lor [\exists x \ni C(x)]$. Either no one in this room is guilty, or someone in this room will be charged.

7. $\forall x, \sim [L(x) \land \sim G(x)]$ or $\forall x, [\sim L(x) \lor \sim\sim G(x)]$ or $\forall x, [L(x) \Rightarrow G(x)]$. Everyone waiting in line for the show will get in.

9. $\exists x \ni\sim [G(x) \Rightarrow C(x)]$ or $\exists x \ni [G(x) \land \sim C(x)]$. There is someone in the room who is guilty but need not confess now.

11. $\exists x \ni x^2 + 3x - 1 = 0$. The equation $x^2 + 3x - 1 = 0$ has a real number solution.

13. $\forall x, A(x)$. I can agree with all of your ideas.

15. $\sim [\forall x, E(x)] \lor \sim [\sim \forall x, S(x)]$ or $[\exists x \ni \sim E(x)] \lor [\forall x, S(x)]$. Either someone is not eligible to try, or everyone will succeed.

17. $\exists x \ni \sim [E(x) \Rightarrow \exists y \ni \sim H(y)]$ or $\exists x \ni [E(x) \land \sim \exists y \ni \sim H(y)]$ or $\exists x \ni [E(x) \land \forall y, H(y)]$. There is someone who could get elected and with whom everyone would be happy.

19. $\exists x, \exists y \ni \sim B(x,y)$. There exists a pair of men who are not brothers.

21. $\exists x, \exists y \ni [x \neq y \land L(x,y)]$. There are two people who look exactly alike.

23. $\forall x, \forall y, \sim [S(x,y) \land \sim C(x,y)]$ or $\forall x, \forall y, [\sim S(x,y) \lor C(x,y)]$ or $\forall x, \forall y, [S(x,y) \Rightarrow C(x,y)]$. If triangles are similar, they are congruent.

25. $\forall x, \exists y \ni \sim [(y \neq x) \Rightarrow x < y]$ or $\forall x, \exists y \ni [(y \neq x) \land x \geq y]$. For every member of this set there is a number in the set smaller than it. (No number in the set is smaller than all the rest.)

27. $\exists x \ni \forall y, x \geq y$. There is a largest real number.

Part C:

1. (a) For every integer there is an odd integer that, when added to it, yields an even sum. False: take $n = 0$.

Negation: $\exists x \in I \ni \forall y \in O$, $x + y \notin E$. There is an integer such that no odd integer added to it yields an even sum.

(b) For every odd integer there is an odd integer that, when added to it, yields an even sum. True. (Odd + odd = even.)

Negation: $\exists x \in O \ni \forall y \in O$, $x + y \notin E$. There is an odd integer such that no odd integer added to it yields an even sum.

(c) There is an integer that, when added to any other integer, yields the other integer as sum. True: that integer is 0.

Negation: $\forall y \in I$, $\exists x \in I \ni x + y \neq x$. There is no integer that, when added to any other integer, yields the other integer as sum.

(d) For every integer, there is an integer that, when added to it, yields 0 as sum. True: the second integer is the negative of the first.

Negation: $\exists x \in I \ni \forall y \in I$, $x + y \neq 0$. There is an integer that has no "additive inverse."

EXERCISE SET B.1

1.

	$A \cap B$	$A \cup B$	A^c	B^c
(a)	$\{4,5\}$	$\{1,2,3,4,5,6,7\}$	$\{6,7,8,9,10\}$	$\{1,2,3,8,9\}$
(b)	$\varnothing$	$\{1,2,3,4,5,6\}$	$\{4,5,6,7,8,9,10\}$	$\{1,2,3,7,8,9,10\}$
(c)	$[3,4)$	$(0,6]$	$(-\infty,0] \cup [4,+\infty)$	$(-\infty,3) \cup (6,+\infty)$
(d)	$[1,2)$	$(-\infty,+\infty)$	$[2,+\infty)$	$(-\infty,1)$

	$A - B$	$B - A$	$A \cup (B \cap C)$	$A \cap (B \cup C)$
(a)	$\{1,2,3\}$	$\{6,7\}$	$\{1,2,3,4,5\}$	$\{3,4,5\}$
(b)	$\{1,2,3\}$	$\{4,5,6\}$	$\{1,2,3,4,6\}$	$\{2\}$
(c)	$(0,3)$	$[4,6]$	$(0,5)$	$(2,4)$
(d)	$(-\infty,1)$	$[2,+\infty)$	$(-\infty,2)$	$(-1,2)$

3. $x \in (A \cap B)^c \Leftrightarrow x \notin (A \cap B) \Leftrightarrow \sim (x \in A \text{ and } x \in B)$
$\Leftrightarrow x \notin A$ or $x \notin B$ (by de Morgan's law in logic)
$\Leftrightarrow x \in A^c$ or $x \in B^c \Leftrightarrow x \in A^c \cup B^c$

5. $x \in (A \cap B) \cup (A \cap C) \Leftrightarrow x \in A \cap B$ or $x \in A \cap C$
$\Leftrightarrow (x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$
$\Leftrightarrow x \in A$ and $(x \in B \text{ or } x \in C)$ by the distributive law in logic
$\Leftrightarrow x \in A$ and $x \in B \cup C$
$\Leftrightarrow x \in A \cap (B \cup C)$

7.

(a) $\bigcap_{n\in\mathbb{N}} (-n,n) = (-1,1)$ $\quad$ $\bigcup_{n\in\mathbb{N}} (-n,n) = (-\infty,+\infty)$

$\bigcup_{n\in\mathbb{N}} (-n,n)^c = (-\infty,-1] \cup [1,+\infty)$ $\quad$ $\bigcap_{n\in\mathbb{N}} (-n,n)^c = \varnothing$

(b) $\bigcap_{n\in\mathbb{N}} (-\infty,n) = (-\infty,1)$ $\quad$ $\bigcup_{n\in\mathbb{N}} (-\infty,n) = (-\infty,+\infty)$

$\bigcup_{n\in\mathbb{N}} (-\infty,n)^c = [1,+\infty)$ $\quad$ $\bigcap_{n\in\mathbb{N}} (-\infty,n)^c = \varnothing$

(c) $\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n},\frac{1}{n}\right) = \{0\}$ $\quad$ $\bigcup_{n\in\mathbb{N}} \left(-\frac{1}{n},\frac{1}{n}\right) = (-1,1)$

$\overset{c}{\bigcup_{n\in\mathbb{N}}} \left(-\frac{1}{n},\frac{1}{n}\right)^c = (-\infty,0) \cup (0,+\infty)$ $\quad$ $\overset{c}{\bigcap_{n\in\mathbb{N}}} \left(-\frac{1}{n},\frac{1}{n}\right)^c = (-\infty,-1] \cup [1,+\infty)$

(d) $\bigcap_{n\in\mathbb{N}} \left[-2+\frac{1}{n},2-\frac{1}{n}\right] = [-1,1]$ $\quad$ $\bigcup_{n\in\mathbb{N}} \left[-2+\frac{1}{n},2-\frac{1}{n}\right] = (-2,2)$

$\bigcup_{n\in\mathbb{N}} \left[-2+\frac{1}{n},2-\frac{1}{n}\right]^c = (-\infty,-1) \cup (1,\infty)$

$\bigcap_{n\in\mathbb{N}} \left[-2+\frac{1}{n},2-\frac{1}{n}\right]^c = (-\infty,-2] \cup [2,\infty)$

(e) $\bigcap_{n\in\mathbb{N}} (n,n+1) = \varnothing$ $\quad$ $\bigcup_{n\in\mathbb{N}} (n,n+1) = (1,+\infty) - \mathbb{N}$

$\bigcup_{n\in\mathbb{N}} (n,n+1)^c = \mathbb{R}$ $\quad$ $\bigcap_{n\in\mathbb{N}} (n,n+1)^c = (-\infty,-1] \cup \mathbb{N}$

9. $x \in \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right)^c \Leftrightarrow x \notin \bigcup_{\lambda\in\Lambda} A_\lambda$
$\Leftrightarrow \sim \exists\lambda \in \Lambda \ni x \in A_\lambda \Leftrightarrow \forall\lambda \in \Lambda,\ x \notin A_\lambda$
$\Leftrightarrow \forall\lambda \in \Lambda,\ x \in A_\lambda^c \Leftrightarrow x \in \bigcap_{\lambda\in\Lambda} A_\lambda^c.$

11. $x \in B \cap \left(\bigcup_{\lambda\in\Lambda} A_\lambda\right) \Leftrightarrow x \in B$ and $x \in \bigcup_{\lambda\in\Lambda} A_\lambda$
$\Leftrightarrow x \in B$ and $\exists\lambda \in \Lambda \ni x \in A_\lambda \Leftrightarrow \exists\lambda \in \Lambda \ni x \in B$ and $x \in A_\lambda$
$\Leftrightarrow \exists\lambda \in \Lambda \ni x \in B \cap A_\lambda \Leftrightarrow x \in \bigcup_{\lambda\in\Lambda} (B \cap A_\lambda).$

EXERCISE SET B.2

1. (a) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = \mathbb{R}$; 1-1; onto
(b) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = [-2,\infty)$; not 1-1; not onto
(c) $\mathcal{D}(f) = \left[\frac{4}{3},\infty\right)$; $\mathcal{R}(f) = [0,\infty)$; 1-1; not onto
(d) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = [0,\infty)$; not 1-1; not onto
(e) $\mathcal{D}(f) = \mathbb{R} - \{0\}$; $\mathcal{R}(f) = \{-1,1\}$; not 1-1; not onto
(f) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = (0,1)$; not 1-1; not onto

2. (a) $\mathbb{R}$ (b) $(-\infty, 4]$ (c) $[3, 4]$ (d) $[-2, -\sqrt{3}] \cup [\sqrt{3}, 2]$
(e) $(2, 4)$ (f) $(-2, 2)$ (g) $[-\sqrt{2}, \sqrt{2}]$ (h) $[-\sqrt{8}, -2] \cup [2, \sqrt{8}]$
(i) $(-2, 2)$ (j) $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$ (k) $\{-2, 2\}$ (l) $\{\pm\sqrt{5}\}$

3. (a) $\mathbb{R}$ (b) $(0, \infty)$ (c) $[1, 2]$ (d) $(0, 4)$ (e) $(1, \infty)$
(f) $\left[\frac{1}{2}, \sqrt{2}\right]$ (g) $[0, 1]$ (h) $(1, 3)$ (i) $(-\infty, 0)$ (j) $\varnothing$

5. (a) $(-4, 0)$, $(-4, 0)$ (b) $\varnothing$, $(-4, 0)$ (c) $(-4, 0]$, $(-4, -2]$
(d) $(-\infty, 16)$, $(-\infty, 0] \cup \left(f^{-1}(2), f^{-1}(4)\right)$
(e) $\left(f^{-1}(2), f^{-1}(3)\right]$, $\left(f^{-1}(2), f^{-1}(3)\right]$, (f) $\left(-\infty, f^{-1}(2)\right]$, $\left(-\infty, f^{-1}(2)\right]$

7. Let $y \in f(C_1) - f(C_2)$. Then, $\exists x \in C_1 \ni y = f(x)$. But $x \notin C_2$ since $f(x) = y \notin f(C_2)$. $\therefore x \in C_1 - C_2$, so $y \in f(C_1 - C_2)$.

9. $x \in f^{-1}(D_1 - D_2) \Leftrightarrow f(x) \in D_1 - D_2 \Leftrightarrow f(x) \in D_1$ and $f(x) \notin D_2 \Leftrightarrow$ $x \in f^{-1}(D_1)$ and $x \notin f^{-1}(D_2) \Leftrightarrow x \in f^{-1}(D_1) - f^{-1}(D_2)$.

11.
$$\begin{aligned}
&y \in f\left(\bigcap_{\lambda\in\Lambda} C_\lambda\right) \Rightarrow \exists x \in \bigcap_{\lambda\in\Lambda} C_\lambda \ni f(x) = y\\
&\Rightarrow \exists x \ni \forall\lambda \in \Lambda,\ x \in C_\lambda \text{ and } f(x) = y\\
&\Rightarrow \forall\lambda \in \Lambda,\ \exists x \ni x \in C_\lambda \text{ and } f(x) = y \quad \text{(Note: this step is not } \Leftrightarrow\text{)}\\
&\Rightarrow \forall\lambda \in \Lambda,\ y \in f(C_\lambda) \quad \Rightarrow \quad y \in \bigcap_{\lambda\in\Lambda} f(C_\lambda).
\end{aligned}$$

13.
$$\begin{aligned}
&x \in f^{-1}\left(\bigcap_{\lambda\in\Lambda} D_\lambda\right) \quad \Leftrightarrow \quad f(x) \in \bigcap_{\lambda\in\Lambda} D_\lambda \quad \Leftrightarrow \quad \forall\lambda \in \Lambda,\ f(x) \in D_\lambda\\
&\Leftrightarrow \forall\lambda \in \Lambda,\ x \in f^{-1}(D_\lambda) \quad \Leftrightarrow \quad x \in \bigcap_{\lambda\in\Lambda} f^{-1}(D_\lambda).
\end{aligned}$$

EXERCISE SET B.3

1. $(f+g)(x) = x^2+2x-1$ $(f-g)(x) = -x^2+2x+3$ $f(x+2) = 2x+5$
$f(x) + 2 = 2x + 3$ $g(x + 2) = x^2 + 4x + 2$ $g(x) + 2 = x^2$
$3f(x) = 6x + 3$ $f(3x) = 6x + 1$ $3g(x) = 3x^2 - 6$ $g(3x) = 9x^2 - 2$
$(fg)(x) = 2x^3 + x^2 - 4x - 2$ $(f/g)(x) = (2x + 1)/(x^2 - 2)$

$|f|(x) = |2x + 1|$ $\max\{f, g\}(x) = \left\{\begin{array}{l} x^2 - 2 \text{ if } x \le -1 \text{ or } x \ge 3 \\ 2x + 1 \text{ if } -1 \le x \le 3 \end{array}\right\}$

$\min\{f, g\}(x) = \left\{\begin{array}{l} 2x + 1 \text{ if } x \le -1 \text{ or } x \ge 3 \\ x^2 - 2 \text{ if } -1 \le x \le 3 \end{array}\right\}$

$(f \circ g)(x) = 2x^2 - 3$ $(g \circ f)(x) = 4x^2 + 4x - 1$

3. (a) $\mathcal{D}(f) = \mathcal{R}(f) = \mathbb{R}$; 1-1 (b) $\mathcal{D}(f) = [-1, \infty)$; $\mathcal{R}(f) = [0, \infty)$; 1-1
(c) $\mathcal{D}(f) = (-\infty, -1] \cup [1, \infty)$; $\mathcal{R}(f) = [0, \infty)$; not 1-1
(d) $\mathcal{D}(f) = (0, \infty)$; $\mathcal{R}(f) = \mathbb{R}$; 1-1 (e) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = (0, \infty)$; 1-1
(f) $\mathcal{D}(f) = (-\infty, -1) \cup (-1, \infty)$; $\mathcal{R}(f) = (-\infty, 1) \cup (1, \infty)$; 1-1
(g) $\mathcal{D}(f) = \mathbb{R}$; $\mathcal{R}(f) = [-1, 1]$; not 1-1 (h) $\mathcal{D}(f) = \mathcal{R}(f) = \mathbb{R}$; 1-1

5. Let $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$, $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$ and $(fg)(x) = f(x)g(x) = g(x)f(x) = (gf)(x)$. $\therefore f+g = g+f$ and $fg = gf$.

7. Let $f, g \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$, $[r(f+g)](x) = r[(f+g)(x)] = r[f(x)+g(x)] = rf(x)+rg(x) = (rf)(x)+(rg)(x) = (rf+rg)(x)$. $\therefore r(f+g) = rf+rg$.

9. Let $f \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$, $(1f)(x) = 1 \cdot f(x) = f(x)$. $\therefore 1f = 1$.

11. Let $f, g, h \in \mathcal{F}(\mathcal{S}, \mathbb{R})$. Then, $\forall x \in \mathcal{S}$, $[f(g+h)](x) = f(x)(g+h)(x) = f(x)[g(x)+h(x)] = f(x)g(x)+f(x)h(x) = (fg)(x)+(fh)(x) = (fg+fh)(x)$. $\therefore f(g+h) = fg+fh$.

13. $f(x) = x^2$, $g(x) = x^3$.

15. (a) $\forall a \in A$, $(f \circ i_A)(a) = f\,(i_A(a)) = f(a)$, so $f \circ i_A = f$.
(b) $\forall b \in B$, $(i_A \circ g)(b) = i_A\,(g(b)) = g(b)$, so $i_A \circ g = g$.

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Glossary of Symbols

GREEK ALPHABET

α A	alpha	ν N	nu
β B	beta	ξ Ξ	xi
γ Γ	gamma	o O	omicron
δ Δ	delta	π Π	pi
ε E	epsilon	ρ P	rho
ζ Z	zeta	σ Σ	sigma
η H	eta	τ Υ	tau
θ Θ	theta	υ Y	upsilon
ι I	iota	ϕ, φ Φ	phi
κ K	kappa	χ X	chi
λ Λ	lambda	ψ Ψ	psi
μ M	mu	ω Ω	omega

GENERAL

Symbol	Meaning	Defined on page
$*$	indicates optional material in 1-term course	ix
$\blacksquare$	end of proof	xxi
$\square$	end of example or remark	xxi
$[\cdots]$	reference to item in bibliography	2

LOGIC

Symbol	Meaning	Defined on page
$\wedge, \vee$	and, or	584–585
$\Rightarrow$	implies (if...then...)	586–587
$\Leftarrow$	the converse of $\Rightarrow$	593
$\Leftrightarrow$	if and only if (iff)	587–588
$\sim$	not	588
$\equiv$	is logically equivalent to	590–591
$\forall x$	for all x	597–598
$\exists x \ni$	there exists an x such that	598–599

SETS

Symbol	Meaning	Defined on page
$\in$	belongs to (is a member of)	613
$\{a, b, c, \cdots\}$	set containing $a, b, c, \cdots$	613
$\{x : P(x)\}$	set of all x such that $P(x)$	614
$\cup$, $\cap$	union, intersection	614–617
A^c	complement of A	614–617
$B - A$	complement of A in B	614–617
$\mathcal{U}$	the universal set	614
$\varnothing$	the empty set	614
$\subseteq$	is a subset of	615
$\{A_\lambda : \lambda \in \Lambda\}$	family of sets A_λ, indexed by $\lambda \in \Lambda$	617
$\bigcup_{\lambda \in \Lambda} A_\lambda$	union of sets A_λ, $\lambda \in \Lambda$	617
$\bigcap_{\lambda \in \Lambda} A_\lambda$	intersection of sets A_λ, $\lambda \in \Lambda$	617
$A \cong B$	A and B are equivalent sets.	126
$x + A$	$\{x + a : a \in A\}$	357
xA	$\{xa : a \in A\}$	357
$-A$	$\{-a : a \in A\}$	357
$A + B$	$\{a + b : a \in A,\ b \in B\}$	358

FUNCTIONS

Symbol	Meaning	Defined on page
$f : A \to B$	f is a function from A to B	619
$\mathcal{D}(f)$, $\mathcal{R}(f)$	domain of f, range of f	619
$f(C)$	$\{f(x) : x \in C\}$	188, 620
$f^{-1}(D)$	$\{x : f(x) \in D\}$	188, 620
$\mathcal{F}(\mathcal{S}, \mathbb{R})$	{all functions $f : \mathcal{S} \to \mathbb{R}$}	541, 625
$f \pm g$, rf, fg, f/g	algebraic combinations of f and g	626
$\|f\|$	absolute value of a function	626
$\min\{f, g\}$, $\max\{f, g\}$	minimum (maximum) of f and g	626, 627
$g \circ f$	composite of f and g	629
i_A	identify function on A	631
f^{-1}	inverse function of f	631

THE REAL NUMBER SYSTEM

Symbol	Meaning	Defined on page
x^{-1}	multiplicative inverse of x	6
$\mathcal{P}$	set of all positive elements of an ordered field	11
$<$, $>$, $\leq$, $\geq$	less than, greater than, etc.	11
$\|x\|$	absolute value of x	16
$[a, b]$, (a, b), etc.	(bounded) intervals	18, 19
$(-\infty, a)$, $[b, +\infty)$, etc.	(unbounded) intervals	18, 19
$\mathbb{N}_F$	set of all natural numbers of ordered field F	21
$n!$	n factorial	27
$\binom{n}{k}$	binomial coefficient, for $0 \leq k \leq n \in \mathbb{N}$	27
$\mathbb{Z}_F$	set of all integers of ordered field F	28
$\mathbb{Q}_F$	set of all rational numbers of ordered field F	28
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$	natural numbers, integers, rational numbers	4, 24, 29
$\min A$, $\max A$	minimum and maximum elements of A	35
$\sup A$	least upper bound of A	37
$\inf A$	greatest lower bound of A	37
$\mathbb{R}$	set of all real numbers	42
$+\infty$, $-\infty$	sup or inf of unbounded sets	43
e	$\lim\limits_{n\to\infty} (1 + 1/n)^n$, often called Euler's number	96
π	$2 \sin^{-1} 1$	429
γ	Euler's constant	475

SEQUENCES

Symbol	Meaning	Defined on page
$\{x_n\}$	a sequence of real numbers	50
$\lim_{n\to\infty} x_n = L$	The sequence $\{x_n\}$ has limit L.	52
$x_n \to L$	The sequence $\{x_n\}$ converges to L.	52
T_m	m-tail of a sequence $\{x_n\}$	69
$\lim_{n\to\infty} x_n = \pm\infty$	The sequence $\{x_n\}$ has limit $+\infty$ or $-\infty$.	82
$x_n \to \pm\infty$	The sequence $\{x_n\}$ diverges to $+\infty$ or $-\infty$.	82
$\underline{\lim}_{n\to\infty} x_n$, $\overline{\lim}_{n\to\infty} x_n$	lower and upper limits of $\{x_n\}$	131, 132

TOPOLOGY OF $\mathbb{R}$

Symbol	Meaning	Defined on page
$N_\varepsilon(x)$	ε-neighborhood of x	138
A^o, A^{ext}, A^b	interior, exterior, and boundary of A	141–143
$\overline{A}$, A^{cl}	closure of A	149
A'	set of all cluster points of A	150
$d(A)$	diameter of A	163
$\mu(A)$	measure of A	171
$\mathcal{M}$	class of all μ-measurable sets	171

LIMITS OF FUNCTIONS

Symbol	Meaning	Defined on page
$\lim_{x\to x_0} f(x) = L$	f has limit L as x approaches x_0.	178
$N'_\varepsilon(x_0)$	deleted ε-neighborhood of x_0	188
$\lim_{x\to x_0^+} f(x)$ or $f(x_0^+)$	limit of f as x approaches x_0 from the right	203, 237
$\lim_{x\to x_0^-} f(x)$ or $f(x_0^-)$	limit of f as x approaches x_0 from the left	203, 237
$\lim_{x\to x_0} f(x) = +\infty$	f has limit $+\infty$ as x approaches x_0.	209
$\lim_{x\to x_0} f(x) = -\infty$	f has limit $-\infty$ as x approaches x_0.	209
$\lim_{x\to +\infty} f(x) = L$	f has limit L as x approaches $+\infty$.	217
$\lim_{x\to -\infty} f(x) = L$	f has limit L as x approaches $-\infty$.	217

CONTINUOUS FUNCTIONS

Symbol	Meaning	Defined on page
sgn (x)	signum function	227
$T(x)$	Thomae's function	229
$\lfloor x \rfloor$	greatest integer (floor) function	241
$\chi_A(x)$	characteristic function of (the set) A	244
$f\|_A$	f restricted to the set A	246
$\sqrt[n]{x}$	unique nonnegative n^{th} root of $x \geq 0$	253
φ	Cantor's function	270–273
a^x	a^x for $a > 1$ and $x \in \mathbb{R}$	280
x^t	x^t for $x \in \mathbb{R}$ and $t > 0$	283
$\log_a x$	$\log_a x$ for $a, x > 0$	288
$\psi_f(A)$	oscillation of f on the set A	292
$\psi_f(x)$	oscillation of f at x	292
F_σ-set	a union of countably many closed sets	293

DIFFERENTIABLE FUNCTIONS

Symbol	Meaning	Defined on page
$f'(x_0)$	derivative of f at x_0	297
$f'_-(x_0)$, $f'_+(x_0)$	derivative of f from the left (right) at x_0	302
$D_x f(x)$, $\frac{df(x)}{dx}$, $\frac{d}{dx} f(x)$	alternative notation for the derivative of f	306
y', $\frac{dy}{dx}$, $\frac{d}{dx} y$	alternative notation for the derivative of f	306
$f^{(k)}(x)$	nth derivative of f at x	329
$T_n(x)$	nth Taylor polynomial for f	$329, 519$
$R_n(x)$	nth Taylor remainder for f	$329, 519$

THE RIEMANN INTEGRAL

Symbol	Meaning	Defined on page
$\mathcal{P}$	partition of $[a,b]$	360
m_i	$\inf\{f(x) : x \in [x_{i-1}, x_i]\}$	361
M_i	$\sup\{f(x) : x \in [x_{i-1}, x_i]\}$	361
$\triangle_i$	$x_i - x_{i-1}$	361
$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} m_i \triangle_i$	lower Darboux sum of f over $\mathcal{P}$	361
$\overline{S}(f,\mathcal{P}) = \sum_{i=1}^{n} M_i \triangle_i$	upper Darboux sum of f over $\mathcal{P}$	361
$\underline{\int}_a^b f,\ \overline{\int}_a^b f$	lower (upper) Darboux integrals of f over $[a,b]$	363
$\int_a^b f$	Riemann integral of f over $[a,b]$	364
$\Vert\mathcal{P}\Vert$	mesh of the partition $\mathcal{P}$	372
$\mathcal{P}^*$	tagged partition of $[a,b]$	374
$R(f,\mathcal{P}^*) = \sum_{i=1}^{n} f(x_i^*)\triangle_i$	Riemann sum of f over tagged partition $\mathcal{P}^*$	374
$\mathcal{Q}_n$	regular partition of $[a,b]$ into n subintervals	379
$j(f,x_0)$	jump of f at x_0	392
$\int_a^{+\infty} f,\ \int_{-\infty}^{b} f,\ \int_{-\infty}^{+\infty} f$	(improper) integrals of f over infinite intervals	438, 439

SERIES OF REAL NUMBERS

Symbol	Meaning	Defined on page
$\sum_{k=1}^{\infty} a_k \ (= S)$	an infinite series of numbers with sum S	453
$S_n = \sum_{k=1}^{n} a_k$	n^{th} partial sum of $\sum_{k=1}^{\infty} a_k$	453
$a_n^+,\ a_n^-$	$\max\{a_n, 0\},\ \max\{-a_n, 0\}$	480
$\overrightarrow{x} = (x_1, x_2, \cdots x_n)$	an n-vector	498
$\mathbb{R}^n$	Euclidean n-space	498
$\overrightarrow{x} \cdot \overrightarrow{y}$	dot product of $\overrightarrow{x}$ and $\overrightarrow{y}$	499
$\sum_{k=1}^{\infty} a_k (x-c)^k$	a power series in $(x-c)$	504
ρ	radius of convergence of a power series	506
$\binom{\alpha}{k}$	binomial coefficient for arbitrary $\alpha \in \mathbb{R},\ n \in \mathbb{N}$	523
$\sum_{i,j=1}^{\infty} a_{ij}$	a double series	529

SEQUENCES AND SERIES OF FUNCTIONS

Symbol	Meaning	Defined on page
$\mathcal{F}(\mathcal{S}, \mathbb{R})$,	set of all functions $f : \mathcal{S} \to \mathbb{R}$	541, 625
$B(\mathcal{S})$	set of all bounded functions on $[a, b]$	541
$C(\mathcal{S})$	set of all continuous functions on $[a, b]$	541
$D(\mathcal{S})$	set of all differentiable functions on $[a, b]$	541
$C^k(\mathcal{S})$	set of all f for which $f^{(k)}$ is continuous on $[a, b]$	541
$C^\infty(\mathcal{S})$	set of all $f \ni \forall k \in \mathbb{N}$, $f^{(k)}$ is continuous on $[a, b]$	541
$R[a, b]$	set of all f that are Riemann integrable on $[a, b]$	541
$\{f_n\}$	a sequence of functions	541
$\lim\limits_{n\to\infty} f_n = f$	$\{f_n\}$ converges pointwise to f	541
$f_n \to f$	$\{f_n\}$ converges pointwise to f	541
$\|f\|$	sup norm of f	548
$d(f, g)$	$\|f - g\|$, the distance between f and g	548
$\zeta(x)$	Riemann's zeta-function	568
$P[a, b]$	set of all polynomials on $[a, b]$	573
$CAP[a, b]$	all continuous f approximable by polynomials on $[a, b]$	573
$(x - c)^+$	$\max\{0,\ x - c\}$	575

Index

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